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Équations aux dérivées partielles à conditions initiales aléatoires

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Résumé

Cette thèse porte sur des équations aux dérivées partielles hamiltoniennes à conditions initiales aléatoires. En effet, on étudie ici l'évolution de certaines mesures à travers le flot de telles équations. Cette étude suit deux axes.

Premièrement, on considère le caractère globalement bien posé de l'équation d'onde non linéaire quand la donnée initiale est de faible régularité. Cette donnée initiale est une variable aléatoire et on obtient le caractère globalement bien posé de façon presque sûre par rapport à la mesure induite par cette variable. La faible régularité fait référence à l'espace auquel appartient les valeurs de la variable aléatoires et dénote une régularité moins contraignante que celle requise par la théorie déterministe.

Dans certaines conditions, des propriétés d'invariance de la loi de la donnée initiale sont nécessaires à la démonstration du caractère bien posé. C'est pourquoi le deuxième axe comprend la question de l'invariance de mesures et leurs stabilités à travers le flot d'EDPs.

On donne ainsi une loi invariante à travers le flot de l'équation d'onde cubique et une autre à travers celui de l'équation de Benjamin-Bona-Mahony (BBM). la mesure invariante pour BBM est telle que les amplitudes associées à chaque longueur d'onde de la solution sont des variables aléatoires indépendantes les unes des autres. On considère alors la stabilité de l'invariance pour BBM lorsqu'on ajoute des corrélations entre ces amplitudes.

Enfin, en s'inspirant de la littérature physique à propos de la turbulence faible, on s'est demandé ce qu'il advenait de l'indépendance entre les amplitudes dans un contexte plus général. Plus précisément, on a cherché à si les covariances des amplitudes restent petites lorsque celles-ci sont initialement indépendantes et que le terme non quadratique de l'énergie associée à l'équation étudiée est très petit devant l'énergie totale.

Abstract

This thesis is about Hamiltonian partial differential equations with random initial data. Indeed, the evolution of particular measures are studied here through the flow of such equations. This study is done along two axis.

First, the global well-posedness with initial data with low regularity is considered for the non linear wave equation. The initial datum is a random variable and the global well-posedness is obtained almost surely with regard to the measure induced by this variable. The low regularity refers to the space which the values of the random initial datum belong to and means a regularity under the one given by deterministic theory.

Some properties of invariance of the law of the initial datum are required in the proof of the global well-posedness under certain conditions. Hence, the second axis is the invariance of measures through the flow of PDEs and their stability.

An invariant law is given for the cubic non linear wave equation and for the Benjamin-Bona-Mahony equation (BBM). The invariant measure for BBM is such that the amplitudes associated to each wavelength

of the solution are random variables independent from each other. The stability of the invariance for BBM is considered when one adds correlations between these amplitudes.

Finally, inspired by the Physics literature about wave turbulence, the stability of the independence between the amplitudes is investigated about. Namely, we tried to know if the covariances of the amplitudes remain small when they are initially independent and when the quadratic term of the energy associated to the equation is small compared to the total energy.

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Chapitre 1

Introduction

Cette introduction comporte cinq sections. La première regroupe quelques motivations physiques et mathématiques à l'étude de l'aléa dans l'analyse des équations aux dérivées partielles. La seconde présente des résultats d'existence et unicité de solutions, ainsi que de scattering relatifs à l'équation d'onde non linéaire. La troisième résume des techniques concernant l'invariance de statistique par le flot d'EDPs hamiltoniennes. La quatrième présente un résultat de stabilité pour une mesure invariante par le flot de l'équation de Benjamin-Bona- Mahony. Enfin, la cinquième introduit un problème apparaissant dans la théorie de la turbulence faible statistique et s'intéressant aux corrélations impliquées dans l'étude de l'évolution de statistiques.

1.1 Introduction générale

Cette thèse porte sur l'étude de l'évolution de statistiques, ou de mesures sur des espaces de dimension infinie par le flot d'équations aux dérivées partielles hamiltoniennes.

1.1.1 Motivations physiques

Plusieurs aspects de la physique mettent en exergue l'utilisation de statistiques ou de probabilités. D'une façon générale, on peut considérer que dès lors qu'un système physique est constitué d'un grand nombre de particules, l'approche statistique est pertinente.

Considérons tout d'abord quelques aspects de la physique statistique, ou thermodynamique statistique. On y distingue la notion de micro-état et de macro-état. Un micro-état décrit le système étudié dans son intégralité, c'est-à-dire, l'état de chaque particule du système est déterminé. À l'inverse, un macro-état ne décrit que certains paramètres macroscopiques, tels que la température, l'énergie, la pression totale, etc. Ces grandeurs sont équivalentes à une certaine moyenne sur toutes les particules. Un exemple notable en est le calcul de l'énergie cinétique totale d'un gaz parfait via la statistique de Maxwell-Boltzmann. On cherche alors à connaître l'évolution de ces paramètres sans avoir à connaître l'évolution du micro-état du système. Un des enjeux de la thermodynamique est de déterminer le macro-état du système à l'équilibre, on parle alors de physique statistique à l'équilibre. On se base sur le postulat suivant : dans un système isolé à l'équilibre, chacun des micro-états possibles se réalise avec la même probabilité.

La formulation de cette théorie passe par les notions d'ensembles statistiques (introduits par Gibbs en 1878) : il s'agit de variables aléatoires décrivant le système. On distingue l'ensemble micro-canonique, canonique, et grand-canonique. Plus particulièrement, l'ensemble canonique est utilisé dans la cas d'un système fermé en équilibre thermique avec son environnement, le seul flux d'énergie possible avec l'extérieur est un flux d'énergie thermique. Dans un tel système, la probabilité d'être dans un état i est donnée par

$$p_i = \frac{1}{Z(\beta)} e^{-\beta E_i}$$

où E_i est l'énergie de l'état i, β est l'inverse de la température et $Z(\beta)$ est un facteur de renormalisation.

Les mesures de Gibbs sont une généralisation de la notion d'ensemble canonique. Pour un système donné, la mesure de Gibbs est l'unique mesure qui maximise l'entropie :

$$S = -\sum p_i \ln p_i$$

à énergie totale fixée,[38]. Dans le cadre précédent et à température donnée, il s'agit bien de :

$$p_i = \frac{1}{Z(\beta)} e^{-\beta E_i} \ .$$

Pour un système hamiltonien, on constate que cette mesure est invariante dans le temps, puisque l'énergie d'un tel système est constante.

La mesure de Gibbs correspond ainsi à l'équilibre thermodynamique d'un système.

Néanmoins, elle n'est pas nécessairement l'unique mesure invariante dans le temps, ou du moins, l'unique mesure dont certaines caractéristiques sont préservées. En particulier, dans le cadre de la turbulence faible statistique, on observe d'autres mesures présentant une notion d'invariance par le flot d'équations d'hydrodynamique, telle que l'équation d'ondes capillaires,[56]. Les ensembles statistiques sont des distributions de probabilité sur la vitesse d'un fluide. On parle alors d'équilibre statistique. Ces mesures ne caractérisent pas l'équilibre thermodynamique dans le sens où elles ne maximisent pas l'entropie, allant ainsi à l'encontre du deuxième principe de la thermodynamique.

Soyons plus précis. En turbulence faible, le système considéré est représenté comme une fonction d'onde décomposable sur chacun de ses modes de Fourier, et dont l'amplitude de chaque mode de Fourier est une variable aléatoire, toutes ces variables étant liées les unes aux autres par l'équation d'évolution du système. On parle d'équilibre statistique quand la moyenne de chaque amplitude au carré est invariante dans le temps. Si u est la fonction d'onde du système, on a donc

$$u(t) = \sum_n A_n(t) e_n$$

où A_n est l'amplitude du mode e_n et vérifie pour tout n

$$\dot{E}(|A_n(t)|^2) = 0$$

avec E l'espérance.

L'introduction d'aléa dans les modèles physiques peut recouvrir d'autres formes. En particulier, lorsqu'on considère l'évolution d'une particule ou la propagation d'une onde dans un médium. Dans un médium avec impuretés, on peut modéliser la présence de celles-ci de façon aléatoire, exprimant ainsi l'incertitude de l'observateur vis-à-vis de leur nature ou de leur position. Cela génère des équations présentant un potentiel aléatoire et les solutions de telles équations sont bien des variables aléatoires. On trouve de telles équations dans les cadres des condensats de Bose-Einstein (Schrödinger non-linéaire,[23]), de la propagation d'onde dans des fluides de compressibilité variable ou inconnue (Burgers,[4]), et autres. D'autres modèles apparaissent avec l'introduction de bruits blanc. Dans le cadre d'équations paraboliques, on peut trouver les travaux de Kuksin et Shirikyan, [36, 37].

1.1.2 Motivations mathématiques

L'objet de cette thèse est d'étudier des équations aux dérivées partielles dont la condition initiale est une variable aléatoire. Deux optiques sont à envisager.

La première consiste à trouver presque sûrement des solutions quand la condition initiale est moins régulière que ce qu'impose la théorie déterministe. On s'est inspiré des travaux de Burq et Tzvetkov, [15, 16, 17]. En notant $H^s(G)$ l'espace de Sobolev de la variété riemannienne G, c'est-à-dire l'espace topologique sur les distributions induit par la norme

$$||.||_{H^s} = ||(1-\triangle_G)^{s/2}\ .\ ||_{L^2}$$

où $\triangle_G$ est l'opérateur de Laplace-Beltrami sur G, ou encore l'espace des distributions s fois dérivables dans L^2, la théorie déterministe peut fournir des solutions à une équation $*$ quand la condition initiale u_0 est dans H^s avec $s \geq s_0$ ou $s > s_0$. On voudrait alors montrer qu'on peut trouver des solutions quand la condition initiale est dans un H^σ, avec $\sigma < s_0$, c'est-à-dire en imposant moins de conditions de régularité à la donnée initiale.

On veut donc trouver une variable aléatoire u_0 sur un espace de probabilité $\mathbb{F}$ telle que pour presque tout $\omega \in \mathbb{F}$:

- il existe une unique (dans un sens à déterminer) solution à l'équation * avec donnée initiale $u_0(\omega)$,
- $u_0(\omega)$ appartient à H^σ, $\sigma < s_0$,
- $u_0(\omega)$ n'appartient pas à H^{s_0}.

Plusieurs idées permettent d'envisager l'existence de telles variables aléatoires. Tout d'abord, certaines variables aléatoires permettent de gagner en intégrabilité si ce n'est en régularité. En effet, si u_0 est un vecteur gaussien à valeurs dans H^σ, il sera presque sûrement dans un $W^{\sigma',p}$ (les fonctions σ' fois dérivables dans L^p) permettant plus de liberté sur p qu'une injection de Sobolev. Dans certains cas, on a même $\sigma = \sigma'$, et $p \in [1, \infty[$. Cette première étape permet de résoudre le problème de Cauchy localement en temps. En effet, en séparant la solution u de * en une partie linéaire $S(t)u_0$ et une partie non linéaire v, on résoudra un problème de point fixe sur v dans des espaces de régularité en accord avec la théorie déterministe, mais non pas en imposant à la partie linéaire d'être très régulière mais très intégrable. Or, on imagine que la loi de $S(t)u_0$ présente des similarités avec celle de u_0.

Ensuite, une des possibilités pour montrer l'existence et l'unicité de solutions globales passe par l'utilisation de mesures au moins formellement invariantes par des équations hamiltoniennes. En effet, pour une équation hamiltonienne :

$$u_t = J \nabla_u H(u)$$

avec J un opérateur antisymétrique et H une fonctionnelle s'écrivant $H_K + H_p$, avec H_k l'énergie cinétique, quadratique, ie de de la forme

$$H_K = \langle u \ , \ Ku \rangle$$

où K est un opérateur auto adjoint sur L^2, la mesure de Gibbs, qui est une renormalisation de

$$e^{-H(u)} dL(u)$$

avec L la mesure de Lebesgue sur un espace approprié est formellement invariante par le flot de l'équation, puisque $H(u)$ est lui-même invariant. Notons que l'on a besoin d'un cadre hamiltonien pour que la mesure de Lebesgue soit invariante (théorème de Liouville). Cette mesure est dans certains cas la mesure image d'une variable aléatoire à valeurs dans un espace de faible régularité. On pose pour cela $(g_n)_n$ une suite de gaussiennes réelles centrées normalisées indépendantes et $(e_n)_n$ une base de L^2 formée de vecteurs propres de l'opérateur K d'énergie cinétique associés aux valeurs propres $\lambda_n > 0$, c'est-à-dire

$$H_K(u) = \langle u, Ku \rangle_{L^2} = \sum_n \lambda_n |\langle u, e_n \rangle|^2 \ .$$

La variable aléatoire qui nous intéresse est donnée par :

$$u_0 = \sum_n \lambda_n^{-1} g_n e_n \ .$$

En effet, c'est une variable gaussienne de matrice de covariance K^{-1}. En posant μ la mesure induite sur L^2 par une telle variable, on a de façon heuristique que μ est une renormalisation de

$$e^{-\langle u, Ku \rangle} dL(u) = e^{-H_K(u)} dL(u)$$

puis en multipliant cette mesure par $e^{-H_p(u)}$, on obtient la mesure de Gibbs.

En utilisant une telle mesure comme condition initiale, on peut alors contrôler la norme de la solution à des temps discrets sans nuire à la valeur de la mesure de l'espace sur lequel on propage la solution de proche en proche grâce à la théorie locale, et ainsi obtenir presque sûrement des solutions globales. Autrement dit, pour un $R > 0$, et des suites R_n et $T_n \to \infty$ appropriées (qui dépendent de R), en posant

$$\begin{aligned} E_n(R) = \{\omega \mid & \text{ la solution } u \text{ avec condition initiale } u_0(\omega) \text{ existe} \\ & \text{et est unique jusqu'au temps } T_n \text{ et } \|u(T_n)\| \leq R_n\} \end{aligned}$$

on peut propager la solution de proche en proche et vérifier que

$$E = \bigcup_R \bigcap_n E_n(R)$$

est de mesure pleine, d'où l'existence et l'unicité presque sûre de solutions

Par ailleurs, pour étudier l'équation d'onde sur $\mathbb{R}^3$, on utilise une transformation conforme qui rend compact l'espace de phase, ce qui fournit de surcroît des résultats de scattering.

Comme il a été mentionné, on peut trouver des mesures formellement invariantes par des équations hamiltoniennes. Un autre objectif est alors d'étudier de tels objets. Tout d'abord, il s'agit de lever le caractère formel de cette invariance, dans l'esprit du travail de Bourgain [7, 8, 9]. Pour des équations en dimension finie, l'invariance s'obtient en utilisant le théorème de Liouville (qui assure l'invariance de la mesure de Lebesgue) puis l'invariance de l'énergie. En effet, en dimension finie, la mesure de Lebesgue et l'énergie sont bien définies, ce qui permet de dériver l'invariance de façon effective et non formelle. Il s'agit alors d'approcher l'équation en dimension infinie par des équations en dimensions finies telles que les flots des équations finies convergent uniformément sur tout compact de l'espace de dimension infinie.

Une fois l'invariance prouvée, on peut se poser la question de la stabilité de telles mesures.

Ici, la stabilité est étudié de deux points de vue différents. Dans le cas de l'équation de Benjamin-Bona-Mahony (BBM), il existe une mesure invariante qui prend la forme d'un vecteur gaussien sur $H^{1/2-}$ dont la matrice de covariance est l'inverse de $1 - \triangle$. Afin de préserver le cadre gaussien, on a modifié la mesure en modifiant la matrice de covariance avec un paramètre de perturbation V. On a alors étudié la loi de la mesure à travers le flot de BBM et constaté qu'elle restait "proche" de sa valeur initiale, dans le sens où la différence de leur fonctionnelles caractéristiques était borné par $|V|$, bien que l'estimée en temps soit exponentielle. L'idée était d'ajouter des corrélations entre les amplitudes associées aux différentes longueurs d'onde. En effet, l'opérateur linéaire correspondant à la linéarisation en 0 de BBM admet les mêmes fonctions propres que la matrice de covariance de la mesure invariante. Les différentes longueurs d'ondes, correspondant à ces fonctions propres restaient donc indépendantes lorsqu'on utilisait la mesure invariante. En modifiant la matrice de covariance (en la dé-diagonalisant) on ajoute donc des corrélations entre les différents modes propres de l'équation. Étudier l'évolution revient alors en partie à étudier l'évolution des corrélations entre les modes propres de l'équation.

La stabilité de l'indépendance des modes propres est alors entrée en jeu. En effet, si l'on considère une équation linéaire :

$$iu_t = Hu$$

avec H un opérateur linéaire hermitien diagonalisable de vecteurs propres e_n et de valeurs propres λ_n, on constate que la loi de

$$u_0(\omega) = \sum_n g_n(\omega) e_n$$

est invariante par le flot

$$S(t)(u_0(\omega)) = \sum_n e^{-i\lambda_n t} g_n(\omega) e_n$$

dès lors que les g_n sont des i.i.d. dont la loi est invariante par multiplication par un complexe de module 1. Les modes propres de cette équation sont indépendants les uns des autres. On a alors cherché à étudier l'évolution de la covariance entre deux modes propres lorsqu'une non linéarité quadratique est ajoutée à l'équation. Pour être exacte, cette étude a été réalisée dans un cadre réel et non complexe mais l'idée reste la même. On peut voir l'étude complexe comme une étude dans le cadre réel en doublant la dimension quoi qu'il en soit. En plaçant un paramètre de perturbation ε devant la non-linéarité, on constate que la correction apportée aux covariances est d'ordre ε^2 et non ε comme on aurait pu l'imaginer.

1.2 Caractère bien posé de l'équation d'onde non linéaire

Cette partie de l'introduction présente les chapitre 2,3.

Les questions de problème de Cauchy bien-posé pour des données initiales étant des variables aléatoires peu régulières ont été initiées par Burq et Tzvetkov, [15, 16], mais on peut aussi remarquer les travaux de Nahmod, Pavlović, et Staffilani, [44] et de Colliander et Oh, [22].

Nous nous sommes intéressés au problème de Cauchy pour l'équation d'onde non linéaire :

$$\begin{cases} (\partial_t^2 - \triangle_G)u + |u|^\alpha u = 0 & \\ u(t=0) = u_0 & \partial_t u(t=0) = u_1 \end{cases} \tag{1.1}$$

où G est soit la sphère de dimension 3, S^3, soit l'espace euclidien $\mathbb{R}^3$, $\alpha \in [2,3[$ et u_0 et u_1 sont deux variables aléatoires à définir plus tard.

On a distingué deux cas : celui où la condition initiale présente une symétrie (zonale pour la sphère, sphérique pour $\mathbb{R}^3$), et le cas général. Dans le cas général, il existe une base de $L^2(S^3)$ de fonctions propres du laplacien uniformément bornées dans L^p quel que soit p. Ce n'est pas vrai dans le cas à symétrie. Cela permet de donner des propriétés sur une variable aléatoire correctement choisie que l'on n'a pas dans le cas symétrique. Aussi, les stratégies utilisées sont différentes. Ces travaux sont largement inspirés de ceux de Burq et Tzvetkov, [15, 16, 17].

Remarquons que cette équation admet pour invariant, en notant $\Psi(t)(u_0,u_1) = (u(t), \partial_t u(t))$:

$$E(\Psi(t)(u_0,u_1)) = \frac{1}{2}\int (\partial_t u)^2 + \frac{1}{2}\int |\nabla u|^2 + \frac{1}{\alpha+2}\int |u|^{\alpha+2}$$

$$= E(u_0,u_1) = \frac{1}{2}\int (u_1)^2 + \frac{1}{2}\int |\nabla u_0|^2 + \frac{1}{\alpha+2}\int |u_0|^{\alpha+2} .$$

1.2.1 Cas symétrique

Nous présentons ici le chapitre 2.

Sur la sphère Décrivons les objets mis en jeu dans ce cadre.

Sur la sphère, on remplace $\triangle$ par $\triangle - 1$ de façon à pouvoir traiter la fréquence 0. L'énergie devient :

$$E(v_0,v_1) = \frac{1}{2}\left(\|\nabla v_0\|_{L^2}^2 + \|v_0\|_{L^2}^2 + \|v_1\|_{L^2}^2\right) + \frac{1}{\alpha+2}\|v_0\|_{L^{\alpha+2}}^{\alpha+2} .$$

Tout d'abord, une distribution u de S^3 est dite à symétrie zonale si elle ne dépend que de la distance à un pôle de la sphère. Si on note (θ, φ, R) les coordonnées sphériques de S^3 alors u ne dépend que de R. On dispose alors d'une base orthogonale des fonctions L^2 à symétrie sphérique :

$$e_n(R) = \frac{1}{\sqrt{2}\pi} \frac{\sin(nR)}{\sin(R)}$$

étant également des vecteurs propres de l'opérateur de Laplace-Beltrami de S^3 :

$$\triangle_{S^3} e_n = -n^2 + 1 \ .$$

Notons que les normes L^p de ces fonctions vérifient :

$$\|e_n\|_{L^p} \leq \begin{cases} C_p & \text{si } \ p < 3 \\ C \ln n & \text{si } p = 3 \\ C_p n^{1-3/p} & \text{sinon,} \end{cases}$$

ce qui n'assure pas une borne uniforme quand p est supérieur à 3.

Soit $(h_n)_n$ et $(l_n)_n$ deux suites de variables aléatoires sur un espace de probabilité $\mathbb{F}$ réelles indépendantes de loi normale $\mathcal{N}(0, 1/2)$.

On pose alors

$$u_0 = \sum_{n \geq 1} \frac{h_n}{|n|} e_n \text{ et } u_1 = \sum_n l_n e_n \ .$$

Ces séries convergent dans $L^2(\mathbb{F}, H^{1/2-})$ et $L^2(\mathbb{F}, H^{-1/2-})$. Elles induisent une mesure de probabilité μ sur $H^{1/2-} \times H^{-1/2-}$. Les espaces $H^{1/2-}$ et $H^{-1/2-}$ désignent :

$$H^{1/2-} = \bigcup_{s<1/2} H^s \text{ et } H^{-1/2-} = \bigcap_{s<-1/2-} H^s \ .$$

Notons que μ est formellement égal à

$$d\mu(v_0, v_1) = e^{-\frac{1}{2}(\|\nabla v_0\|^2_{L^2} + \|v_0\|^2_{L^2} + \|v_1\|^2_{L^2})} dL(v_0) \otimes dL(v_1) \ .$$

En effet, μ est un vecteur gaussien de matrice de covariance $(-\triangle)^{-1} \oplus Id$, $-(\triangle)^{-1}$ agissant sur la première composante v_0 et Id sur la seconde v_1. En multipliant μ par $e^{-\frac{1}{\alpha+2}\|v_0\|^{\alpha+2}_{L^{\alpha+2}}}$, on forme la mesure

$$d\rho(v_0, v_1) = e^{-E(v_0, v_1)} dL(v_0, v_1)$$

où E est l'énergie initiale de l'équation d'onde avec pour donnée initiale (v_0, v_1). La mesure ρ est formellement invariante par le flot de l'équation d'onde dans le sens où en appelant $\Psi(t)(v_0, v_1) = (v(t), \partial_t v(t))$ le flot, on a

$$d\rho(\Psi(t)(v_0, v_1)) = e^{-E(\Psi(t)(v_0, v_1))} dL(\Psi(t)(v_0, v_1)) = e^{-E(v_0, v_1)} dL(v_0, v_1) = d\rho(v_0, v_1)$$

puisque la mesure de Lebesgue est invariante par les flots hamiltoniens et l'énergie ne dépend pas du temps.

En utilisant cette mesure ρ, on obtient l'existence d'un ensemble $E \subseteq H^{1/2-} \times H^{-1/2-}$ de fonctions à symétrie zonale tel que :

- E est de mesure ρ pleine : $\rho(E^c) = 0$,
- quel que soit $(v_0, v_1) \in E$, v_0, v_1 n'appartient pas à $H^{1/2} \times H^{-1/2}$,
- le problème de Cauchy $v+\partial_t^2 v - \triangle_{S^3} v + |v|^{\alpha-1} v = 0$ avec condition initiale v_0, v_1 admet une unique solution dans $S(t)(v_0, v_1) + C(\mathbb{R}, H^s)$ pour un $s > 1/2$ où $S(t)$ est le flot de l'équation linéaire $\partial_t^2 - \triangle_{S^3} = 0$.

Remarquons que par des arguments de scaling, on obtient que le problème de Cauchy est critique posé dans H^s lorsque $s = 3/2 - 2/\alpha \leq 1/2$ (pour $\alpha \geq 2$). On a donc ici des résultats de faible régularité surcritique.

On observe que la solution est dans un espace particulier. En effet, on passe par la séparation de la solution en une partie linéaire $S(t)(v_0, v_1)$ et une partie non linéaire. La partie linéaire est moins régulière que la partie non linéaire. Cette partie non linéaire vérifie :

$$w + \partial_t^2 w - \triangle_{S^3} w + |S(t)(v_0, v_1) + w|^\alpha (S(t)(v_0, v_1) + w) - |S(t)(v_0, v_1)|^\alpha S(t)(v_0, v_1) = 0$$

avec condition initiale $(0, 0)$. En utilisant une inégalité de Strichartz, on constate qu'on a existence locale de la solution pour la partie non linéaire dès lors que $S(t)(v_0, v_1)$ appartient à un L^p, où p dépend de s, localement en temps et globalement sur la sphère S^3. Or, cette propriété est vérifiée ρ-presque sûrement grâce à la forme gaussienne de (u_0, u_1).

Pour démontrer l'existence globale, on réduit l'équation à la dimension finie en utilisant le projecteur orthogonal Π_N dans L^2 sur l'espace E_N engendré linéairement par l'ensemble $\{e_n \mid n \leq N\}$. L'équation devient :

$$\partial_t^2 v - \triangle_{S^3} v + \Pi_N (|v|^\alpha v) = 0$$

avec pour condition initiale $(\Pi_N v_0, \Pi_N v_1)$. En modifiant ρ, on obtient une mesure ρ_N sur E_N invariante par le flot de cette équation. On peut aussi montrer que les solutions (globales) v_N de ces ODE convergent uniformément sur tout compact de L^2 et en particulier sur les boules de H^σ vers la solution (locale) de l'équation en dimension infinie et que les mesures ρ_N convergent en un sens vers ρ. En choisissant de contrôler les normes :

$$\|S(t)v_N\|_{L^p}$$

pour tout N à des temps discrets appropriés, on propage la solution en dimension infinie de proche en proche. Grâce à l'invariance des ρ_N par le flot des équations en dimension finie, on obtient un ensemble E de mesure ρ pleine sur lequel on a existence et unicité globale de la solution. L'ensemble E est contenu dans $H^{1/2-} \times H^{-1/2-}$ mais ses éléments ne sont presque sûrement pas dans $H^{1/2} \times H^{-1/2-}$.

Sur l'espace euclidien $\mathbb{R}^3$ Pour étendre les solutions à l'équation d'onde sur $\mathbb{R}^3$, on utilise la transformée de Penrose, qui injecte $\mathbb{R} \times \mathbb{R}^3$ dans $[-\pi, \pi] \times S^3$ par le changement de variables :

$$(t, r, \omega) \mapsto (T = \text{Arctan}\,(t + r) + \text{Arctan}\,(t - r), R = \text{Arctan}\,(t + r) - \text{Arctan}\,(t - r), \omega)\ .$$

Avec $\Omega = \cos R + \cos T = \frac{2}{\sqrt{(1+(t+r)^2)(1+(t-r)^2)}}$, et $f(t, r) = \Omega^{-1} v(T, R)$, si v vérifie l'équation

$$\partial_T^2 v + (1 - \triangle_{S^3}) v + \Omega^{\alpha-2} |v|^\alpha v = 0 \tag{1.2}$$

avec pour condition initiale (v_0, v_1), alors f vérifie

$$\partial_t^2 f - \triangle_{\mathbb{R}^3} f + |f|^\alpha f = 0$$

avec pour condition initiale

$$f_0(r) = \Omega^{-1}(0,r)v_0(R)\ ,\ f_1(r) = \Omega^{-2}(0,r)v_1(R)\ .$$

L'équation ci-dessus (1.2) est certes différente de l'équation d'onde non linéaire mais le même travail peut être fait sur cette équation et on obtient ainsi une mesure ρ' sur les mêmes espaces et un ensemble E' de ρ' mesure pleine dans lequel on a existence et unicité de l'équation. Notons que les fonctions zonales sont envoyées sur des fonctions radiales.

À travers la transformée de Penrose, on obtient une mesure η et un ensemble F de η mesure pleine, dans lequel on a existence de solution pour l'équation d'onde non linéaire. La norme L^p, $p \geq 4$ en temps et en en espace de la transformée f de v est inférieure à la norme L^p de v sur $[-\pi, \pi] \times S^3$. On obtient ainsi que f appartient à $L(t)(f_0, f_1) + C(\mathbb{R}, H^s)$, pour un $s > 1/2$ et où $L(t)$ est le flot de l'équation linéaire $\partial_t^2 - \triangle_{\mathbb{R}^3} = 0$ par des méthodes locales et on obtient l'unicité par récurrence et propagation des estimées sur les normes.

Cette solution satisfait des propriétés de scattering dans les espaces de Lebesgue L^q avec q suffisamment grand.

En étudiant également les propriétés de la mesure η image par Penrose de μ, on obtient le résultat démontré au chapitre 2, théorèmes 1, 2 :

Théorème 1.2.1. *Pour η-presque tout (f_0, f_1),*
- $(f_0, f_1) \in L^p \times W^{-1,p}$ *pour tout* $p \in]2, 6[$,
- (f_0, f_1) *n'appartient pas à* $L^p \times W^{-1,p}$, *pour tout* $p < 2$ *et* $p > 6$,
- f_0 *est localisée en espace, dans le sens où* $|f_0(r)|$ *est un* $O(r^{-(1+\gamma)})$ *pour* $\gamma \in]0, 1/2[$ *quand* r *tend vers* $+\infty$.

On a aussi, à propos de l'étude de l'équation d'onde non linéaire :

Théorème 1.2.2. *Il existe un ensemble F de η mesure pleine tels que :*
- *l'équation d'onde non linéaire admet une unique solution f dans $L(t)(f_0, f_1) + C(\mathbb{R}, H^s)$ pour un $s > 1/2$,*
- $\|f - L(t)(f_0, f_1)\|_{L^q}$ *tend vers* 0 *quand* t *tend vers* $\pm\infty$ *pour tout* $q \in]\max(4, \frac{3\alpha}{2}), \frac{9}{2}[$.

L'espace $W^{-1,p}$ désigne l'espace topologique induit par la norme :

$$\|\ .\ \|_{W^{-1,p}} = \|(1 - \triangle_{\mathbb{R}^3})^{-1}\ .\ \|_{L^p}\ .$$

La non-appartenance aux espaces mentionnés dérivent du théorème de Fernique, [30] et du fait que ρ est absolument continue par rapport à une variable gaussienne :

Théorème 1.2.3 (Fernique)**.** *Soit X un vecteur gaussien sur un espace de Banach B et N une semi-norme sur B, c'est-à-dire une application de B dans $\mathbb{R}^+ \cup \{+\infty\}$, homogène, définie positive et vérifiant l'inégalité triangulaire, si $N(X)$ est finie presque sûrement alors pour tout $p \geq 0$, on a*

$$E(N^p(X)) < \infty \ .$$

Quant à l'appartenance de f_1 à $W^{-1,p}$, $p \in]2, 6[$ et la localisation de f_0, elles sont issues d'une description de la dérivation fractionnaire dûe à Zygmund, [58] comme le produit de convolution avec une fonction régulière. En voyant f_0 et f_1 comme les dérivées fractionnaires de certaines applications, on en déduit des propriétés de régularités sur ces fonctions, qui entraîne l'appartenance aux espaces mentionnés et la localisation de f_0.

1.2.2 Cas non symétrique

Le cas général peut sembler a priori plus difficile à traiter que le cas symétrique. Mais comme il a été dit plus haut, le fait d'avoir "plus" de fonctions propres du laplacien permet d'obtenir une base orthogonale de L^2 constituée de tels fonctions uniformément bornées dans L^p. Ceci a pour conséquence de changer considérablement notre stratégie pour l'existence globale. Nous n'utiliserons plus une invariance de mesure mais une estimée d'énergie. On allégera alors les hypothèses sur la condition initiale, elle n'aura plus besoin d'être une gaussienne mais de vérifier certaines estimées dites de large déviation.

Sur la sphère On utilise un théorème de Burq et Lebeau, [12] :

Théorème 1.2.4 (Burq, Lebeau)**.** *Il existe une base orthogonale de $L^2(S^3)$,*

$$(e_{n,k})_{n \geq 1, 1 \leq k \leq (n+1)^2}$$

telle qu'il existe C_p tel que

$$(1 - \triangle_{S^3}) e_{n,k} = n^2 e_{n,k} \text{ et } \|e_{n,k}\|_{L^p} \leq C_p \ .$$

On pose également $(h_{n,k})_{n,k}$ et $(l_{n,k})_{n,k}$ deux suites d'i.i.d. réelles satisfaisant des estimées gaussiennes de grandes déviations, conséquentes du fait qu'il existe c tel que pour tout $\gamma \in \mathbb{R}$,

$$E(e^{\gamma h_{n,k}}), E(e^{\gamma l_{n,k}}) \leq e^{c\gamma^2} \ .$$

On peut supposer que $E(h_{n,k}^2) = E(l_{n,k}^2) = 1$.

Enfin, on pose $(\lambda_{n,k})_{n,k}$ et $(\mu_{n,k})_{n,k}$ deux suites de réels vérifiant :

$$\sum_{n,k} |n|^{2\sigma} |\lambda_{n,k}|^2 < \infty \text{ et } \sum_{n,k} |n|^{2(\sigma-1)} |\mu_{n,k}|^2 < \infty$$

pour un $\sigma \geq 0$ donné et

$$\sum_{n,k} |n|^{2s} |\lambda_{n,k}|^2 = \infty \text{ et } \sum_{n,k} |n|^{2(s-1)} |\mu_{n,k}|^2 = \infty$$

pour tout $s > \sigma$.

La condition initiale est alors donnée par les variables aléatoires :

$$u_0 = \sum_{n,k} \lambda_{n,k} h_{n,k} e_{n,k} \ , \ u_1 = \sum_{n,k} \mu_{n,k} l_{n,k} e_{n,k} \ .$$

Ces variables appartiennent presque sûrement à H^σ et $H^{\sigma-1}$ respectivement et, dans le cas où les $h_{n,k}$ et $l_{n,k}$ sont des gaussiennes, elles n'appartiennent presque sûrement pas à H^s et H^{s-1} respectivement. Elles induisent une mesure μ sur $H^\sigma \times H^{\sigma-1}$. Le support de cette mesure est dense dans $H^\sigma \times H^{\sigma-1}$, [17], lorsque les coefficients $\mu_{n,k}$ et $\lambda_{n,k}$ sont non nuls et que les distributions de $h_{n,k}, l_{n,k}$ satisfont certaines propriétés vérifiées par exemple par des variables gaussiennes.

Avec une telle mesure initiale on étudie l'équation d'onde non linéaire cubique :

$$\begin{cases} \partial_t^2 v - \triangle_{S^3} v + v^3 = 0 \\ v(t=0) = v_0 \qquad\qquad \partial_t v(t=0) = v_1 \end{cases} . \tag{1.3}$$

On note $U(t)$ le flot de l'équation linéaire $\partial_t^2 - \triangle_{S^3} = 0$.

On a le résultat suivant (qu'on retrouvera au Chapitre 3, Theorem 3.1.1) :

Théorème 1.2.5. *Il existe un ensemble E de μ-mesure pleine tel que pour tout $(v_0, v_1) \in E$:*
- $(v_0, v_1) \in H^\sigma \times H^{\sigma-1}$,
- *l'équation* (1.3) *admet une unique solution dans $U(t) + \mathcal{C}(\mathbb{R}, H^1(S^3))$,*
- *si les $h_{n,k}$ et $l_{n,k}$ sont des gaussiennes, (v_0, v_1) n'est pas dans $H^s \times H^{s-1}$ pour $s > \sigma$.*

L'existence locale vient du fait que $U(t)(u_0, u_1)$ appartient presque sûrement à $L^p(S^3)$ pour $p < \infty$ si $\sigma = 0$ et $p \in [1, \infty]$ si $\sigma > 0$. En effet, en étudiant l'équation que vérifie la partie non linéaire de la solution de (1.3), c'est-à-dire :

$$\partial_t^2 w - \triangle_{S^3} w + (U(t)(v_0, v_1) + w)^3 - (U(t)(v_0, v_1))^3 = 0$$

on obtient l'existence et unicité locale dans $\mathcal{C}([-T, T], H^1(S^3))$ dès lors que
$U(t)(v_0, v_1)$ appartient à $L^3([-T, T], L^6(S^3))$. Or, on a existence presque sûre de $U(t)(u_0, u_1)$ à cet espace pour les raisons suivantes : pour tout couple de suites $((a_{n,k})_{n,k}, (b_{n,k})_{n,k})$ dans l^2, on a

$$|| \sum_{n,k} a_{n,k} h_{n,k} + b_{n,k} l_{n,k} ||_{L_P^q} \leq C \left(q \sum_{n,k} a_{n,k}^2 + b_{n,k}^2 \right)^{1/2}$$

où L_P^q désigne la norme L^q sur l'espace de probabilité.

On en déduit par, entre autres, une inégalité de Minkowski, que pour tout $q \geq p$,

$$||U(t)(u_0, u_1)||_{L_P^q, L^p(S^3)} \leq C|t| \sqrt{q} \left(\sum_{n,k} |\lambda_{n,k}|^2 ||e_{n,k}||_{L^p}^2 + |\frac{\mu_{n,k}}{n}|^2 ||e_{n,k}||_{L^p}^2 \right)^{1/2} < \infty$$

d'où l'intérêt d'avoir des bornes uniformes en n pour les normes L^p des $e_{n,k}$. On a donc que la norme L^p de $U(t)(u_0, u_1)$ est finie presque sûrement.

Lorsque $\sigma > 0$, on utilise une inégalité de Sobolev pour borner presque sûre- ment la norme

$$\|U(t)(u_0, u_1)\|_{L^\infty(S^3)} .$$

On obtient ainsi existence et unicité locale en temps presque sûrement.

Pour l'existence globale, on utilise une inégalité d'énergie sur la partie non linéaire de la solution :

$$\mathcal{E}(w) = \frac{1}{2}\|\partial_t w\|_{L^2}^2 + \frac{1}{2}\| \nabla w\|_{L^2}^2 + \frac{1}{4}\|w\|_{L^4}^4 .$$

On montre par un lemme de Gronwall que cette énergie reste bornée, ce qui assure l'existence.

Remarquons que pour $\sigma > 0$, le fait que $U(t)(u_0, u_1)$ appartienne à L^1 localement en temps et L^∞ en espace permet une preuve relativement rapide tandis le cas $\sigma = 0$ requiert une définition précise de l'ensemble E sur lequel l'équation (1.3) est globalement bien posée ainsi qu'un argument de bootstrap.

Notons que ces arguments sont valables lorsqu'on remplace le laplacien $\triangle_{S^3}$ par $\triangle_{S^3} - 1$.

Sur $\mathbb{R}^3$ Pour prouver l'existence presque sûre de solutions sur $\mathbb{R}^3$ on utilise une fois de plus la transformée de Penrose. On se place dans le cas $\sigma = 0$.

On pose :

$$\mathrm{PT}_0^{-1}(v_0, v_1)(r, \omega) = \Big(\Omega(t = 0, r)v_0(R(t = 0, r), \omega), \Omega^2(t, r)v_1(R(t = 0, r), \omega)\Big) ,$$

$$\mathrm{PT}^{-1}v(t, r, \omega) = \Omega(t, r)v(T, R, \omega) ,$$

de sorte que si v est une solution sur la sphère avec pour condition initiale (v_0, v_1) alors $\mathrm{PT}^{-1}v$ est une solution sur $\mathbb{R}^3$ avec pour condition initiale $\mathrm{PT}^{-1}(v_0, v_1)$.

$$\mathcal{H}_0^s = \{f\ ,\ \|f\|_{\mathcal{H}_0^s} := \|\left(\frac{2}{1 + r^2}\right)^{1/2} (1 - H_0)^{s/2} f\|_{L^2}\}$$

$$\mathcal{H}_1^s = \{f\ ,\ \|f\|_{\mathcal{H}_1^s} := \|\left(\frac{2}{1 + r^2}\right)^{-1/2} (1 - H_1)^{s/2} f\|_{L^2}\}$$

où H_0 et H_1 sont des opérateurs différentiels d'ordre 2 et $\|f\|_{\mathcal{H}_0^s}$,$\|f\|_{\mathcal{H}_1^s}$ sont bien des normes. Ces espaces et opérateur son introduits à cause du changement de variable impliqué dans la transformation de Penrose.

L'application PT_0^{-1} est continue de $L^2 \times H^{-1}$ dans $\mathcal{H}_0^0 \times \mathcal{H}_1^{-1}$. On pose pour tout A borélien de $\mathcal{H}_0^0 \times \mathcal{H}_1^{-1}$,

$$\nu(A) = \mu(\mathrm{PT}_0(A))$$

et

$$F = \mathrm{PT}_0^{-1}(E)$$

de sorte que F est de ν mesure pleine et que pour tout $(f_0, f_1) \in F$, il existe $(v_0, v_1) \in E$ et donc une solution v de

$$\partial_t^2 u + (1 - \triangle_{S^3}) u + u^3 = 0$$

avec pour condition initiale (v_0, v_1). La fonction $f = \mathrm{PT}^{-1} v$ est alors solution de

$$\partial_t^2 f - \triangle_{\mathbb{R}^3} f + f^3 = 0$$

avec pour condition initiale (f_0, f_1) ce qui assure l'existence presque sûre de solutions.

L'unicité vient du fait que la norme L^p de f est bornée par celle de v en temps et en espace dès lors que $p \geq 4$. On a :

$$\|f\|_{L^p(\mathbb{R}\times\mathbb{R}^3)} \leq \|v\|_{L^p([-\pi,\pi]\times S^3)} .$$

On en déduit que f appartient à L^p, puis que f appartient à

$$L(t)(f_0, f_1) + C(\mathbb{R}, H^1(\mathbb{R}^3))$$

où $L(t)$ est le flot de l'équation linéaire $\partial_t^2 - \triangle_{\mathbb{R}^3} = 0$. On obtient l'unicité par un lemme de Gronwall sur l'énergie de la partie non linéaire $f(t) - L(t)(f_0, f_1)$. On a également un résultat de scattering. On obtient le résultat (qu'on retrouvera au Chapitre 3, Theorem 3.1.2) suivant :

Théorème 1.2.6. *Pour tout (f_0, f_1) dans F, on a :*
- $(f_0, f_1) \in \mathcal{H}_0^0 \times \mathcal{H}_1^{-1}$,
- *il existe une unique solution à l'équation d'onde cubique avec pour donnée initiale (f_0, f_1),*
- *pour tout $q \in]\frac{18}{5}, 6]$, la norme L^q de $f(t) - L(t)(f_0, f_1)$ est un $O((1+t^2)^{-1/6})$,*
- *si les $h_{n,k}$ et $l_{n,k}$ sont des gaussiennes alors (f_0, f_1) n'est pas dans $\mathcal{H}_0^s \times \mathcal{H}_1^{s-1}$.*

1.3 Invariance de mesure

On a vu que l'existence et unicité de solutions presque sûre pour l'équation d'onde non linéaire dans le cas à symétrie radiale est en partie dû à l'invariance de mesures en dimension finie. On s'est également demandé comment étendre un résultat d'invariance en dimension finie à la dimension infinie. Remarquons qu'il s'agit d'étendre en dimension infinie l'invariance de la mesure de Gibbs par des équations hamiltoniennes, [8, 7, 42, 47, 43]. Ce travail est inspiré par celui de Burq, Thomann et Tzvetkov, [13].

Cette partie de l'introduction se rapporte au chapitre 4.

1.3.1 Invariance de la mesure gaussienne par le flot linéaire

À présent , on étudie l'équation d'onde cubique sur la boule unité de $\mathbb{R}^3$ avec condition au bord de Dirichlet et à symétrie radiale :

$$\begin{cases} \partial_t^2 u - \triangle_{B^3} u + u^3 = 0 \\ u(t=0) = u_0 \qquad \partial_t u(t=0) = u_1 \end{cases} . \tag{1.4}$$

On pose e_n, $n \geq 1$ la fonction propre radiale du laplacien associée à la valeur propre $-n^2$ et $(h_n)_n$ et $(l_n)_n$ deux suites de gaussiennes centrées normalisées réelles et indépendantes les unes des autres. On pose alors :

$$u_0 = \sum_n \frac{h_n}{n} e_n \text{ et } u_1 = \sum_n l_n e_n$$

et μ la mesure (gaussienne) induite sur $H^\sigma \times H^{\sigma-1}$ pour un σ dans $]0, \frac{1}{2}[$.

On voudrait montrer l'invariance de cette mesure par le flot $S(t)$ de l'équation :

$$\partial_t^2 u - \triangle_{B^3} u = 0 .$$

On note également μ_N la mesure induite par :

$$\sum_{n \leq N} \frac{h_n}{n} e_n \text{ et } \sum_{n \leq N} l_n e_n .$$

Cette mesure est invariante par $S(t)$ car elle est de la forme :

$$d\mu_N((v_0, v_1) = e^{-\|\nabla v_0\|_{L^2}^2 - \|v_1\|_{L^2}^2} dL(v_0, v_1)$$

où L est a mesure de Lebesgue et $\| \nabla v_0\|_{L^2}^2 + \|v_1\|_{L^2}^2$ est l'énergie initiale (et donc en tout temps) associée à l'équation d'onde linéaire avec pour condition initiale (v_0, v_1).

La démonstration utilise plusieurs ingrédients :

- le fait que pour tout $s < \frac{1}{2}$, il existe C_s, c_s telles que $\mu(\|u_0\|_{H^s} + \|u_1\|_{H^{-1}} \geq R) \leq C_s e^{-c_s R^2}$, ce qui est dû au fait que la mesure est gaussienne, et donc vérifie des estimées gaussiennes de grande déviation,
- les boules fermés de H^s sont compactes dans H^σ, pour $s > \sigma$,
- l'espace vectoriel E_N engendré par $\{e_n \mid n \geq N\}$ est stable par le flot $S(t)$.
- $S(t)$ est une isométrie de H^s,
- le flot est réversible ($S(t) \circ S(-t) = Id$).

En utilisant les deux premiers arguments, on montre que pour tout ouvert U de H^σ, on a :

$$\mu(U) \leq \liminf_N \mu_N(U \cap E_N) ,$$

et de même, pour tout fermé F,

$$\mu(F) \geq \limsup \mu_N(F \cap E_N) .$$

Une suite d'inéquations sur $\mu(S(t)F)$ permet de montrer que pour tout fermé, on a $\mu(S(t)F) \geq \mu(F)$, la réversibilité du flot donne l'égalité et on déduit l'égalité sur tout ensemble mesurable car $S(t)$ préserve la topologie de H^σ.

1.3.2 Invariance par le flot non linéaire

Pour avoir une mesure dont on peut espérer avoir l'invariance par le flot de l'équation non linéaire, on va multiplier la mesure $d\mu$ par $e^{-\frac{1}{2}\|v_0\|_{L^4}^4}$. En effet, $\frac{1}{2}\|v_0\|_{L^4}^4$ est le terme non-quadratique (et donc non issu de l'équation linéaire) de l'énergie.

$$d\rho((v_0, v_1) = e^{-\frac{1}{2}\|v_0\|_{L^4}^4} d\mu(v_0, v_1) .$$

On obtient le résultat suivant (qu'on retrouvera au Chapitre 4, Theorem 4) :

Théorème 1.3.1. *Il existe un ensemble E de ρ-mesure pleine tel que pour tout ensemble $A \subseteq E$, ρ mesurable de H^σ, le flot $\Psi(t)$ de* (1.4)*, $\Psi(t)(v_0, v_1) = (v(t), \partial_t v(t))$ est globalement bien défini dans $S(t)(v_0, v_1) + C(\mathbb{R}, H^s)$, $s > 1/2$ et pour tout $t \in \mathbb{R}$:*

$$\rho(\Psi(t)A) = \rho(A) .$$

On va également utiliser l'invariance de mesures par des flots en dimension finie.

Ce travail se distingue de précédents résultats d'invariance dû au fait que la projection L^2-orthogonale Π_N sur l'espace de dimension finie engendré par les N premiers modes de l'équation n'est pas continue sur les espaces où le problème est localement bien posé. Le travail de Burq, Thomann et Tzvetkov [13] résout cette difficulté mais le point de vue adopté ici est légèrement différent.

Pour cela, on approche l'identité de H^σ par des opérateurs S_N à valeurs dans E_N.

On pose

$$d\rho_N((v_0, v_1) = e^{-\frac{1}{2}\|S_N v_0\|_{L^4}^4} d\mu(v_0, v_1) .$$

et v_N la solution de

$$\partial_t^2 v_N - \triangle_{B^3} v_N + S_N (S_N v_N)^3 = 0 \tag{1.5}$$

avec pour condition initiale (v_0, v_1).

La mesure ρ_N est un produit cartésien de mesures :

$$\rho_N = \rho'_N \otimes \mu^N ,$$

où $d\rho'_N(v_0, v_1) = e^{-\frac{1}{2}\|S_N v_0\|_{L^4}^4} d\mu_N(v_0, v_1)$ est une mesure sur E_N invariante par l'équation $\partial_t^2 v - \triangle_{B^3} v + S_N(S_N v)^3 = 0$, cette équation étant hamiltonienne sur $E_N \times E_N$, et μ^N est la mesure induite par

$$\sum_{n>N} \frac{h_n}{n} e_n \text{ et } \sum_{n>N} l_n e_n$$

invariante par $S(t)$ sur le supplémentaire dans H^σ de E_N. On en déduit que ρ_N est invariante par le flot de l'équation (1.5) puisque v_N s'écrit comme la somme d'une solution de (1.5) avec condition initiale dans $E_N \times E_N$ et d'une solution de l'équation linéaire dans le supplémentaire.

L'invariance de ρ dépend de deux résultats de convergence :

Tout d'abord la suite ρ_N tend vers ρ en variation totale, ce qui signifie ici que :

$$\sup_A |\rho_N(A) - \rho(A)| \to_{N\to\infty} 0 .$$

Puis on a un résultat de convergence uniforme de v_N vers la solution v de l'équation d'onde non linéaire. Avec D un paramètre de contrôle sur les normes de v_0, v_1 et $A(D)$ l'ensemble des (v_0, v_1) satisfaisant ces propriétés de contrôles, on a que, pour tout $\epsilon > 0$, il existe $N_0 \in \mathbb{N}$ tel que pour tout $N \geq N_0$ et tout $(v_0, v_1) \in A(D)$,

$$\|v - v_N\|_{C([-T,T],H^s)} \leq \epsilon$$

où le temps de validité de la convergence T est proportionnel à une puissance négative de D et $s > \frac{1}{2}$.

On obtient ainsi l'invariance locale de la mesure ρ et on obtient l'invariance globale en propageant les propriétés locales sur un ensemble de mesure pleine composé d'ensembles $A(D)$ avec des choix appropriés pour les D.

1.4 Stabilité de la mesure invariante

Cette partie de l'introduction se rapporte au chapitre 5.

1.4.1 Choix de l'équation

Pour étudier stabilité de la mesure invariante, on a choisi une équation qui admet un invariant quadratique : l'équation de Benjamin-Bona-Mahony, qui est une alternative plus régularisante de KdV,

$$\begin{cases} (1-\partial_x^2)u_t + u_x + uu_x = 0 \\ u(t=0) = u_0 \end{cases} \tag{1.6}$$

dont la variable spatiale se déplace dans le tore de dimension 1, T.

Le caractère bien posé dans L^2 de cette équation a été démontré par Bona et Tzvetkov [5].

Il s'agit d'une équation hamiltonienne dont

$$\int_T u(1-\partial_x^2)u$$

est un invariant a priori. Cet invariant correspond à l'invariant de masse (la norme L^2) dans d'autres équations hamiltoniennes comme celle de Schrödinger.

Avec $(h_n)_{n\geq 0}$ et $(l_n)_{n\geq 1}$ deux suites de gaussiennes réelles indépendantes les unes des autres centrées et normalisées, on estime donc que la mesure μ induite sur L^2 par la variable aléatoire :

$$h_0 c_0 + \sum_{n\geq 1} \frac{h_n}{\sqrt{1+n^2}} c_n + \frac{l_n}{\sqrt{1+n^2}} s_n$$

est invariante par le flot de BBM, où $(c_n)_{n\geq 0}, (s_n)(n \geq 1)$ est la base orthonormée de $L^2(T)$ formée à partir des $\cos(nx)$ et $\sin(nx)$. C'est en effet un vecteur gaussien de matrice de covariance $(1-\triangle)^{-1}$, c'est-à-dire une renormalisation de la mesure formelle :

$$e^{-\frac{1}{2}\int u(1-\triangle)u} dL(u) .$$

On peut montrer l'invariance de cette mesure en utilisant les mêmes techniques que pour l'invariance de la mesure décrite dans le cas de l'équation d'onde non linéaire cubique. C'est-à-dire, on montre d'abord l'invariance de cette mesure par le flot de l'équation linéaire $(1-\partial_x^2)u_t + u_x = 0$ puis on approche l'équation par une autre dont la partie non linéaire est dans un espace de dimension finie.

1.4.2 Perturbation de la mesure invariante

On a alors cherché à mesurer la stabilité de cette mesure μ. Afin de pouvoir utiliser des techniques déjà développées précédemment, on a choisi de conserver la structure gaussienne. Il s'agissait donc de modifier la matrice de covariance du vecteur gaussien. L'idée sous-jacente est que μ laisse les valeurs propres du laplacien indépendantes, et donc il semblait intéressant de construire une mesure μ_V telle que les amplitudes des différentes longueurs d'onde soient corrélées. Pour cela, on introduit V (le paramètre perturbatif), une fonction suffisamment régulière (C^2 par exemple) et de norme suffisamment petite et on construit un vecteur gaussien de matrice de covariance

$$(1+V)^{-1/2}(1-\partial_x^2)^{-1}(1+V)^{-1/2}$$

en appliquant à la variable aléatoire induisant μ l'opérateur

$$(1+V)^{-1/2} \ ,$$

on obtient ainsi une seconde variable aléatoire et une seconde mesure μ_V.

Notons que de par la régularité de V, plus deux longueurs d'ondes sont proches, plus elles sont corrélées (dans le sens où les covariances sont plus grandes).

On veut donc étudier l'évolution de μ_V à travers le flot de BBM, ou plutôt de sa loi. On note $\Psi(t)$ le flot de BBM. Pour mesurer la distance entre deux mesures ν_1 et ν_2 sur L^2, on pose :

$$d(\nu_1,\nu_2) = \sup_{\|\lambda\|_{L^2}=1} \left(|E_{\nu_1}(e^{i\langle u,\lambda\rangle}) - E_{\nu_2}(e^{i\langle u,\lambda\rangle})| \right)$$

où E_{ν_i} dénote l'espérance par rapport à la mesure ν_i et $\langle\ ,\ \rangle$ est le produit scalaire dans L^2. Cette distance est utilisée dans la littérature physique à propos de turbulence faible, elle ne mesure que la distance entre les lois et non entre les mesures elles-mêmes. On cherche à borner :

$$d(\mu, \Psi(t)\mu_V) \ ,$$

où $\Psi(t)\mu_V$ est définie comme la mesure telle que pour tout ensemble mesurable A de L^2 :

$$\Psi(t)\mu_V(A) = \mu_V(\Psi(t)(A)) \ .$$

Notons que

$$d(\mu,\mu_V) = O(\|V\|)$$

où $\|\ .\ \|$ est par exemple la norme C^2 de V et mesure en tous cas la perturbation.

1.4.3 Stabilité de la mesure invariante

L'intérêt d'avoir gardé une structure gaussienne est qu'il existe une équation proche de BBM dont le flot laisse invariant la mesure μ_V. En effet, en remplaçant u par $(1+V)^{1/2}u$ dans BBM, on obtient l'équation :

$$(1+V)^{1/2}(1-\partial_x^2)(1+V)^{1/2}u_t + (1+V)^{1/2}\partial_x\left((1+V)^{1/2}u + \frac{((1+V)^{1/2}u)^2}{2}\right) = 0$$

qui admet comme invariant a priori

$$\int_T u(1+V)^{1/2}(1-\partial_x^2)(1+V)^{1/2}u$$

et comme forme hamiltonienne :

$$u_t = J_V \nabla_u H_V(u)$$

avec

$$J_V = -(1+V)^{-1/2}(1-\partial_x^2)^{-1}\partial_x(1+V)^{-1/2}$$

antisymétrique et

$$H_V = \frac{1}{2}\int_T ((1+V)^{1/2}u)^2 + \frac{1}{3}((1+V)^{1/2}u)^3 \ .$$

Ceci permet de montrer que la mesure μ_V est invariante par le flot Ψ_V de cette équation. On obtient alors :

$$d(\mu_V, \Psi_V(t)\mu_V) = 0 \ .$$

On vérifie également que d'une part la seconde équation est bien posée dans L^2, d'autre part, que son flot est très proche de celui de BBM dans L^2. Ceci permet d'obtenir une estimée de

$$d(\Psi(t)\mu_V, \Psi_V(t)\mu_V) \ .$$

En combinant ces résultats, on obtient l'estimée recherchée en considérant le fait que :

$$\begin{aligned} d(\Psi(t)\mu_V, \mu) &\leq d(\Psi(t)\mu_V, \Psi_V(t)\mu_V) + d(\Psi_V(t)\mu_V, \mu_V) + d(\mu_V, \mu) \\ &= d(\Psi(t)\mu_V, \Psi_V(t)\mu_V) + d(\mu_V, \mu) \ . \end{aligned}$$

On obtient le résultat suivant (qu'on retrouvera au Chapitre 5, Theorem 6) :

Théorème 1.4.1. *Soit $\varepsilon \in]0, 1/2]$, il existe deux constantes C, c telles que pour tout V de régularité C^2 et de norme inférieure à $1/2$ et pour tout $t \in \mathbb{R}$, on ait :*

$$d(\Psi(t)\mu_V, \mu) \leq C\|V\|e^{c|t|^{6/\varepsilon-2}} \ .$$

On obtient donc un résultat de stabilité pour μ, même s'il est restreint à une certaine déviation dans un cadre gaussien : en ajoutant des corrélations d'ordre V sur la mesure initiale, on garde des corrélations d'ordre V (bien que l'estimée en temps soit exponentielle) lorsqu'on fait agir le flot de BBM sur de telles mesures. Notons que ce résultat présente de l'intérêt dans le cas de temps d'ordre 1 et pour $V \to 0$.

1.5 Stabilité de l'indépendance des modes propres d'une équation

Cette partie de l'introduction se rapporte au chapitre 6.

1.5.1 Turbulence faible

Cette étude est inspirée de la théorie de la turbulence faible statistique.

Cette théorie se place dans le cadre d'une équation dont la variable d'espace se déplace sur le tore de dimension d, T^d, et admettant la forme hamiltonienne

$$iu_t = Hu + V(u, \overline{u})$$

où H est un opérateur hermitien de $L^2(T^d)$, et même un multiplicateur de Fourier, et V est une fonctionnelle.

On suppose que cette équation est bien-posée dans un espace donné, et on écrit sa solution

$$u(t) = \sum_{n \in \mathbb{Z}^d} g_n(t) e^{inx}$$

avec $nx = \sum n_i x_i$. Les $g_n(t)$ sont des variables aléatoires.

On suppose qu'initialement

$$g_n(t = 0) = \chi_n A_n$$

avec χ_n des variables aléatoires indépendantes les unes des autres uniformément réparties sur S^1, A_n des variables aléatoires indépendantes les unes des autres à valeurs dans $\mathbb{R}_+$. On suppose également que A_m est indépendante de χ_n pour tout n et m.

Un des objectifs de la turbulence statistique est de trouver des solutions $g_n(t)$ telles que $E(|g_n(t)|^2)$ ne dépende pas du temps. Néanmoins, cet objectif passe par l'hypothèse que les $g_n(t)$ restent indépendantes les unes des autres en tout temps, [56], du moins en première approximation.

Remarquons que quand le terme non linéaire V est nul, on a, avec $He^{inx} = \omega_n e^{inx}$,

$$u(t) = \sum_n e^{i\omega_n t} \chi_n A_n$$

ce qui permet à u de conserver ses propriétés stochastiques : $e^{i\omega_n t}\chi_n$ est une variable uniformément répartie dans S^1 et on garde les propriétés d'indépendance.

Le fait que l'on suppose que les g_n reste indépendantes est en réalité une approximation de la covariance de g_n et g_m pour tout n et m, lorsqu'on considère que le terme non linéaire V est très petit devant le terme linéaire Hu, [18]. Cela revient par ailleurs à supposer que V est du même ordre de grandeur que H mais que la donnée initiale est très petite (dans la norme appropriée) devant 1.

Nous avons donc choisi d'explorer le cadre mathématique sous-jacent à cette approximation.

1.5.2 Définition du problème

Le cadre qui nous a semblé le plus abordable (on voulait pouvoir appliquer ce cadre à l'équation BBM de la section précédente) était celui d'une équation dont la solution est à valeurs réelles, la variable d'espace dans un tore T^d et qui présente une non-linéarité quadratique. Plus précisément, on étudie des équations de la forme :

$$u_t + Lu + \varepsilon J(u^2) = 0$$

où L et J sont des multiplicateurs de Fourier de la forme

$$Le^{inx} = i\omega_n e^{inx} \text{ et } Je^{inx} = i\varphi(n)e^{inx}$$

et $\varepsilon \ll 1$ pour étudier l'approximation. On suppose que $\omega_{-n} = -\omega_n \in \mathbb{R}$ et $\varphi(-n) = -\varphi(n)$ pour assurer le caractère réel des solutions. On note $u(\varepsilon, t, x)$ une solution (dans un espace où l'équation est bien posée) de cette équation.

Les équations auxquelles nous nous intéressons en particulier sont BBM :

$$L = J = (1 - \partial_x^2)^{-1}\partial_x \ , \ \omega_n = \varphi(n) = \frac{n_1}{1 + n_1^2}$$

et les équations de Kadomtsev-Petviashvili, KP-I ($\kappa = -1$) et KP-II ($\kappa = +1$) :

$$L = \partial_{x_1}^3 + \kappa\partial_{x_1}^{-1}\partial_{x_2}^2 \ , \ J = \partial_{x_1}\omega_n = -n_1^3 + \kappa\frac{n_2^2}{n_1} \ , \ \varphi(n) = n_1 \ .$$

De façon à ce que ces équations soient bien définies, on supposera

$$\int_{T^1} u(\varepsilon, t = 0, x)dx_1 = 0$$

cette propriété étant préservée par le flot des équations mentionnées. Autrement dit, on étudie ces équations sur des espaces H^s de la forme

$$H^s = \{u = \sum_{n_1 \neq 0} u_n e^{inx} \mid u_{-n} = \overline{u}_n \ , \ \|u\|_{H^s} := \sum_n |u_n|^2|n|^{2s} < \infty\}$$

avec $|n| = \sum |n_i|$.

En admettant que les $g_n(t)$ soient des variables aléatoires, on cherche à développer la covariance de u_n et u_m en les différents ordres possibles en ε.

1.5.3 Développement de la solution

On commence par faire un travail purement déterministe en développant

$$u(\varepsilon, t, x)$$

en ses différentes interactions de Picard. On écrit

$$u(\varepsilon, t, x) = a(t, x) + \varepsilon b(t, x) + \varepsilon^2 c(\varepsilon, t, x)$$

avec a la solution de

$$a_t + La = 0 \ , \ a(t = 0) = u(t = 0) \ ,$$

b la solution de

$$b_t + Lb + J(a^2) = 0 \ , \ b(t = 0) = 0 \ ,$$

et c la solution de

$$c_t + Lc + +J(2ab + \varepsilon(b^2 + 2ac) + \varepsilon^2 2bc + \varepsilon^3 c^2) = 0 \ , \ c(t = 0) = 0 \ .$$

Pour estimer a, on utilise le fait que e^{-tL} est une isométrie de H^s. Pour estimer b, on doit distinguer deux cas : celui où l'équation ne présente pas de résonance, c'est-à-dire

$$\omega_n - \omega_k - \omega_l \neq 0$$

pour tout $n = k + l$, il s'agit de BBM et KP-II, et celui où il y a des résonances KP-I. Lorsqu'il n'y a pas de résonance, on peut borner b indépendamment du temps, sinon, b est d'ordre 1 en t, ce qui génère différentes estimées pour c.

Pour borner c en tout temps, on utilise les invariants de masse (la norme $\| \, . \, \|_{L^2}$ pour KP et la norme $\| \, . \, \|_{H^1}$ pour BBM). En utilisant des estimées d'énergie, on borne ainsi la norme L^2 ou H^1 de c. Il reste maintenant à prendre la moyenne de telles estimées de façon à borner la covariance de c_n et de c_m, a_m ou b_m (les autres covariances ayant une formule explicite).

1.5.4 Description de la condition initiale

Pour ce faire, on a besoin d'une donnée initiale "très intégrable". En effet, l'estimée de c fait apparaître $e^{ct^2\|u(t=0)\|^2_{H^s}}$ dans le pire des cas (avec résonance). On pose $(g_n)_n$ une suite de variables aléatoires complexes indépendantes de même loi invariante par une rotation d'angle θ (multiplication par $e^{i\theta}$ tel que $12\theta \ \neq 0 \ [2\pi]$, et qui vérifient des estimées de grande déviation, conséquence de l'hypothèse

$$E(e^{\gamma \mathrm{Re} g_n}), E(e^{\gamma \mathrm{Im} g_n}) \leq e^{\alpha\gamma^2}$$

où α est fixé et $\gamma \in \mathbb{R}$. L'invariance par rotation implique que $E(g_n) = E(g_n^2) = E(g_n^3) = E(g_n^4) = 0$. Remarquons que l'hypothèse physique pour ce type de statistique est que g_n s'écrit $g_n = \chi_n A_n$ avec χ_n distribuée uniformément sur S^1 et indépendante de A_n, elle-même à valeurs réelles positives. De la sorte, g_n est invariante par n'importe quelle rotation.

On suppose, quitte à renormaliser g_n que $E(|g_n|^2) = 1$. On pose $(\lambda_n)_n$ une suite de complexes telle que

$$\lambda_{-n} = \overline{\lambda}_n \ , \ \sum_n |\lambda_n|^2 |n|^{2s} < \infty$$

avec $s > 3$ pour KP-I, $s > 2$ pour KP-II et $s > 3/8$ pour BBM et on pose

$$u(t=0) = \sum_n g_n \lambda_n e^{inx}$$

de sorte que $u(t = 0)$ appartient presque sûrement à H^s. Remarquons que les s choisis sont bien supérieurs à ceux pour lesquels ces équations sont globalement bien posées.

On obtient alors le résultat suivant (qu'on retrouvera au Chapitre 6, Theorem 7) :

Théorème 1.5.1. *Il existe C tel que pour tout $n, m, \varepsilon \geq 1$ et $|t| \leq \frac{1}{C\varepsilon}$*

$$E(\overline{u_m(\varepsilon;t)}u_n(\varepsilon;t)) = \delta_n^m |\lambda_n|^2 + \delta_n^m \varepsilon^2 G_n(\lambda, t) + \varepsilon^3 R(\varepsilon; t, m, n)$$

où $G_n(\lambda, t)$ est donné par $G_n(\lambda, 0) = 0$ et

$$\partial_t G_n(\lambda,t) = 4\varphi(n) \sum_{k+l=n} Re(-F_n^{k,l}(-t))\Big(\varphi(n)|\lambda_k|^2|\lambda_l|^2 - \varphi(k)|\lambda_n|^2|\lambda_l|^2 - \varphi(l)|\lambda_n|^2|\lambda_k|^2\Big)$$

$$+(E(|g_n|^4) - 2)\Big(2\delta_n^{2q} Re(-F_n^{q,q}(-t))\varphi^2(n)|\lambda_q|^4 - 4Re(-F_n^{2n,-n}(-t))\varphi(2n)\varphi(n)|\lambda_n|^4\Big) .$$

Par ailleurs $G_n(t)$ et $R(\varepsilon; t, m, n)$ satisfont les estimées suivantes. Il existe $C > 0$ tel que pour tout $\varepsilon \in (0, 1]$, tout $|t| \leq \frac{1}{C\varepsilon}$, tout m, n,

$$|G_n(\lambda,t)| \leq C|t|\,|n|^{-\beta(s)}, \qquad |R(\varepsilon;t,m,n)| \leq C \min(|n[,|m|)^{-1}\,(1+|t|)|t|$$

dans le cas de BBM, avec $\beta(s) = 2 + 2s$ si $s \geq 1/2$ et $\beta(s) = 1 + 4s$ sinon,

$$|G_n(\lambda,t)| \leq C|t|\,|n|^{-2s}, \qquad |R(\varepsilon;t,m,n)| \leq C \max(|n|,|m|)|t|(1+|t|)$$

dans le cas de KP-II, et

$$|G_n(\lambda,t)| \leq Ct^2|n|^{2-2s}, \qquad |R(\varepsilon;t,m,n)| \leq C \max(|n|,|m|)|t|^3$$

dans le cas de KP-I.

Ce résultat stipule que lorsque le temps reste d'ordre ε^{-1}, l'ordre en ε de la déviation de la covariance par rapport à sa valeur initiale est en ε^2 et non ε comme ce que l'on pourrait attendre. Par ailleurs, on obtient une equation sur la donnée initiale (les $|\lambda_n|^2$) qui permet d'annuler le terme d'ordre 2.

Enfin, remarquons que dans le cas BBM, la mesure invariante, qui correspond à $\lambda_n = \frac{1}{\sqrt{1+n_1^2}}$, soit $|\lambda_n|^2 = \frac{\varphi(n)}{n_1}$, et $E(|g_n|^4) = 2$ annule bien le terme d'ordre 2.

Les cinq chapitres suivants sont chacun issus d'un article.

Le premier s'intéresse au caractère bien posé de l'équation d'onde non linéaire sur la sphère et sur $\mathbb{R}^3$ dans un espace à symétrie zonale ou radiale. Les espaces auxquels appartiennent la condition initiale sont de faible régularité.

Le deuxième traite le cas où l'espace ne présente pas de symétrie.

Le troisième est une preuve de l'invariance de la mesure introduite plus tôt par l'équation d'onde non linéaire sur la boule unité de $\mathbb{R}^3$. On utilisera des résultats d'existence et d'unicité de solutions préexistants.

Le quatrième s'intéresse à la stabilité d'une mesure invariante par le flot de BBM, il donne une définition possible de la notion de stabilité pour ces statistiques. Notons que l'on se place ici sur des espaces où l'équation est globalement bien posée dans un cadre déterministe, nous n'avons pas cherché à alléger les hypothèses de régularité sur la condition initiale.

Le cinquième étudie l'évolution des covariances entre les amplitudes des modes de Fourier de solutions d'équations hamiltoniennes à non linéarité quadratique. Cela correspond aux équations cinétiques de la théorie de la turbulence faible statistique.

Chapitre 2

Large data low regularity scattering results for the wave equation on the Euclidean space

Ce chapitre est largement inspiré de l'article [24].

2.1 Introduction

The main point of this paper is to show the existence of globally defined solution for the non linear wave equation with localized large initial data displaying low regularity.

Indeed, the lowest regularity one can obtain in the cubic case using what one would call deterministic tools is, for this equation, and under the assumption of spherical data, $H^s \times H^{s-1}$ with $s > \frac{7}{10}$, see [50], which means one must be able to differentiate s times one's initial data and still be in L^2 to get a globally well-posed problem. With the tools used here, it will be possible to get globally defined solution with initial data which is spatially localized but not in L^2.

In order to gain a release on regularity, that is to say, to get global solutions with low-regularity initial data, we will use a probabilistic point of view. Our initial data will be a random variable which, as afore-mentioned, is not in L^2, but which satisfies certain properties almost surely. These properties will enable us to extend local well-posedness to global well-posedness. This methods requires to reduce our problem to both a problem on a compact manifold and in finite dimension before extending it to the non linear wave equation on the Euclidean space.

Nicolas Burq and Nikolay Tzvetkov have built in [15, 16] a measure on a Cartesian product of Sobolev spaces $H^\sigma \times H^{\sigma-1}$ with $\sigma < \frac{1}{2}$ and showed the existence of a set of full probability such that the problem was globally well-posed for any initial data in this set. However, the support of the space variable $x \in \mathbb{R}^3$ was a compact (the unit ball of $\mathbb{R}^3$), allowing the Laplace-Beltrami operator with Dirichlet boundary conditions (and so the wave equation) to have a discrete spectrum and eigenfunctions satisfying nice properties regarding their L^p-norms.

Our strategy will be to use a space-time compactification, called the Penrose transform, in order to change the non linear wave equation on the space $\mathbb{R}^3$ into a non linear wave equation on the sphere S^3. Indeed, the eigenfunctions of the Laplace-Beltrami operator on the sphere have properties very similar to the ones on the unit ball of $\mathbb{R}^3$. Thus, we shall use the same tools and methods to study the dynamics of the equation on the sphere. Then, we will return to the equation on $\mathbb{R}^3$ and study its large time dynamics, and the order of regularity of the initial data for which the problem is well-posed.

Considering the reduced problem, we will build a measure on the complex Sobolev space H^σ, let us call it ρ, such that the non linear wave equation is well-posed for all $T \in [-\pi, \pi]$ ρ almost surely, and show this well-posedness. The inverse of the Penrose transform will provide a measure image η of ρ on a space of radial functions of $\mathbb{R}^3$ very similar to $L^2 \times H^{-2}$ (where the Sobolev spaces are real and not complex) such that the non linear wave equation is globally well-posed almost surely.

The key to prove almost sure global well-posedness on the sphere is the fact that, at least formally, the image-measure ρ_t of ρ through the flow satisfies that for all measurable set A :

$$\rho_t(A) \leq \rho(A) \ .$$

However, there is no a priori reason why the image measure of η (the one on $\mathbb{R}^3$) through the flow of the non linear wave equation on $\mathbb{R}^3$ should satisfy similar properties. Hence, though we do not exclude the possibility of a direct proof (without using the Penrose transform), we think that if one wants to use the techniques presented here, the measure built for the problem on $\mathbb{R}^3$ should be different.

We will also determine the spaces which the initial data belongs to and investigate on the regularity of the solution. As we will see, it is almost surely (with regards to the above mentioned measure) not in L^2. Nonetheless, it remains localized initial data and keeps a certain regularity, only not an L^2 or an H^s one. Furthermore, we do not need to assume that their existing norms are taken small. Hence, they are said 'large' initial data.

Finally, there is a scattering result on the solutions that will be stated. Indeed, when $t \to \pm\infty$, the global solutions behave as a free evolution (that is to say a linear one) with a suitable initial data.

Let us now describe more precisely the results we obtain. The equation studied here is :

$$\begin{cases} \partial_t^2 f - \Delta_{\mathbb{R}^3} f + |f|^\alpha f = 0 & (t,x) \in \mathbb{R} \times \mathbb{R}^3 \\ f|_{t=0} = f_0 & \partial_t f|_{t=0} = f_1 \end{cases} \tag{2.1}$$

where f is real and $\alpha \in [2, 3[$. The function f is also radial, namely when using the spherical coordinates $(r, \omega) \in \mathbb{R}^+ \times S^2$, $x = r\omega$, f depends only on r. We will transform (2.1) into

$$\begin{cases} i\partial_T u + \sqrt{1 - \Delta_{S^3}}(u) + (1 - \Delta_{S^3})^{-1/2} (\cos T + \cos R)^{\alpha - 2} |\mathrm{Re} u|^\alpha \mathrm{Re} u = 0 \\ (T, R) \in \{(T, R) \in [-\pi, \pi] \times [0, \pi] \mid \cos T + \cos R > 0\} \\ u|_{T=0} = u_0 \end{cases} \tag{2.2}$$

thanks to the Penrose transform :

$$\begin{cases} T = \text{Arctan }(t+r) + \text{Arctan }(t-r) & \in]-\pi,\pi[\\ R = \text{Arctan }(t+r) - \text{Arctan }(t-r) & \in [0,\pi[\end{cases} \tag{2.3}$$

In (2.2), Δ_{S^3} is the Laplace-Beltrami operator on S^3. The sphere S^3 is parametrized by $(\theta \sin R, \cos R)$ with $R \in [0,\pi]$, and $\theta \in S^2$. The vector θ is the spherical coordinate on which f did not depend and which was not modified by the Penrose transform. The function u_0 is entirely defined by f_0 and f_1. The correspondence between u_0 and f_0, f_1 is given by :

$$u_0 = PT^{-1}(f_0, f_1) = \frac{f_0(\tan(R/2))}{1+\cos R} + i(1-\Delta_{S^3})^{-1/2}\left(\frac{f_1(\tan(R/2))}{(1+\cos R)^2}\right) .$$

The measure we will introduce is defined thanks to the zonal eigenfunctions of $1-\Delta_{S^3}$, written $(e_n)_n$, where Δ_{S^3} is the Laplace-Beltrami operator on the sphere S^3. We set Ω, P a probability space and $(g_n)_n$ a sequence of independent complex Gaussian variables of law $\mathcal{N}(0,1)$. The map

$$\varphi : \begin{cases} \Omega \to \mathcal{D}(S^3) \\ \omega \mapsto \sum_n \frac{\sqrt{2} g_n(\omega)}{n} e_n \end{cases}$$

defines an image-measure μ of P on the zonal distributions of S^3. The term 'zonal' means that the distribution depends only on the distance to a pole of the sphere, data entirely given by the angle $R \in [0,\pi]$ into he parametrization $(\theta \sin R, \cos R)$.

Moreover, the integral

$$\frac{1}{\alpha+2}\int_0^\pi (1+\cos R)^{\alpha-2} |\mathrm{Re} u|^{\alpha+2} \sin^2 R dR$$

is μ almost surely finite, which allow us to define a non-zero measure :

$$d\rho(u) = \exp\left(-\frac{1}{\alpha+2}\int_0^\pi (1+\cos R)^{\alpha-2} |\mathrm{Re} u|^{\alpha+2} \sin^2 R dR\right) d\mu(u) .$$

The problem (2.2) is ρ almost surely well-posed for all $T \in [-\pi,\pi]$.

By using the map PT, we get a measure η on the pairs of radial distributions of $\mathbb{R}^3$ thanks to which we can reach global well-posedness of (2.1). This map is continuous from $L^2(S^3)$ to $L^2_{-1}(\mathbb{R}^3) \times H^2_{-6}(\mathbb{R}^3)$ where L^2_{-1} and H^{-2}_{-6} are the topological spaces induced by the norms :

$$||f||_{L^2_{-1}} := ||\left(\frac{1+r^2}{2}\right)^{-1} f||_{L^2} \text{ and } ||f||_{H^{-2}_{-6}} = ||\left(\frac{1+r^2}{2}\right)^{-6} f||_{H^{-2}} .$$

The support of η is therefore contained on these spaces.

What is more, the support of η satisfies the following properties :

Theorem 1. *For η almost all (f_0, f_1) :*

– *(f_0, f_1) belongs to $L^p \times W^{-1,p}$ for all $p \in]2, 6[$,*
– *f_0 does not belong to L^p, for $p \leq 2$ or $p \geq 6$,*
– *f_0 is localized, that is, for all $\nu \in]0, \frac{1}{2}[$, $f_0(r) = O(r^{-(1+\nu)}$.*

Regarding the problem (2.1) on $\mathbb{R}^3$, we get that, with $L(t)$ the flow of the linear equation $\partial_t^2 - \triangle_{\mathbb{R}^3} = 0$:

Theorem 2. *The non linear wave equation* (2.1) *is η almost surely globally well-posed in*

$$L(t)(f_0, f_1) + C(\mathbb{R}, H^s)$$

for some $s > 1/2$. Then, we have some scattering properties. Let $\max(\frac{3}{2}\alpha, 4) < q < \frac{9}{2}$. *For any solution f of* (2.1)*, there exists f_∞ such that $f(t) - L(t)f_\infty$ is in L^q and its norm converges toward* 0 *when $t \to \infty$.*

These theorems will be proved in different places in the paper : the first one is the subject of section 4, the first part of the second one is proved in sections 2 and 3, and the scattering property is proved in section 5.

Plan of the paper In part two, we will introduce some pre-requisite useful for the sequel : first, Sobolev's embedding theorem and Strichartz inequalities, then the Penrose transform (the one to change the problem on $\mathbb{R}^3$ into a problem on S^3), and finally, general results about Gaussian measures on distribution spaces.

The third part is dedicated to the construction of the measure on the zonal distributions of S^3 and the existence of a set of full measure on these distributions into which the transformed problem is well-defined for all $T \in [-\pi, \pi]$. We will get from these results the immediate existence of a measure on the radial distributions of $\mathbb{R}^3$ and a set Π of full measure satisfying the same properties provided that the reverse Penrose transform is explicit.

In part four, the spaces into which Π is almost surely included and almost surely disjoint from are studied, using theorems given by A.Ayache and N.Tzvetkov in [2].

Finally, in part five, we focus on scattering properties. Since the Penrose transform is a map both on time and space, the time $t = \infty$ doesn't correspond to $T = \infty$ (indeed, $T \in [-\pi, \pi]$). We will then have to study the dynamics of f the solution of (2.1) in a completely different way from the dynamics of the corresponding solution of the reduced problem (2.2).

2.2 Preliminaries

2.2.1 Sobolev spaces and Strichartz inequalities

First, let us consider the Sobolev inequalities on a compact boundary-free manifold.

Theorem 2.2.1 (Sobolev embedding theorem)**.** *Let M be a compact boundary-free manifold of dimension n. Let $s \in \mathbb{R}$ and $p \in [2, \infty[$ such that $\frac{1}{2} = \frac{1}{p} + \frac{s}{n}$. The space H^s is continuously embedded into L^p. That is to say, there exists a constant $C(p, s)$ such that, for all $f \in H^s$,*

$$\|f\|_{L^p} \leq C\|f\|_{H^s}.$$

For $p = \infty$, we have that H^s is continuously embedded into L^∞ if $s > \frac{n}{2}$.

More generally, if $\frac{1}{q} = \frac{1}{p} + \frac{s}{n}$ and $p < \infty$, then there exists C such that

$$\|f\|_{L^p} \leq C\|(1-\Delta)^{s/2} f\|_{L^q}$$

One can find the proof in [1].

Remark 2.2.1. *In fact, the first inequality is also true in $\mathbb{R}^N$. We will use this theorem in the sphere S^3 and the first inequality in the Euclidean space $\mathbb{R}^3$.*

Let us now add a variable of time t, and build the operator $S(t) = e^{-it\sqrt{-\Delta}}$, where Δ is the Laplace Beltrami operator on M, we obtain a Strichartz inequality, see [33]. To reach this goal, we need to introduce the space X_T^s.

Definition 2.2.2. A couple of real numbers (p, q), $2 < p \leq \infty$ is said admissible (on dimension 3) if

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{2}\ .$$

For all admissible couple, and all time T we define :

$$X_T^s = C^0([-T, T], H^s(M)) \cap L^p([-T, T], L^q(M))\ ,\ p = \frac{2}{s}\ .$$

Remark that the definition of X_T^s implies that we choose s in $[0, 1[$.

We then have the following property :

Proposition 2.2.3 (Strichartz inequality)**.** *Let (p, q) an admissible couple and $0 < T \leq \pi$. There exists C independent from T such that for all $f \in H^s$, we have :*

$$\|S(t)f\|_{X_T^s} \leq C\|f\|_{H^s}.$$

2.2.2 The Penrose transform

We will use the Penrose transform to change the problem on $\mathbb{R} \times \mathbb{R}^3$ into a problem on a bounded set included in $]-\pi, \pi[\times S^3$

Definition 2.2.4. For all $t \in \mathbb{R}$ and $r \in \mathbb{R}^+$, we define :

$$\begin{cases} T = \text{Arctan}\,(t+r) + \text{Arctan}\,(t-r) & \in]-\pi, \pi[\\ R = \text{Arctan}\,(t+r) - \text{Arctan}\,(t-r) & \in [0, \pi[\end{cases} \tag{2.4}$$

The transform $\mathbb{R} \times \mathbb{R}^+ \times S^2 \to]-\pi, \pi[\times[0, \pi[\times S^2$, $(t, r, \omega) \mapsto (T, R, \omega)$ sends $\mathbb{R}^4$ in a bounded set of $\mathbb{R}^4$.

Lemma 2.2.5. *The inverse of this transformation on the set* $\{(T,R) \in]-\pi,\pi[\times[0,\pi[\mid \Omega(T,R) = \cos T + \cos R > 0\}$ *is given by :*

$$t = \frac{\sin T}{\Omega} \ , \ r = \frac{\sin R}{\Omega} \ .$$

What's more, it turns the d'Alembertian in $\mathbb{R}_t \times \mathbb{R}^3_x$ into a d'Alembertian in $]-\pi,\pi[\times S^3$, see [55, 20].

Proposition 2.2.6. *Let* f *be a distribution on* $\mathbb{R}_t \times \mathbb{R}^3_x$*. We set* $v(T,R,\omega) = f(t,r,\omega)\Omega^{-1}$*. We have :*

$$\Omega^3(\partial_T^2 - \partial_R^2 - \frac{2}{\tan R}\partial_R + 1)v = (\partial_t^2 - \partial_r^2 - \frac{2}{r}\partial_r)f. \tag{2.5}$$

The Laplacian onto the sphere S^2 doesn't depend on the considered variables t, r, T, R, so we get :

$$\begin{array}{rcl} (\partial_t^2 - \Delta_{\mathbb{R}^3})f &=& (\partial_t^2 - \partial_r^2 - \frac{2}{r}\partial_r - \frac{1}{r^2}\Delta_{S^2})f \\ &=& \Omega^3(\partial_T^2 - \partial_R^2 - \frac{2}{\tan R}\partial_R + 1)v - \frac{\Omega^3}{\sin^2 R}\Delta_{S^2}v \\ &=& \Omega^3(\partial_T^2 + 1 - \Delta_{S^3})v \end{array}$$

Let now see how (2.1) is changed. We have :

$$(\partial_t^2 - \Delta_{\mathbb{R}^3})f + |f|^\alpha f = \Omega^3(\partial_T^2 + 1 - \Delta_{S^3})v + \Omega^{\alpha+1}|v|^\alpha v$$

and

$$v|_{T=0} = \frac{f_0(\tan(R/2))}{(1+\cos R)} \ , \ \partial_T v|_{T=0} = \frac{f_1(\tan(R/2))}{(1+\cos R)^2}$$

The Penrose transform sends $\mathbb{R}^4$ in a set where Ω is always positive. Indeed, in terms of r and t, Ω is :

$$\Omega = \frac{2}{\sqrt{(1+(t+r)^2)(1+(t-r)^2)}} \ .$$

Thus, the original problem (2.1) is transformed by the Penrose transform into :

$$\left\{ \begin{array}{ll} (\partial_T^2 + 1 - \Delta_{S^3})v + \Omega^{\alpha-2}|v|^\alpha v = 0 & T, R \in [-\pi,\pi]\times[0,\pi]\cap\Omega^{-1}(]0,2]) \\ v|_{T=0} = v_0 = \frac{f_0(\tan(R/2))}{1+\cos R} & \partial_T v|_{T=0} = v_1 = \frac{f_1(\tan(R/2))}{(1+\cos R)^2} \end{array} \right. . \tag{2.6}$$

What we will do is replace the domain of definition of the problem by $[-\pi,\pi]\times[0,\pi[$, $\Omega = \cos T + \cos R$ by $\widetilde{\Omega}$ defined as :

$$\widetilde{\Omega} = \left\{ \begin{array}{ll} \cos T + \cos R & \text{when } \cos T + \cos R > 0 \\ 0 & \text{otherwise} \end{array} \right. .$$

The problem is now given by :

$$
\begin{cases} (\partial_T^2 + 1 - \Delta_{S^3})v + \widetilde{\Omega}^{\alpha-2}|v|^\alpha v = 0 & T, R \in [-\pi,\pi] \times [0,\pi[\\ v|_{T=0} = v_0 = \frac{f_0(\tan(R/2))}{1+\cos R} & \partial_T v|_{T=0} = v_1 = \frac{f_1(\tan(R/2))}{(1+\cos R)^2} \end{cases} \tag{2.7}
$$

If v is a solution of (2.7) then it follows that its restriction to the domain where Ω is strictly positive is a solution of (2.6).

But one can see that there is a singularity in the definition of the initial data at $R = \pi$. Indeed, it corresponds to the singularity $r = \infty$ before taking the transformed problem. Dividing by $1+\cos R$ corresponds to multiplying by $\frac{1+r^2}{2}$. Hence, the original initial data f_0, f_1 is "more regular" than the transformed one.

2.2.3 Stochastic tools

Let us define the stochastic objects that we will need.

Definition 2.2.7. Let Ω, P be a probabilistic space. In what follows, $(g_n)_n$ is a sequence of independent random variables of complex normal distribution.

We have the following lemma, see [15].

Lemma 2.2.8. *Let $1 \leq q < \infty$. There exists a constant C independent from q such that for all sequence $(c_n)_n$, we have,*

$$
\|\sum g_n c_n\|_{L^q_\omega} \leq C\sqrt{q\sum |c_n|^2}\ .
$$

Remark 2.2.2. *The same inequality is valid when g_n are replaced by real centred Gaussian variables and c_n is replaced by a sequence of real numbers.*

2.3 Existence of a measure and a set of full measure where the flow of the transformed problem is defined for convenient times

We will rewrite the equation (2.7) into a first order equation with initial datum u_0 in H^σ when (v_0, v_1) belongs to $H^\sigma \times H^{\sigma-1}$. We will prove the existence of a finite measure on the complex Sobolev space H^σ, for any $\sigma < \frac{1}{2}$, such that this rewritten equation has almost surely a unique solution. This is equivalent to building a measure on $H^\sigma \times H^{\sigma-1}$ such that (2.7) has almost surely a unique solution.

We are first going to consider the Sobolev spaces H^σ of the sphere S^3 and the eigenfunctions of $1 - \Delta_{S^3}$ on this sphere. Recall that the radial distributions on $\mathbb{R}^3$ are transformed into what are called zonal distributions on S^3, that is to say, distributions that only depend on the angle R, or on the distance to a pole of S^3. The integration unit becomes then $4\pi \sin^2 R dR$.

The restriction on σ is due to the construction of the measure. The deterministic analysis which is done here would allow bigger σs.

2.3.1 Norms of the Laplace Beltrami operator's eigenfunctions

Proposition 2.3.1. *The functions* $e_n(R) = \frac{1}{\sqrt{2}\pi}\frac{\sin nR}{\sin R}$, $n \geq 1$ *form an orthonormal basis of* $L^2(S^3)$ *with zonal symmetry and are eigenfunctions of* $\Delta_{S^3} - 1 = \partial_R^2 + \frac{2}{\tan R}\partial_R - 1$ *with eigenvalues* $-n^2$.

The proof of this proposition can be found in [52].

Proposition 2.3.2. *Let* $1 \leq p \leq \infty$. *There exists* C_p *such that for all* n *:*

$$\|e_n\|_{L^p} \leq \begin{cases} C_p & \text{if } p < 3\,, \\ C_p \log n & \text{if } p = 3 \\ C_p n^{1-3/p} & \text{otherwise} \end{cases} \tag{2.8}$$

Proof

We have to sum $|\sin nR|^p|\sin R|^{2-p}$ over $]0,\pi[$. With a change of variable $R \mapsto \pi - R$ on $[\frac{\pi}{2},\pi[$, the integral over $]0,\pi[$ is twice the one over $]0,\pi/2]$. What's more, in $]0,\pi/2]$, $\frac{2R}{\pi} \leq \sin R \leq R$.

For $p < 3$, we do $|\sin nR| \leq 1$ and sum R^{2-p} over $[0,\pi/2]$.

For $p \geq 3$, we have to consider separately the integral over $]0,\frac{1}{n}]$ in which we do $\frac{\sin nR}{\sin R} \leq n$, and the one over $[\frac{1}{n},\pi/2]$ in which we use $|\sin nR| \leq 1$ and $\sin^{2-p} R \leq (\frac{2R}{\pi})^{2-p}$.

◊

Now that the e_n are introduced, we can give some definitions.

Definition 2.3.3. We set :

- E_N the finite dimensional complex vector space linearly spanned by $\{e_n \mid n = 1,\dots,N\}$,
- Π_N the projection of H^σ on E_N,
- χ_S a smooth non-negative function compactly supported, with support included in $[-1,1]$ equal to 1 on $[-1/2,1/2]$,
- S_N the operator from H^σ to E_N defined by $\chi_S(-\frac{\Delta}{N^2})$ that is to say the operator that sends $\sum c_n e_n$ in $\sum \chi_S(\frac{n^2}{N^2})c_n e_n$.

Theorem 2.3.4. *Let* $1 \leq p \leq \infty$, S_N *is continuous from* L^p *to* L^p *and the supremum over* N *of the norms of* S_N *as operators is finite. In other terms, there exists* C *(independent from* N*) such that for all* N *and all* $f \in L^p$, $\|S_N f\|_{L^p} \leq C\|f\|_{L^p}$. *What is more, for all* $f \in L^p$, *the sequence* $S_N f$ *converges in norm* L^p *toward* f.

The proof of this theorem can be found in [11].

2.3.2 "Hamiltonian" problem

Let us now replace the notation $\widetilde{\Omega}$ by merely Ω.

The first step consists in changing the equation obtained after applying the Penrose transform, that is

$$\begin{cases} \partial_T^2 v + H^2 v + \Omega^{\alpha-2}|v|^\alpha v = 0 \\ v|_{T=0} = v_0\,, \qquad\qquad \partial_T v|_{T=0} = v_1 \end{cases} \tag{2.9}$$

into a "Hamiltonian" form. The operator H is the strictly positive square root of $1 - \Delta_{S^3}$ where, again, Δ_{S^3} is the Laplace-Beltrami operator restricted to zonal functions.

For this purpose, we write $u = v - i\partial_T H^{-1} v$. Let us remark that since we assume that f, and so v, are real distributions, we have $v = \mathrm{Re}u$. What is more, H is strictly positive, which makes $H^s v$ a real distribution for all $s \in \mathbb{R}$. Also, ∂_T and H commute. The equation (2.9) is equivalent to :

$$\begin{cases} i\partial_T u + Hu + H^{-1}(\Omega^{\alpha-2}|\mathrm{Re}u|^\alpha \mathrm{Re}u) = 0 \\ u|_{T=0} = u_0 = v_0 + iH^{-1}v_1 \end{cases} \tag{2.10}$$

Let us prove this fact.

Indeed,

$$\begin{aligned} i\partial_T u &= i\partial_T v + \partial_T^2 H^{-1} v \\ Hu &= Hv - i\partial_T v \\ i\partial_T u + Hu &= \partial_T^2 H^{-1} v + Hv = H^{-1}(\partial_T^2 v + H^2 v) = -H^{-1}(\Omega^{\alpha-2}|v|^\alpha v). \end{aligned}$$

Therefore, (2.9) reduces to (2.10).

So now, we are provided with an almost-Hamiltonian formulation on u of the equation (2.9). The energy defined as :

$$\mathcal{E}(T,u) = \frac{1}{2}\|Hu\|_{L^2}^2 + \frac{1}{\alpha+2}\int \Omega^{\alpha-2}|\mathrm{Re}u|^{\alpha+2} \tag{2.11}$$

is "formally" decreasing under the flow for $T \in [0,\pi]$, increasing for $T \in [-\pi, 0]$. Indeed, by differentiating $\mathcal{E}(T, u(T,.))$, we get, as a formal computation, that :

$$\begin{aligned} d_T\mathcal{E} &= \int \mathrm{Re}(\partial_T \overline{u}(H^2 u + \Omega^{\alpha-2}|\mathrm{Re}u|^\alpha \mathrm{Re}u)) \sin^2 R dR - \sin T \frac{\alpha-2}{\alpha+2}\int \Omega^{\alpha-1}|\mathrm{Re}u|^{\alpha+2}\sin^2 R dR \\ &= \mathrm{Re}\left(\int \partial_T \overline{u}(iH\partial_T u)\sin^2 R dR\right) - \sin T\frac{\alpha-2}{\alpha+2}\int \Omega^{\alpha-1}|\mathrm{Re}u|^{\alpha+2}\sin^2 R dR \\ &= \mathrm{Re}\left(i\int |H^{1/2}\partial_T u|^2 \sin^2 R dR\right) - \sin T\frac{\alpha-2}{\alpha+2}\int \Omega^{\alpha-1}|\mathrm{Re}u|^{\alpha+2}\sin^2 R dR \\ &= -\sin T\frac{\alpha-2}{\alpha+2}\int \Omega^{\alpha-1}|\mathrm{Re}u|^{\alpha+2}\sin^2 R dR\,. \end{aligned}$$

Hence, we can see that since $\frac{\alpha-2}{\alpha+2}\int \Omega^{\alpha-1}|\mathrm{Re}u|^{\alpha+2}\sin^2 R dR \geq 0$, the energy reaches its maximum at $T = 0$.

2.3.3 "Hamiltonian" equations and approximation

In order to get a well posed problem, we will approach it with ODEs, and so restrict ourselves to finite dimensions.

This is where we will use the definitions of E_N, Π_N, and S_N.

By replacing the energy $\mathcal{E}$ with

$$\mathcal{E}_N(T,u) = \frac{1}{2}\|Hu\|_{L^2}^2 + \frac{1}{\alpha+2}\int \Omega^{\alpha-2}|S_N\mathrm{Re}u|^{\alpha+2}$$

the corresponding equation on u becomes :

$$i\partial_T u + Hu + H^{-1}S_N(\Omega^{\alpha-2}(|S_N(\mathrm{Re}u))|^\alpha S_N(\mathrm{Re}u)) = 0$$

We then consider the following equation :

$$\begin{cases} i\partial_T u + Hu + H^{-1}S_N(\Omega^{\alpha-2}(|S_N(\mathrm{Re}u))|^\alpha S_N(\mathrm{Re}u)) = 0 \\ u|_{T=T_0} = u_0 \in E_N \end{cases} \tag{2.12}$$

Proposition 2.3.5. *For all $u_0 \in E_N$, the equation* (2.12) *has a unique global solution in $C(\mathbb{R}, E_N)$.*

Proof Given the structure of the non linearity in dimension N, the local well-posedness is obtained by applying Cauchy-Lipschitz theorem. In order to extend the local result to a global one, we have to consider both the equivalence of norms in finite dimension, and the fact that the energy $\mathcal{E}_N$, which controls the L^2 norm of Hu and so the L^2 norm of u, reaches its maximum at $T = 0$ (same computation as for $\mathcal{E}$, but it is not only formal in finite dimension). We will then denote by $\Psi_N(T_0, T)$ the flow of (2.12) for all $T_0, T \in [-\pi, \pi]^2$. $\diamond$

Remark 2.3.1. *Although we can globally define the flow of these equations as they are on a finite dimensional subset of H^s, we will only consider this flow on the times in $[-\pi, \pi]$. This is due to the fact that the Cauchy problem on the Euclidean space we are interested in reduces to a problem on a compact (even in time) set.*

The local well-posedness in low regularity spaces will be proved in the next subsection. Also, we will see that this property is uniform in N, that is, we get time of existence and controls on L^p norms independent from N thanks to the uniformity of the norms of S_N.

The point of using approximated equations is to allow us to approach the global flow of the wave equation. Moreover, the measure on E_N that it will induce converges towards the measure that we will consider on H^σ.

2.3.4 Local well-posedness

We will now prove the local well-posedness of the approached and general form of the pseudo-Hamiltonian equations and give some inequalities regarding the norms of their solutions.

Definition 2.3.6. We set

$$Y_T^s = L^1([-T,T],H^{-s}) + L^{p'}([-T,T],L^{q'})$$

where p' and q' are the respective conjugate numbers of $p = \frac{2}{s}$ and q such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$.

Remark 2.3.2. *The set Y_T^s is not the dual of the above-mentioned X_T^s but since for all $1 \leq p \leq \infty$, $\|f\|_{L^p} = \sup\{\int f\overline{g} \mid \|g\|_{L^{p'}} \leq 1\}$, we get that if F maps continuously H^σ to X_T^s, then its adjoint is continuous from Y_T^s to the dual of H^σ, namely, $H^{-\sigma}$.*

The following proposition regroups some consequences of the Sctrichartz inequality.

Proposition 2.3.7. *For all $0 < s < s_1 < 1$ there exists $C > 0$ depending only on s such that for all $T \in]0,\pi]$, $f \in H^s$, $g \in Y_T^{1-s}$ and $h \in Y_T^{1-s_1}$, we have :*
- $\|S(t)f\|_{X_T^s} \leq C\|f\|_{H^s}$,
- $\| \int_0^t S(t-t')H^{-1}g(t')dt'\|_{X_T^s} \leq C\|g\|_{Y_T^{1-s}}$,
- $\|(1-S_N)\int_0^t H^{-1}S(t-t')h(t')dt'\|_{X_T^s} \leq CN^{s-s_1}\|h\|_{Y_T^{1-s_1}}$.

Proof

The first inequality is the Strichartz inequality already mentioned in 2.2.3. One can find its proof in [33]. It is equivalent to the continuity of $S : f \mapsto (t \mapsto S(t)f)$ from H^s to X_T^s with a constant independent from T (as long as T is taken in a compact of $\mathbb{R}^+$).

We deduce from that that its adjoint : $S^*g \mapsto \int_{-T}^T S(-t')g(t')dt'$ is continuous from Y_T^{1-s} to H^{s-1} with a constant independent from T. Hence, as H^{-1} is continuous from H^{s-1} to H^s, and S from H^s to X_T^s, we get that $S \circ H^{-1} \circ S^*$ is continuous from Y_T^{1-s} to X_T^s. Then, we get from M. Christ and A. Kiselev lemma (see [19]) the continuity of

$$g \mapsto \int_0^t S(t-t')H^{-1}g(t')dt'$$

from Y_T^{1-s} to X_T^s with a constant independent from T. That is to say :

$$\| \int_0^t S(t-t')H^{-1}g(t')dt'\|_{X_T^s} \leq C\|g\|_{Y_T^{1-s}} .$$

At last, $(1-S_N)$ is continuous from H^{s_1} to H^s and its norm is less than CN^{s-s_1}. Indeed,

$$\|(1-S_N)\sum c_n e_n\|_{H^s} = \sqrt{\sum(1-\chi_S(\frac{n^2}{N^2}))n^{2s}|c_n|^2}$$

$$\leq CN^{s-s_1}\sqrt{\sum n^{2s_1}|c_n|^2} = CN^{s-s_1}\|\sum c_n e_n\|_{H^{s_1}}$$

We get then that $S \circ (1 - S_N) \circ H^{-1} \circ S^*$ is continuous from $Y_T^{1-s_1}$ to X_T^s and its norm is less than CN^{s-s_1}, which leads to the third inequality, using, once again M. Christ and A. Kiselev lemma.

◊

Proposition 2.3.8. *Let (p, q) be such that $\frac{1}{p} + \frac{3}{q} = \frac{3}{2} - s$ and p', q' their respective conjugate numbers, ($p \geq \frac{2}{s}$). Again, these definitions imply that s belongs to* $[0, 1[$. *Then,*

$$\|f\|_{L_t^p, L_x^q} \leq C\|f\|_{X_T^s} \text{ and } \|f\|_{Y_T^s} \leq C\|f\|_{L_t^{p'}, L_x^{q'}} .$$

Proof We get the inequalities for the extremal couples $p = \frac{2}{s}, q = \frac{2}{1-s}$ by definition of the norm X_T^s and $p = \infty, \frac{1}{q} = \frac{1}{2} - \frac{s}{3}$ thanks to Sobolev embedding theorem. Regarding the other couples, we deduce the result from Hölder inequalities, indeed, if $\frac{6}{3-2s} \leq q \leq \frac{2}{1-s}$ then $\frac{1}{q}$ can be written as $\frac{1}{q} = \theta\frac{1-s}{2} + (1-\theta)\frac{3-2s}{6}$ with $\theta \in [0, 1]$, hence :

$$\|f\|_{L_x^q} \leq C\|f\|^\theta_{L^{\frac{2}{1-s}}} \|f\|^{1-\theta}_{L^{\frac{6}{3-2s}}} .$$

We deduce that :

$$\|f\|_{L_t^p, L_x^q} \leq C\| \, \|f\|^\theta_{L^{\frac{2}{1-s}}} \, \|_{L_t^p}\| \, \|f\|^{1-\theta}_{L^{\frac{6}{3-2s}}} \, \|_{L_t^\infty} = C\|f\|^\theta_{L_t^{p\theta}, L^{\frac{2}{1-s}}} \|f\|^{1-\theta}_{L_t^\infty, L^{\frac{6}{3-2s}}} .$$

Since $\frac{1}{p} = \frac{3-2s}{2} - \frac{3}{p} = \theta(\frac{3-2s}{2} - \frac{3-3s}{2}) = \theta\frac{s}{2}$ and so $p\theta = \frac{2}{s}$, we get :

$$\|f\|_{L_t^p, L_x^q} \leq C\|f\|^\theta_{X_T^s} \|f\|^{1-\theta}_{X_T^s} = C\|f\|_{X_T^s} .$$

◊

Proposition 2.3.9. *Let $s \in]0, 1[$ and p defined as $s = \frac{3}{2} - \frac{4}{p}$. (Let us note that $p \in]\frac{8}{3}, 8[$.) There exists C such that for all $T \in [0, \pi]$, we have :*

$$\|S(t)f\|_{L^p([-T,T]\times S^3)} \leq C\|f\|_{H^s}$$

Proof Let $s' = \frac{3}{2} - \frac{6}{p} = s - \frac{2}{p}$ and q such that $\frac{1}{q} = \frac{1}{p} + \frac{s'}{3} = \frac{1}{2} - \frac{1}{p}$. By Sobolev embedding theorem,

$$\|S(t)f\|_{L_x^p} \leq C\|(1-\Delta)^{s'/2} S(t)f\|_{L_x^q}$$

and so

$$\|S(t)f\|_{L^p([-T,T]\times S^3)} \leq C\|S(t)(1-\Delta)^{s'/2} f\|_{L_t^p, L_x^q} .$$

Since $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$, thanks to Strichartz inequality 2.2.3,

$$\|S(t)f\|_{L^p([-T,T]\times S^3)} \leq C\|S(t)(1-\Delta)^{s'/2} f\|_{X^{2/p}} \leq C\|(1-\Delta)^{s'/2} f\|_{H^{2/p}} .$$

As $\|(1-\Delta)^{s'/2}f\|_{H^{2/p}} = \|(1-\Delta)^{s'/2+1/p}f\|_{L^2} = \|(1-\Delta)^{s/2}f\|_{L^2} = \|f\|_{H^s}$, we get the result. $\diamond$

We will deduce from this properties the local well-posedness of the equations. In order to do so, we write $F(t,u) = \Omega^{\alpha-2}(t,.)|\text{Re}u|^\alpha\text{Re}u$, and we decompose the solutions with initial data $u|_{T=t_0} = u_0$ by writing them as $u(t+t_0) = S(t)u_0 + v(t)$, $v_{t=0} = 0$.

Proposition 2.3.10. *Let $p \in]2\alpha, 6[$ and s defined as previously as $s = \frac{3}{2} - \frac{4}{p}$. We choose an initial data u_0 such that $\|S(t)u_0\|_{L^p([-\pi,\pi]\times S^3)} \leq A$, with A a finite positive constant. The following problems :*

$$\begin{cases} i\partial_t u - Hu - H^{-1}F(t,u) = 0 \\ u|_{t=t_0} = u_0 \end{cases}$$

and

$$\begin{cases} i\partial_t u - Hu - S_N H^{-1}F(t,S_N u) = 0 \\ u|_{t=t_0} = S_N u_0 \end{cases}$$

have unique local solutions u and u_N on $[t_0-\tau, t_0+\tau]$ with $\tau = c(1+A)^{-\gamma}$ where c and γ are constant depending only on s (and in particular, are independent from A). The functions u and u_N, $N \geq 1$ can be written as $u(t_0+t) = S(t)u_0 + v(t)$ and $u_N(t_0+t) = S(t)S_N u_0 + v_N(t)$ with v and v_N in X^s_τ. Furthermore, there exists C such that $\|v\|_{X^s_\tau}, \|v_N\|_{X^s_\tau} \leq CA$. We deduce immediately from the Proposition 2.2.3 and the periodicity of S that

$$\sup_{t'\in[-\tau,\tau]} \|S(t)u(t_0+t')\|_{L^p([-\pi,\pi]\times S^3)} \leq CA$$

and that if $u_0 \in H^\sigma$, for any $\sigma < s$ (and so for any $\sigma < \frac{1}{2}$) then

$$\|u\|_{H^\sigma} \leq \|u_0\|_{H^\sigma} + CA \ .$$

Proof We turn the problem on u into a fixed point one on v that depends on u_0. We are now looking for a v (resp. v_N) satisfying :

$$v = K(v) \text{ (resp.} v_N = K_N(v_N) \text{)}$$

with

$$K(v) = -i\int_0^t S(t-t')H^{-1}F(t_0+t', S(t')u_0 + v(t'))dt'$$

and

$$K_N(v_N) = -i\int_0^t S(t-t')H^{-1}S_N F(t_0+t', S(t')S_N u_0 + S_N v_N)dt' \ .$$

Since the operator norms $L^p \to L^p$ of the S_N, $N \geq 1$ are bounded by a constant independent from N, we can do the proof only for u.

We have then to apply the fix point theorem to K in X^s_τ with τ small enough.

$$\|K(v)\|_{X^s_\tau} \leq C\|F(t_0+t, S(t)u_0+v(t))\|_{Y^{1-s}_\tau} \leq C\|F(t_0+t, S(t)u_0+v(t))\|_{L^{q'}([-\tau,\tau]\times S^3)}$$

with q satisfying $\frac{1}{q}+\frac{3}{q}=\frac{3}{2}-(1-s)$, that is to say $q=\frac{8}{1+2s}$ and $q'=\frac{8}{7-2s}=\frac{2p}{p+2}$. Finally, as $\alpha \geq 2$ and $0 \leq \Omega \leq 2$, for all w, we have :

$$\|F(t_0+t,w)\|_{L^{q'}} \leq \|\Omega^{\alpha-2}\|_{L^\infty}\|w^{\alpha+1}\|_{L^{q'}} \leq C\|w\|^{\alpha+1}_{L^{(\alpha+1)q'}},$$

and so

$$\|K(v)\|_{X^s_\tau} \leq C(\|S(t)u_0\|^{\alpha+1}_{L^{(\alpha+1)q'}} + \|v\|^{\alpha+1}_{L^{(\alpha+1)q'}})$$

What is more, $(\alpha+1)q' = (\alpha+1)\frac{2p}{p+2} < (\alpha+1)\frac{2p}{2\alpha+2} = p$ so, as the integration is done over compacts with size τ in time, $\|f\|_{L^{(\alpha+1)q'}} \leq C\tau^{\delta/(\alpha+1)}\|f\|_{L^p}$, with $\delta/(\alpha+1) = \frac{1}{(\alpha+1)q'} - \frac{1}{p} > 0$.

$$\|K(v)\|_{X^s_\tau} \leq C\tau^\delta(A^{\alpha+1} + \|v\|^{\alpha+1}_{X^s_\tau})$$

For all C', by choosing $\tau \leq (\frac{C'}{C(1+(C')^{\alpha+1})})^{1/\delta}(1+A)^{-\alpha/\delta}$, if $\|v\|_{X^s_\tau} \leq C'A$, then $\|K(v)\|_{X^\tau_s} \leq C'A$. So, the ball $B(0,C'A)$ is stable under K.

Let us do the same with $K(v_1)-K(v_2)$.

$$\|K(v_1)-K(v_2)\|_{X^s_\tau} \leq C\|F(t_0+t, S(t)u_0+v_1) - F(t_0+t, S(t)u_0+v_2)\|_{L^{q'}}$$

Since $|F(t_0+t,v)-F(t_0+t,w)| \leq C\Omega^{\alpha-2}(t_0+t)|v-w|(|v|^\alpha+|w|^\alpha)$, and thanks to a Hölder inequality, given that $\frac{1}{q'} = \frac{1}{(\alpha+1)q'} + \frac{\alpha}{(1+\alpha)q'}$,

$$\begin{aligned}\|K(v_1)-K(v_2)\|_{X^s_\tau} \leq \quad & C\|v_1-v_2\|_{L^{(\alpha+1)q'}}(2\| \, |S(t)u_0|^\alpha\|_{L^{(\alpha+1)q'/\alpha}} + \\ & \| \, |v_1|^\alpha\|_{L^{(\alpha+1)q'/\alpha}} + \| \, |v_2|^\alpha\|_{L^{(\alpha+1)q'/\alpha}})\end{aligned}$$

$$\|K(v_1)-K(v_2)\|_{X^s_\tau} \leq C\tau^\delta(A^\alpha + \|v_1\|^\alpha_{X^s_\tau} + \|v_2\|^\alpha_{X^s_\tau})\|v_1-v_2\|_{X^s_\tau}$$

For $\tau = c(1+A)^{-\alpha/\delta}$ with c small enough, we get that K is contracting on the ball of X^s_τ with centre 0 and radius $C'A$, and so we can apply the fix point theorem in this ball, which is also stable under K. There exists a unique solution v of the equation, and it is such that $\|v\|_{X^s_\tau} \leq C'A$.

$\diamond$

2.3.5 Measure construction

We would like to build a measure of the form $(\exp -\mathcal{E}(u))du$, where du is "morally" speaking a Lebesgue measure.

Definition 2.3.11. We write μ_N the image measure on E_N by $\varphi_N : \omega \mapsto \sum_{n=1}^N \frac{\sqrt{2}}{n} g_n(\omega) e_n(.)$.

Proposition 2.3.12. *We have*

$$d\mu_N\Big(\sum_{n=1}^N (a_n + ib_n)e_n\Big) = d_N e^{-\sum \frac{n^2}{2}(a_n^2+b_n^2)} \prod_{n=1}^N da_n db_n$$

where d_N is a factor such that $\mu_N(E_N) = 1$.

What is more, the sequence $(\varphi_N)_N$ is a Cauchy sequence in $L^2(\Omega, H_R^\sigma)$ for all $\sigma < 1/2$, hence it converges toward a function φ in $L^2(\Omega, H_R^\sigma)$. We denote by μ the image measure on H^σ by φ.

Proof The g_n being independent, the vector $(\sqrt{2}g_1, \ldots, \frac{\sqrt{2}}{N} g_N)$ is a complex Gaussian random variable with average 0 and covariance matrix :

$$\begin{pmatrix} 1/2 & & \\ (0) & \ddots & (0) \\ & & N^2/2 \end{pmatrix}$$

that is to say a real Gaussian random variable of covariance matrix :

$$\begin{pmatrix} 1 & 0 & & & \\ 0 & 1 & & & \\ & (0) & \ddots & (0) & \\ & & & N^2 & 0 \\ & & & 0 & N^2 \end{pmatrix}$$

hence the result.

Let us show now that φ_N is a Cauchy sequence. Let $N \geq M \geq 0$ be two integers, we have :

$$\|\varphi_N - \varphi_M\|_{H_R^\sigma}^2 = 2 \sum_{n=M}^N |g_n|^2 \frac{1}{n^2(1-\sigma)}$$

$$\|\varphi_N - \varphi_M\|_{L_\omega^2, H_R^\sigma}^2 = 2 \sum_{n=N}^M \frac{1}{n^{2(1-\sigma)}}$$

as $\sigma < 1/2$ implies $2(1-\sigma) > 1$, the series of general term $n^{-2(1-\sigma)}$ converges and we get the result.

◊

Proposition 2.3.13. *Almost surely, $\int_0^\pi \Omega^{\alpha-2}|Re\varphi(\omega)|^{\alpha+2}\sin^2 R dR$ is finite.*

Proof Indeed, since $\Omega \leq 2$, to show that the probability of the event "$\int \Omega^{\alpha-2}|\mathrm{Re}\varphi(\omega)|^{\alpha+2}\sin^2 R dR = \infty$" is zero, we only have to prove that $E(\|\varphi\|^{\alpha+2}_{L_R^{\alpha+2}})$ is finite.

Since $E(\|\varphi\|^{\alpha+2}_{L_R^{\alpha+2}}) = \int \|\varphi\|^{\alpha+2}_{L_\omega^{\alpha+2}} \sin^2 R dR$ and, for all q, thanks to Lemma 2.2.8, we have

$$\|\sum c_n g_n\|_{L^q_\omega} \leq C\sqrt{q}(\sum |c_n|^2)^{1/2}$$

the norm $L^{\alpha+2}$ of φ is less than $C(\sum \frac{2}{n^2}|e_n(x)|^2)^{1/2}$. We deduce from that :

$$E(\|\varphi\|^{\alpha+2}_{L_R^{\alpha+2}}) \leq C\|\sum \frac{2}{n^2}|e_n|^2\|^{(\alpha+2)/2}_{L^{(\alpha+2)/2}} \leq C\left(\sum \frac{2}{n^2}\|e_n\|^2_{L^{\alpha+2}}\right)^{(\alpha+2)/2}$$

But we have seen that $\|e_n\|_{L^{\alpha+2}} = O(n^{1-\frac{3}{\alpha+2}}) = O(n^{2/5})$ so the series converges. We get that $E(\|\varphi\|^{\alpha+2}_{L_R^{\alpha+2}}) < \infty$, hence the result.

$\diamond$

We can now define a new measure ρ on H^σ.

Definition 2.3.14. We define on H^σ with $\sigma < \frac{1}{2}$ the measure ρ by :

$$d\rho(u) = e^{-\frac{1}{\alpha+2}\int \Omega^{\alpha-2}|\mathrm{Re}u|^{\alpha+2}} d\mu(u)$$

This measure is the limit of a sequence of measure on E_N with the following meaning :

Definition 2.3.15. We denote by ρ_N the measure on E_N such that :

$$d\rho_N(u) = \exp\left(-\frac{1}{\alpha+2}\int \Omega^{\alpha-2}|S_N \mathrm{Re}u|^{\alpha+2}\right) d\mu_N(u) \ .$$

Proposition 2.3.16. *The map $u \mapsto e^{-\frac{1}{\alpha+2}\int \Omega^{\alpha-2}|S_N Reu|^{\alpha+2}}$ converges in norm $L^1_{d\mu}$ towards $e^{-\frac{1}{\alpha+2}\int \Omega^{\alpha-2}|Reu|^{\alpha+2}}$. We deduce from that : $\lim \rho_N(E_N) = \rho(H^\sigma)$.*

Proof Convergence of the $f_N(u) = \exp(-\frac{1}{\alpha+2}\int \Omega^{\alpha-2}|S_N \mathrm{Re}u|^{\alpha+2})$.

The proof uses the following lemma :

Lemma 2.3.17. *Let $p \geq 2$, $\sigma < 1/2$. And let $s < 1/2$ if $p \leq 3$, $s < \frac{3}{p} - \frac{1}{2}$ otherwise. There exists $\beta(s), \lambda_0(p) > 0$ such that for all $N \geq N_0 \geq 0$, we have :*

$$\mu(\{u \in \mathcal{H}^\sigma \mid \|S_N u - S_{N_0} u\|_{\mathcal{W}^{s,p}} > \lambda\}) \leq \begin{cases} \exp(-cN_0^{\beta(s)}\lambda^2) & \text{if } \lambda > \lambda_0(p) \\ C\lambda^{-p} N_0^{-p\beta(s)/2} & \text{otherwise} \end{cases}$$

In particular, for $N_0 = 1$, we get the property : there exists C, c such that for all $\lambda \geq 1$ and $N \geq 1$

$$\mu(u \in \mathcal{H}^\sigma \mid \|S_N u\|_{\mathcal{W}^{s,p}} > \lambda) \leq C e^{-c\lambda^2}.$$

And by making N go to ∞,

$$\mu(\{u \in \mathcal{H}^\sigma \mid \|u - S_{N_0} u\|_{\mathcal{W}^{s,p}} > \lambda\}) \leq \begin{cases} \exp(-cN_0^{\beta(s)}\lambda^2) & \text{if } \lambda > \lambda_0(p) \\ C\lambda^{-p} N_0^{-p\beta(s)/2} & \text{otherwise} \end{cases}$$

We have

$$\int d\mu |f(u) - f_N(u)| = \int d\lambda \mu(|f(u) - f_N(u)| > \lambda)$$

But,

$$|f(u) - f_N(u)| \leq C \Big| \Big(\int \Omega^{\alpha-2} |\mathrm{Re} u|^{\alpha+2}\Big)^{1/(\alpha+2)} - \Big(\int \Omega^{\alpha-2} |\mathrm{Re} S_N u|^{\alpha+2}\Big)^{1/(\alpha+2)} \Big|$$
$$\leq C \|u - S_N u\|_{L^{\alpha+2}}$$

so

$$\int |f(u) - f_N(u)| d\mu \leq C \int \mu(\|u - S_N u\|_{L^{\alpha+2}} > \lambda) d\lambda \ .$$

We apply the lemma for $p = \alpha + 2$, $s = 0$ (which is possible since $\alpha < 3$). As $S_N u$ converges towards u in $L^1_\omega, L^{\alpha+2}_R$-norm , (see the proof of Proposition 2.3.13 and replace φ by $S_N \varphi$), we can do $N \to \infty$ in the lemma and replace the notation N_0 by the notation N. Then, we divide the integral into three parts : between 0 and $N^{-\gamma}$ with $0 < \gamma < \frac{\beta(\alpha+2)}{2(\alpha+1)}$, then between $N^{-\gamma}$ and λ_0, and at last λ_0 and ∞.

For the first part of the integral, we bound

$$\mu(\|u - S_N u\|_{L^{\alpha+2}} > \lambda)$$

by 1 which gives :

$$\int_0^{N^{-\gamma}} \mu(\|u - S_N u\|_{L^{\alpha+2}} > \lambda) d\lambda \leq \frac{1}{N^\gamma} \ ,$$

which converges towards 0 when N goes to ∞.

For the second part of the integral, we use the third inequality of Lemma 2.3.17, with λ less than λ_0, and we have $p = \alpha + 2$.

$$\int_{N^{-\gamma}}^{\lambda_0} \mu(\|u - S_N u\|_{L^{\alpha+2}} > \lambda) d\lambda \leq C \int_{N^{-\gamma}}^{\lambda_0} \frac{N^{-(\alpha+2)\beta/2}}{\lambda^{\alpha+2}} d\lambda$$

Computing the integral gives

$$\int_{N^{-\gamma}}^{\lambda_0} \mu(\|u - S_N u\|_{L^{\alpha+2}} > \lambda) d\lambda \leq \frac{C}{\alpha+1} N^{-(\alpha+2)\beta/2 + \gamma(\alpha+1)}$$

and this converges towards 0 with the choice we have made for γ, as

$$\frac{(\alpha+2)\beta}{2} - \gamma(\alpha+1) < 0 \ .$$

For the third part of the integral, we use Lemma 2.3.17 with $\lambda \geq \lambda_0$, so

$$\mu(\|u - S_N u\|_{L^{\alpha+2}} > \lambda) \leq e^{-cN^\beta \lambda^2}$$

which gives

$$\int_{\lambda_0}^{\infty} \mu(\|u - S_N u\|_{L^{\alpha+2}} > \lambda) d\lambda \leq \int e^{-cN^\beta \lambda^2} d\lambda$$

and with a change of variable $\lambda \leftarrow N^{\beta/2}\lambda$,

$$\int_{\lambda_0}^{\infty} \mu(\|u - S_N u\|_{L^{\alpha+2}} > \lambda) d\lambda \leq \frac{1}{N^{\beta/2}} \int e^{-c\lambda^2} d\lambda$$

which converges towards 0 when $N \to \infty$.

End of the proof of the L^1 convergence of f_N.

Let us show that $\rho_N(E_N)$ tends to $\rho(H^\sigma)$.

$$\rho_N(E_N) = \int f_N(\varphi_N(\omega)) d\omega = \int f_N(\varphi(\omega)) d\omega \to \int f(\varphi(\omega)) d\omega = \rho(H^\sigma)$$

◊

Let us prove the Lemma 2.3.17.

Proof By rewriting μ as the image measure of P by the map φ, we get that

$$\mu(u \in H^\sigma \mid \|S_N u - S_{N_0} u\|_{W^{s,p}} > \lambda)$$

is equal to

$$P(\|\sum_n (\chi_S(\frac{n^2}{N^2}) - \chi_S(\frac{n^2}{N_0^2}))\frac{\sqrt{2}}{n} g_n(\omega) e_n(x)\|_{W_x^{s,p}} > \lambda) \ .$$

and by using the facts that the $W^{s,p}$ norm is given by

$$\|G\|_{W^{s,p}} = \|H^s G\|_{L^p}$$

and that

$$H^s\left(\sum_n (\chi_S(\frac{n^2}{N^2}) - \chi_S(\frac{n^2}{N_0^2}))\frac{\sqrt{2}}{n} g_n(\omega) e_n(x)\right) = \sum_n (\chi_S(\frac{n^2}{N^2}) - \chi_S(\frac{n^2}{N_0^2}))\frac{\sqrt{2}}{n^{1-s}} g_n(\omega) e_n(x)$$

we get that the probability

$$\mu(u \in H^\sigma \mid \|S_N u - S_{N_0} u\|_{W^{s,p}} > \lambda) = P(\|\sum_n (\chi_S(\frac{n^2}{N^2}) - \chi_S(\frac{n^2}{N_0^2}))\frac{\sqrt{2}}{n^{1-s}} g_n(\omega) e_n(x)\|_{L^p_x} > \lambda) \ .$$

Set $f(\omega, x) = \sum_n (\chi_S(\frac{n^2}{N^2}) - \chi_S(\frac{n^2}{N_0^2}))\frac{\sqrt{2}}{n^{1-s}} g_n(\omega) e_n(x)$.

Let $q \geq p$. Thanks to a convexity inequality,

$$\|f\|_{L^q_\omega, L^p_x} \leq \|f\|_{L^p_x, L^q_\omega}.$$

We have seen that

$$\|\sum_n c_n g_n\|_{L^q_\omega} \leq C_1 \sqrt{q} (\sum |c_n|^2)^{1/2}$$

so

$$\|f\|_{L^q_\omega} \leq C_1 \sqrt{q} (\sum_n |\chi_S(\frac{n^2}{N^2}) - \chi_S(\frac{n^2}{N_0^2})|^2 \frac{2}{n^{2(1-s)}} |e_n(x)|^2)^{1/2}$$

With a triangle inequality,

$$\|f\|_{L^p_x, L^q_\omega} \leq C_1 \sqrt{q} (\sum_n |\chi_S(\frac{n^2}{N^2}) - \chi_S(\frac{n^2}{N_0^2})|^2 \frac{2}{n^{2(1-s)}} \| \, |e_n|^2 \|_{L^{p/2}})^{1/2}$$

As $\| \, |e_n|^2 \|_{L^{p/2}} = \|e_n\|^2_{L^p}$ is less than a constant independent from n for $p < 3$, than $C_p \log^2 n$ if $p = 3$ and by $C_p n^{2-6/p}$ otherwise, we have

$$\sum_n |\chi_S(\frac{n^2}{N^2}) - \chi_S(\frac{n^2}{N_0^2})|^2 \frac{2}{n^{2(1-s)}} \| \, |e_n|^2 \|_{L^{p/2}} \leq \begin{cases} C_p \sum_n |\chi_S(\frac{n^2}{N^2}) - \chi_S(\frac{n^2}{N_0^2})|^2 \frac{2}{n^{2(1-s)}} & \text{if } \ p < 3 \\ C_p \sum_n |\chi_S(\frac{n^2}{N^2}) - \chi_S(\frac{n^2}{N_0^2})|^2 \frac{2 \log^2 n}{n^{2(1-s)}} & \text{if } \ p = 3 \\ C_p \sum_n |\chi_S(\frac{n^2}{N^2}) - \chi_S(\frac{n^2}{N_0^2})|^2 \frac{2}{n^{6/p-2s}} & \text{otherwise} \end{cases}$$

The real number s being strictly less than $1/2$ for $p \leq 3$ and than $\frac{3}{p} - \frac{1}{2}$ otherwise, there exists $\beta(s) > 0$ such that for all $N \geq N_0 \geq 1$,

$$\sum_n |\chi_S(\frac{n^2}{N^2}) - \chi_S(\frac{n^2}{N_0^2})|^2 \frac{2}{n^{2(1-s)}} \| \, |e_n|^2 \|_{L^{p/2}} \leq C_3 N_0^{-\beta(s)}.$$

Finally,

$$\|f\|_{L^q_\omega, L^p_x} \leq C_4 \sqrt{q} N_0^{-\beta(s)/2}.$$

We get then

$$P(\|f\|_{L^p_x} > \lambda) = P(\|f\|^q_{L^p_x} > \lambda^q) \le \lambda^{-q}\|f\|^q_{L^q_\omega, L^p_x}$$

$$P(\|f\|_{L^p_x} > \lambda) \le \left(\frac{C_4(p)}{\lambda N_0^{\beta(s)/2}}\sqrt{q}\right)^q$$

For all $\lambda \ge \lambda_0(p) := 2p/C_4(p)$, the real number $q = \frac{\lambda^2 N_0^{\beta(s)}}{4C_4^2}$ is more than p, which means that for $\lambda \ge \lambda_0 := 2p/C_4$ and $N \ge N_0 \ge 1$, by choosing $q = \frac{\lambda^2 N_0^{\beta(s)}}{4C_4^2}$,

$$P(\|f\|_{L^p_x} > \lambda) \le e^{-cN_0^{\beta(s)}\lambda^2}$$

with $c = \frac{log2}{4C_4^2}$.

For $\lambda < \lambda_0$ we choose $p = q$.

In the end, there exists C such that for all $N \ge N_0 \ge 1$,

$$\mu(u \in H^\sigma \mid \|S_N u - S_{N_0} u\|_{W^{s,p}} > \lambda) \le \begin{cases} e^{-cN_0^{\beta(s)}\lambda^2} & \text{if } \lambda \ge \lambda_0 \\ C\lambda^{-p} N_0^{-\beta(s)p/2} & \text{otherwise} \end{cases}$$

◊

Proposition 2.3.18. *For every set $A \subseteq E_N$, the image under the flow of A, i.e. $\Psi_N(0,t)A$ for any time $t \in [-\pi, \pi]$ satisfies :*

$$\mu_N(\Psi_N(0,t)A) \ge \rho_N(A)$$

Proof The Lebesgue measure on E_N is well-defined (since E_N has a finite dimension) and written du.

$$\mu_N(\Psi_N(0,t)A) = \int d_N du 1_{\Psi_N(0,t)A}(u) e^{-\frac{1}{2}\|Hu\|^2_{L^2}}$$

Thanks to a computation similar to the one made in section 2.2 about $\mathcal{E}$, we get that $\mathcal{E}_N$ is decreasing under the flow so, $\mathcal{E}_N(0,u) \ge \mathcal{E}_N(t, \Psi(0,t)u) \ge \frac{1}{2}\|H\Psi_N(0,t)u\|^2_{L^2}$. What is more, the Lebesgue measure is invariant under the flow so :

$$\mu_N(\Psi_N(0,t)A) = \int d_N du 1_A(u) e^{-\frac{1}{2}\|H\Psi_N(0,t)u\|^2_{L^2}}$$

We get

$$\mu_N(\Psi_N(0,t)A) \geq \int d_N du 1_A(u) e^{-\mathcal{E}_N(0,u)} = \rho_N(A).$$

$\diamond$

2.3.6 Well-posedness for all times

Now that we have a sequence of measures ρ_N and ρ with nice properties regarding the pseudo-Hamiltonian flows $\Psi_N(t_0,t)$, we will show the existence of a ρ-full measured set Σ, such that the flow $\Psi(0,t)$ is defined for all $t \in [-\pi,\pi]$ on this set.

Proposition 2.3.19. *Let $p \in]2\alpha, 6[$ and s defined accordingly by $s = \frac{3}{2} - \frac{4}{p}$. Let $\sigma < \frac{1}{2}$. For all integers i and N, there exists a set Σ_N^i, ρ_N-measurable such that $\rho_N(E_N \setminus \Sigma_N^i) = O(2^{-i})$ and $C \geq 0$ (independent from i and from N) such that for all $u_0 \in \Sigma_N^i$ and all $t \in]-\pi,\pi[$,*

$$\|S(t')\Psi_N(0,t)u_0\|_{L^p(t',x\in[-\pi,\pi]\times S^3)} + \|\Psi_N(0,t)u_0\|_{H^\sigma} \leq C\sqrt{i}$$

Proof

Let D be a positive real number and $B_N^i(D) = \{u \in E_N \mid \|S(t)u\|_{L^p_{t,x}} + \|u\|_{H^\sigma} \leq D\sqrt{i}\}$.

Let us study the probability $\mu_N(E_N \setminus B_N^i(D))$.
The set $E_N \setminus B_N^i(D)$ is given by :

$$E_N \setminus B_N^i(D) = \{u \mid \|S(t)u\|_{L^p_{t,x}} + \|u\|_{H^\sigma} > D\sqrt{i}\} .$$

Then, if $\|S(t)u\|_{L^p_{t,x}} + \|u\|_{H^\sigma} > D\sqrt{i}$ either one of the norm is bigger than $\frac{D\sqrt{i}}{2}$. Hence, we have the inclusion

$$\begin{aligned} E_N &\subseteq \{u \in E_N \mid \|S(t)u\|_{L^p} > \frac{1}{2}D\sqrt{i}\} \\ &\quad \cup \{u \in E_N \mid \|u\|_{H^\sigma} > \frac{1}{2}D\sqrt{i}\} \end{aligned}$$

so we get the inequality between the probabilities :

$$\mu_N(E_N \setminus B_N^i(D)) \leq \mu_N(\{u \in E_N \mid \|S(t)u\|_{L^p} > \frac{1}{2}D\sqrt{i}\}) + \mu_N(\{u \in E_N \mid \|u\|_{H^\sigma} > \frac{1}{2}D\sqrt{i}\}) .$$

But we have, by rewriting $S(t)u$:

$$\mu_N(\{u \in E_N \mid \|S(t)u\|_{L^p} > \frac{1}{2}D\sqrt{i}\}) \leq P(\|\sum_{n=1}^N n^{-1} g_n(\omega) e^{-int} e_n\|_{L^p} > D\sqrt{i}/2)$$

and by taking the quantities inside P to the power p

$$\mu_N(\{u \in E_N \mid \|S(t)u\|_{L^p} > \frac{1}{2}D\sqrt{i}\}) \leq P(\|\sum_{n=1}^{N} n^{-1} g_n(\omega)e^{-int}e_n\|_{L^p}^q > (D\sqrt{i}/2)^q)$$

and then applying Markov inequality :

$$\mu_N(\{u \in E_N \mid \|S(t)u\|_{L^p} \leq 2^q D^{-q} i^{-q/2} \|\sum \frac{g_n}{n} e^{int} e_n\|_{L^q_\omega, L^p_{x,t}}^q .$$

For all $q \geq p$, thanks to Minkowski inequality, this implies that :

$$\mu_N(\{u \in E_N \mid \|S(t)u\|_{L^p} > \frac{1}{2}D\sqrt{i}\}) \leq 2^q D^{-q} i^{-q/2} \|\sum \frac{g_n}{n} e^{int} e_n\|_{L^p_{x,t}, L^q_\omega}^q$$

$$\leq 2^q D^{-q} i^{-q/2} \| \sqrt{q \sum \frac{|e_n|^2}{n^2}} \|_{L^p}^q$$

and by Lemma 2.2.8

$$\leq 2^q D^{-q} i^{-q/2} q^{q/2} (\sum \|e_n\|_{L^p}^2 n^{-2})^{q/2}$$

As $p < 6$, there exists $\nu(p)$ and $C(p)$ such that $\|e_n\|_{L^p}^2 < C(p) n^{1-\nu(p)}$ so the sum converges, we get a triangle inequality that can be written

$$\mu_N(\{u \in E_N \mid \|S(t)u\|_{L^p} > \frac{1}{2}D\sqrt{i}\}) \leq (\frac{C\sqrt{q}}{D\sqrt{i}})^q .$$

With $q = \frac{D^2 i}{C^2 e^2} \geq p$, i.e. $D^2 \geq \frac{C^2 e^2 p}{i}$, we get :

$$\mu_N(\{u \in E_N \mid \|S(t)u\|_{L^p} > \frac{1}{2}D\sqrt{i}\}) \leq e^{-\frac{D^2 i}{C^2 e^2}}$$

There exists C, such that for all $D \geq Ce\sqrt{p}$ and all i,

$$\mu_N(\{u \in E_N \mid \|S(t)u\|_{L^p} > \frac{1}{2}D\sqrt{i}\}) \leq Ce^{-cD^2 i}$$

The same argument can be used with the norm H^σ since the general term of the series becomes $n^{2(\sigma-1)}$ and since $\sigma < \frac{1}{2}$, so we have :

$$\rho_N(E_N \setminus B_N^i(D)) \leq Ce^{-cD^2 i}$$

with C and c independent from D, i and N.

Let τ be the time defined in the local well posedness lemma, we have, for all t_0, t such that $t - t_0 \in [-\tau, \tau]$:

$$\Psi_N(t_0, t)(B_N^i(D)) \subseteq \{u \in E_N \mid \|S(t')u\|_{L^p_{t,x}} + \|u\|_{H^\sigma} \leq CD\sqrt{i}\}$$

We set

$$\Sigma_N^i(D) = \bigcap_{k=-[\pi/\tau]}^{[\pi/\tau]} \Psi_N(0,k\tau)^{-1}(B_N^i(D))$$

The probability of its complementary in E_N satisfies :

$$\rho_N(E_N \setminus \Sigma_N^i(D)) \leq \sum_k \rho_N(E_N \setminus \Psi_N(0,k\tau)^{-1} B_N^i(D))$$

Since

$$\rho_N(E_N \setminus \Psi_N(0,k\tau)^{-1} B_N^i(D)) = \rho_N(\Psi_N(0,k\tau)^{-1}(E_N \setminus B_N^i(D)) \leq \mu_N(E_N \setminus B_N^i(D))$$

by Proposition 2.3.18, we get

$$\rho_N(E_N \setminus \Sigma_N^i(D)) \leq (2[\frac{\pi}{\tau}] + 1)Ce^{-cD^2 i}$$

What is more, τ is equal to $c(1 + D\sqrt{i})^{-\delta}$ so

$$\rho_N(E_N \setminus \Sigma_N^i(D)) \leq C(D\sqrt{i})^{\delta} e^{-cD^2 i} .$$

We choose D large enough ($D^2 > \log 2/c$) to have $CD^\delta i^{\delta/2} e^{-cD^2 i} \leq C' 2^{-i}$, C' independent from N and i, and we write $\Sigma_N^i = \Sigma_N^i(D)$. We have $\rho_N(E_N \setminus \Sigma_N^i) \leq C' 2^{-i}$.

One can see that for all t in $[-\pi,\pi]$, t can be written $k\tau + t_1$ with $t_1 \in [-\tau,\tau]$, and $k \in \{-[\pi,\tau],\ldots,[\pi/\tau]\}$. So we get, for $u \in \Sigma_N^i$, $\Psi_N(0,t)u = \Psi_N(k\tau, k\tau + t_1)(\Psi_N(0,k\tau)u)$ and since $\Psi_N(0,k\tau)u \in B_N^i(D)$, $\|S(t')\Psi_N(0,t)u\|_{L^p_{t',x}} + \|\Psi_N(0,t)u\|_{H^\sigma} \leq CD\sqrt{i}$.

◊

Definition 2.3.20. Let $\widetilde{\Sigma}_N^i = \{u \in H^\sigma \mid \Pi_N u \in \Sigma_N^i\}$ and $\Sigma^i = \limsup \widetilde{\Sigma}_N^i$. For all $u \in \Sigma^i$ there exists a sequence $u_k \in \Sigma_{N_k}^i$ ($N_k \to \infty$) such that u_k converges toward u in H^σ.

We also write $\Sigma = \bigcup_i \Sigma^i$.

Proposition 2.3.21. *The set Σ is of full measure.*

Proof We have $\rho(\Sigma_i) \geq \limsup \rho(\widetilde{\Sigma}_N^i)$. By computing

$$\rho_N(\Sigma_N^i) = \int_{\Sigma_N^i} f_N(u) d\mu_N$$

we get

$$\rho_N(\Sigma_N^i) = \int \mu_N \left(\Sigma_N^i \cap f_N^{-1}(]\lambda,\infty])\right) d\lambda$$

and as μ_N is the image measure of μ by Π_N,

$$\rho_N(\Sigma_N^i) = \int \mu\left(\Pi_N^{-1}(\Sigma_N^i \cap f_N^{-1}(]\lambda, \infty]))\right) d\lambda$$

we use then the equality between the sets $\Pi_N^{-1}(\Sigma_N^i \cap f_N^{-1}(]\lambda, \infty]) = \widetilde{\Sigma}_N^i \cap f_N^{-1}(]\lambda, \infty])$

$$= \int \mu\left(\widetilde{\Sigma}_N^i \cap f_N^{-1}(]\lambda, \infty])\right) d\lambda$$

$$= \int_{\widetilde{\Sigma}_N^i} f_N(u) d\mu(u) .$$

Recall that $\rho(\widetilde{\Sigma}_N^i) = \int_{\widetilde{\Sigma}_N^i} f(u) d\mu(u)$.

Since f_N converges in L^1_μ toward f,

$$\begin{aligned} \limsup \rho(\widetilde{\Sigma}_N^i) &= \limsup \int_{\widetilde{\Sigma}_N^i} f_N(u) d\mu \\ &= \limsup \rho_N(\Sigma_N^i) . \end{aligned}$$

What is more,

$$\limsup \rho_N(\Sigma_N^i) \geq \lim \rho_N(E_N) - C2^{-i} = \rho(H^\sigma) - C2^{-i} .$$

We deduce that $\rho(\Sigma) \geq \rho(H^\sigma) - C2^{-i}$ for all i, and so $\rho(\Sigma) = \rho(H^\sigma)$. ◊

Proposition 2.3.22. *Let $p \in]2\alpha, 6[$ and s defined accordingly as $s = \frac{3}{2} - \frac{4}{p}$. Let also $\sigma < \frac{1}{2}$. For all $u_0 \in \Sigma$, there exists a strong solution for all $T \in [-\pi, \pi]$ of the Hamiltonian problem.*

To prove this proposition, we need the following lemma :

Lemma 2.3.23. *Let $t_0 \in [0, \pi]$ and suppose that $\Psi(0, t)u_0$ is defined for all $t \in [0, t_0]$, and satisfies*

$$\|S(t')\Psi(0, t_0)u_0\|_{L^p} + \|\Psi(0, t_0)u_0\|_{H^\sigma} \leq C' \sqrt{i} .$$

We assume that $\Psi(0, t)u_0$ is the unique solution of (2.10) *in the sense that $\Psi(0, t)u_0 - S(t)u_0$ is unique in $X^s_{t_0}$.*

We suppose also that $u_k(t_0)$ converges toward $\Psi(0, t_0)u_0$ in norms $\| \,.\, \|_{H^\sigma_R}$ and $\|S(t). \|_{L^p_{t,R}}$.

Then, there exists a time τ' independent from t_0 such that the solution $\Psi(0, t)u_0$ is defined and unique in the sense that $\Psi(0, t)u_0 - S(t)u_0$ is unique in $X^s_{t_0+\tau'}$ for all $t \in [0, t_0 + \tau']$ and satisfies :

$$\|S(t')\Psi(0, t_0 + \tau')u_0\|_{L^p_{t',R}} + \|\Psi(0, t_0 + \tau')u_0\|_{H^\sigma_R} \leq C' \sqrt{i} .$$

We get the same convergences at time $t_0 + \tau'$.

Proof of Proposition 2.3.22. Let us begin with setting i the integer such that $u_0 \in \Sigma_i$, there exists a sequence $N_k \to \infty$ such that $u_{0,k} := \Pi_{N_k} u_0 \in \Sigma_N^i$. As the norms $\|u_{0,k}\|_{H^\sigma}$ are bounded by $D\sqrt{i}$, and that the sequence $u_{0,k}$ converges toward u_0 in norm H^σ, we have $\|u_0\|_{H^\sigma} \leq D\sqrt{i}$.

We also have that the norms $\|S(t)u_{0,k}\|_{L^p}$ are bounded by $D\sqrt{i}$, and that $S(t)u_{0,k}$ converges as distributions toward $S(t)u_0$ so $S(t)u_{0,k}$ converges toward $S(t)u_0$ in norm L^p.

Set $u_k(t) = \Psi_{N_k}(0,t)u_{0,k}$. For all $t \in [-\pi, \pi]$ and all k we have

$$\|S(t')u_k(t)\|_{L^p_{t',x}} + \|u_k(t)\|_{H^\sigma} \leq C'\sqrt{i}.$$

what is more,

$$\|S(t')u_0\|_{L^p} + \|u_0\|_{H^\sigma} \leq C'\sqrt{i}$$

To finish the proof of the proposition, we can then apply the lemma with
$t_0 = 0, \tau', 2\tau' \ldots, M\tau'$, with $M = [\pi/\tau']$ and use a similar argument for the negative times. ◊

Let us now prove the lemma.

Proof of the Lemma 2.3.23.

Let $A = C'\sqrt{i}$ and τ the corresponding time involved in the local well-posedness lemma. On $[t_0-\tau, t_0+\tau]$, we can define u the local solution and $v(t) = u(t+t_0) - S(t)\Psi(0,t_0)u_0$. As v is unique from the local well-posedness lemma in X^s_τ, we have that

$$v(t) = \Psi(0, t+t_0)u_0 - S(t+t_0)u_0 + S(t)\left(\Psi(0,t_0)u_0 - S(t_0)u_0\right) ,$$

and $S(t)\left(\Psi(0,t_0)u_0 - S(t_0)u_0\right)$ is in $X^s_{\tau+t_0}$ by hypothesis, we have that $\Psi(0,t)u_0 - S(t)u_0$ is unique in $X^s_{t_0+\tau}$.

We write the analogous functions $v_k(t) = u_k(t+t_0) - S(t)u_k(t_0)$, and $w_k = v - v_k$. Remark that from the local well-posedness property, for all $t \in [-\tau, \tau]$:

$$\|v\|_{X^s_\tau} + \|u(t+t_0)\|_{H^\sigma} + \|S(t')u(t+t_0)\|_{L^p_{t',x}} \leq CA,$$

$$\|v_k\|_{X^s_\tau} + \|u_k(t_0+t)\|_{H^\sigma} + \|S(t')u_k(t_0+t)\|_{L^p_{t',x}} \leq CA.$$

The map w_k can be written as :

$$w_k = v - v_k = -i\int_0^t dt' S(t-t')H^{-1}(F(t'+t_0, u) - S_{N_k}F(t'+t_0, S_{N_k}u_k))$$

$$\begin{aligned} w_k = \quad & -i\textstyle\int_0^t dt' S(t-t')H^{-1}S_{N_k}(F(t'+t_0,u) - F(t'+t_0, S_{N_k}u)) \\ & -i\textstyle\int_0^t dt' S(t-t')H^{-1}(1-S_{N_k})F(t'+t_0,u) \end{aligned}$$

For $\tau' \leq \tau$, the norm $X^s_{\tau'}$ of the first integral is less than :

$$\| -i \int_0^t dt' S(t-t') H^{-1} S_{N_k}(F(t'+t_0, u) - F(t'+t_0, S_{N_k} u)) \|_{X^s_{\tau'}}$$

$$\leq C(\tau')^\delta A^2 (\|S(t)(\Psi(0,t_0)u_0 - S_{N_k} u_k(t_0)\|_{L^p} + \|v - S_{N_k} v_k\|_{X^s_{\tau'}})$$

$$\leq C(\tau')^\delta A^2 \|w_k\|_{X^s_{\tau'}}$$

with $\delta > 0$ chosen as previously.

Regarding the second integral, we use the fact that for $s < s_1 < (\alpha+1)s - \frac{3\alpha-4}{2}$ (which is possible since $p > 2\alpha$),

$$\|(1 - S_{N_k}) \int S(t'-t) H^{-1} F(t'+t_0, u) dt' \|_{X^s_{\tau'}} \leq C N_k^{s-s_1} \|F(t'+t_0, u)\|_{Y^{1-s_1}_{\tau'}}$$

By choosing p_1 such that $\frac{4}{p_1} = \frac{3}{2} - 1 + s_1 = \frac{1}{2} + s_1$, that is to say $\frac{4}{p_1'} = \frac{7}{2} - s_1$, we have $\|u\|^{\alpha+1}_{L^{(\alpha+1)p_1'}} \geq C\|F(u)\|_{L^{p_1'}} \geq C\|F(u)\|_{Y^{1-s_1}_{\tau'}}$. But,

$$\frac{4}{(\alpha+1)p_1'} = \frac{7}{2(\alpha+1)} - \frac{s_1}{\alpha+1}$$

$$\frac{4}{(\alpha+1)p_1'} \geq \frac{7}{2(1+\alpha)} - s + \frac{3\alpha-4}{2(1+\alpha)} = \frac{3}{2} - s = \frac{4}{p}$$

So

$$\|(1 - S_{N_k}) \int S(t'-t) H^{-1} F(t'+t_0, u) dt' \|_{X^s_{\tau'}} \leq C N_k^{s-s_1} (\tau')^{\delta_1} (\|\Psi(0,t_0)u_0\|_{H^\sigma} + \|v\|_{X^s_{\tau'}})$$

with $\delta_1 = \frac{1}{(\alpha+1)p_1'} - \frac{1}{p}$.

Hence,

$$\|w_k\|_{X^s_{\tau'}} \leq C(\tau')^\delta A^2 \|w_k\|_{X^s_{\tau'}} + C^2 N_k^{s-s_1} (\tau')^{\delta_1} A \ .$$

By choosing τ' small enough such that $C(\tau')^{\delta_1} A^2 < 1$, we get that the norm of w_k converges toward 0 when $k \to \infty$. We deduce that for all $t \in [-\tau', \tau']$,

$$\begin{aligned} \|u(t_0+t) - u_k(t_0+t)\|_{H^\sigma} \leq \quad & \|S(t)(u(t_0)) - u_k(t_0))\|_{H^\sigma} \\ & + \|u(t_0+t) - u_k(t_0+t) - S(t)(u(t_0) - u_k(t_0))\|_{X^s_{\tau'}} \\ & = \|S(t)\,(u(t_0) - u_k(t_0))\,\|_{H^\sigma} + \|w_k\|_{X^s_{\tau'}} \\ & = \|u(t_0) - u_k(t_0)\|_{H^\sigma} + \|w_k\|_{X^s_{\tau'}} \to 0\ . \end{aligned}$$

What is more, $S(t')(u(t_0+\tau') - u_k(t_0+\tau')) \to 0$ in norm $L^p_{t',R}$. We get that

$$\|S(t')u(t_0+\tau')\|_{L^p} + \|u(t_0+\tau')\|_{H^\sigma} \leq \lim \|S(t')u_k(t_0+\tau')\|_{L^p_{t',R}} + \|u_k(t_0+\tau')\|_{H^\sigma} \leq A = C'\sqrt{i}\ .$$

$\diamond$

2.3.7 Back to the non linear wave equation on the Euclidean space

Let us see now what this result means for the non linear wave equation on the Euclidean space. To understand such a thing, we have to wonder to which spaces belongs the Penrose transform at time $T = 0 \Leftrightarrow t = 0$ of the initial data u_0 that is to say, an initial data taken in L^2.

Let us remark that we will build an image measure of ρ by the Penrose transform at time $t = 0$ and that we need for that to prove the continuity of this map. Unfortunately, the Laplace-Beltrami operator is not turned into a convenient operator by this map. This is the reason why we chose to build the measure for the Euclidean problem on low regularity spaces, knowing that we would be more precise with regards to the spaces the initial datum belongs or does not belong to. This explains why the dependence on σ vanishes from now on.

Definition 2.3.24. Let PT the transform defined on L^2 by :

$$u_0 \in L^2 \mapsto f_0, f_1$$

with

$$f_0(r) = \frac{2}{1+r^2}\mathrm{Re}u_0(2\mathrm{Arctan}\,(r)) \text{ and } f_1(r) = -\left(\frac{2}{1+r^2}\right)^2 (H\Im u_0)(2\mathrm{Arctan}\,(r))\ .$$

Remark 2.3.3. *The transform PT is the Penrose transform taken at time t* $= 0$*. That is to say, if* $f(t,r)$ *is the Penrose transform of* $u(T,R)$ *then* $f(t=0,r) = \frac{2}{1+r^2}Reu_0(2Arctan\,(r))$ *and* $\partial_t f(t=0,r) = -\left(\frac{2}{1+r^2}\right)^2 (H\Im u_0)(2Arctan\,(r))$.

We define now the spaces which the PT sends L^2 to.

Definition 2.3.25. Let $m \in \mathbb{R}$, we define L^2_m (resp. H^{-2}_m) as the set of radial distributions f on $\mathbb{R}^3$ such that the L^2 (resp. H^{-2}) -norm of $(\frac{1+r^2}{2})^{m/2} f$ is finite. This is a normed vector space and its norm is :

$$\|f\|_{L^2_m} = \|\left(\frac{1+r^2}{2}\right)^{m/2} f\|_{L^2}$$

(resp.

$$\|f\|_{H^{-2}_m} = \|\left(\frac{1+r^2}{2}\right)^{m/2} f\|_{H^{-2}}\ ).$$

Proposition 2.3.26. *The transform PT continuously sends $L^2_{S^3}$, meaning the L^2 space of zonal complex functions of the sphere S^3 into $L^2_{-1} \times H^{-2}_{-6}$ which means the L^2 space times the Sobolev space of regularity -2 of radial real functions of the Euclidean space $\mathbb{R}^3$ with weights.*

We will need the following lemma to prove the proposition :

Lemma 2.3.27. *Let u be in $L^2_{S^3}$ and v the radial distribution on the Euclidean space defined by a change of variable as :*

$$v(r) = u(2Arctan\,(r))\ .$$

The change of variable is an isometry between $L^2_{S^3}$ and L^2_{-3}.

Proof Let us compute $\|v\|_{L^2_{-3}}$.

$$\|v\|^2_{L^2_{-3}} = \|\left(\frac{1+r^2}{2}\right)^{-3/2} v\|^2_{L^2}$$

$$\|v\|^2_{L^2_{-3}} = \int_0^\infty |v(r)|^2 \left(\frac{2r}{1+r^2}\right)^2 \frac{2dr}{1+r^2}$$

By the change of variable $R = 2\mathrm{Arctan}\,(r)$, we get :

$$\|v\|^2_{L^2_{-3}} = \int_0^\pi |u(R)|^2 \sin^2(R) dR = \|u\|^2_{L^2_{S^3}}\ .$$

◊

Proof of Proposition 2.3.26.
Let us first deal with the norm of f_0. This distribution is :

$$f_0 = \frac{2}{1+r^2}\mathrm{Re}u_0(2\mathrm{Arctan}\,(r))\ .$$

And let $v_0(r) = \mathrm{Re}u_0(2\mathrm{Arctan}\,(r))$. The L^2_{-1} norm of f_0 is

$$\|f_0\|_{L^2_{-1}} = \|\left(\frac{1+r^2}{2}\right)^{-1/2-1} v_0\|_{L^2} = \|v_0\|_{L^2_{-3}} = \|\mathrm{Re}u_0\|_{L^2}\ .$$

For $f_1 = (\frac{2}{1+r^2})^2(H\mathfrak{I}u_0)(2\mathrm{Arctan}\,(r))$, consider Δ_{S^3} the Laplace-Beltrami operator on the sphere S^3. When acting on zonal functions, this operator reduces into $\partial_R^2 + \frac{2}{\tan R}\partial_R$. Let us now do the change of variable $R = 2\mathrm{Arctan}\,(r)$. First, set u a zonal function of S^3 and v a radial one such that $v(r) = u(2\mathrm{Arctan}\,(r))$ and compute the action of $(1 - \Delta_{S^3})$ after the change of variable $R = 2\mathrm{Arctan}\,(r)$.

$$\frac{du}{dR} = \frac{1+\tan^2(R/2)}{2}\frac{dv}{dr} = \frac{1+r^2}{2}\frac{dv}{dr}\ ,$$

$$\frac{d^2u}{dR^2} = \left(\frac{1+r^2}{2}\right)^2 \frac{d^2v}{dr^2} + \frac{1+r^2}{2} r \frac{dv}{dr} \ ,$$

$$\Delta_{S^3} u = \left(\frac{1+r^2}{2}\right)^2 \frac{d^2v}{dr^2} + \frac{1+r^2}{2r} \frac{dv}{dr} \ .$$

We call

$$D^2 = 1 - \left(\frac{1+r^2}{2}\right)^2 \frac{d^2}{dr^2} - \frac{1+r^2}{2r} \frac{d}{dr} \ .$$

Let φ be a radial function in $H^2(\mathbb{R}^3)$ and compute the distribution bracket :

$$|\langle \left(\frac{1+r^2}{2}\right)^{-3} f_1 \ , \ \varphi \rangle_{\mathbb{R}^3}| \ .$$

By definition, this is equal to :

$$|\int_0^\infty \left(\frac{1+r^2}{2}\right)^{-3} f_1(r)\varphi(r) r^2 dr|$$

and by a change of variable :

$$|\int_0^\pi \left(\frac{1+r^2}{2}\right)^{-2} (H\Im u_0)(R)\psi(R)\sin^2(R) dR|$$

where $\Psi(R) = \varphi(\tan(\frac{R}{2}))$ and $r = \tan(\frac{R}{2})$.

As H is a self-adjoint operator in $L^2_{S^3}$ we get that the bracket is equal to :

$$|\int_0^\pi \Im u_0(R) H \left(\left(\frac{1+r^2}{2}\right)^{-2} \psi \right) \sin^2 R dR|$$

which is less than :

$$||u_0||_{L^2_{S^3}} || \left(\frac{1+r^2}{2}\right)^{-2} \psi ||_{H^1_{S^3}}$$

and also

$$|\langle \left(\frac{1+r^2}{2}\right)^{-3} f_1, \varphi \rangle_{\mathbb{R}^3}| \leq C ||u_0||_{L^2_{S^3}} || \left(\frac{1+r^2}{2}\right)^{-2} \psi ||_{H^2_{S^3}}$$

since $H^2_{S^3}$ is continuously embedded in $H^1_{S^3}$.

So now, we have to prove that $\left(\frac{1+r^2}{2}\right)^{-2}\psi$ is in $H^2_{S^3}$, that is to say that

$$(1-\Delta_{S^3})\left(\left(\frac{1+r^2}{2}\right)^{-2}\psi\right)$$

is in $L^2_{S^3}$.

First, by commuting $(1-\Delta_{S^3})$ and $\left(\frac{1+r^2}{2}\right)^{-2}$, we get that (recall that r is considered as a function of R) :

$$\begin{aligned}(1-\Delta_{S^3})\left(\left(\frac{1+r^2}{2}\right)^{-2}\psi\right)(R) = \left(\frac{1+r^2}{2}\right)^{-2}(1-\Delta_{S^3})\psi(R)\\ +(4-16r^2)\left(\frac{1+r^2}{2}\right)^{-2}\psi(R)\\ +8r\left(\frac{1+r^2}{2}\right)^{-2}\frac{d\psi}{dR}(R)\ .\end{aligned}$$

Hence, since $r=\tan(\frac{R}{2})$ we get that the $L^2_{S^3}$-norm of $(1-\Delta_{S^3})\left(\left(\frac{1+r^2}{2}\right)^{-2}\psi\right)(R)$ is less than the $L^2_{S^3}$ the norm of $C\left(\frac{1+r^2}{2}\right)^{-1/2}(1-\Delta_{S^3})\psi$, C independent from ψ (the multiplying weights are smaller than a certain constant).

Also, we have seen that $(1-\Delta_{S^3})\psi(2\text{Arctan}\,(r))$ was equal to $D^2\varphi(r)$. Let us compute the $L^2_{S^3}$ norm of $\left(\frac{1+r^2}{2}\right)^{-1/2}(1-\Delta_{S^3})\psi$.

$$\|\left(\frac{1+r^2}{2}\right)^{-1/2}(1-\Delta_{S^3})\psi\|^2_{L^2_{S^3}} = \int_0^\pi\left(\frac{1+r^2}{2}\right)^{-1}|(1-\Delta_{S^3})\psi|^2(R)\sin^2 R dR\ .$$

By a change of variable $r=\tan(\frac{R}{2})$, we get :

$$\begin{aligned}\|\left(\frac{1+r^2}{2}\right)^{-1/2}(1-\Delta_{S^3})\psi\|^2_{L^2_{S^3}} &= \int_0^\infty\left(\frac{1+r^2}{2}\right)^{-4}|(1-\Delta_{S^3})\psi|^2(2\text{Arctan}\,(r))r^2dr\\ &= \|\left(\frac{1+r^2}{2}\right)^{-2}D^2\varphi\|^2_{L^2_{\mathbb{R}^3}}\ .\end{aligned}$$

But as

$$\left(\frac{1+r^2}{2}\right)^{-2}D^2\varphi = \left(\frac{1+r^2}{2}\right)^{-2}\varphi-\Delta_{\mathbb{R}^3}\varphi+\frac{2r}{1+r^2}\frac{d\varphi}{dr}$$

we get that :

$$\|\left(\frac{1+r^2}{2}\right)^{-2} D^2\varphi\|_{L^2_{\mathbb{R}^3}} \leq \|\varphi\|_{H^2_{\mathbb{R}^3}} .$$

Let us go back to the distribution bracket. There exists C independent from u_0 such that for all $\varphi \in H^2$, we have :

$$|\langle\left(\frac{1+r^2}{2}\right)^{-3} f_1 \ , \ \varphi\rangle| \leq C\|\Im u_0\|_{L^2}\|\varphi\|_{H^2}$$

so $\left(\frac{1+r^2}{2}\right)^{-3} f_1$ is indeed in H^{-2} with a norm less than $C\|u_0\|_{L^2_{S^3}}$, that is to say f_1 is in H^{-2}_{-6} and its norm satisfies :

$$\|f_1\|_{H^{-2}_{-6}} \leq C\|u_0\|_{L^2}$$

with a constant C independent from u_0.

Hence, the norm of (f_0, f_1) in the Cartesian product $L^2_{-1} \times H^{-2}_{-6}$ satisfies :

$$\|f_0, f_1\| \leq C\|u_0\|_{L^2_{S^3}}$$

with C independent from u_0, so the transform PT is continuous from $L^2_{S^3}$ to $L^2_{-1} \times H^{-2}_{-6}$.

◊

Let us now define a measure η for this set.

Definition 2.3.28. Let $\mathcal{H} = L^2_{-1} \times H^{-2}_{-6}$. We call η the image measure by PT on $\mathcal{H}$ of ρ.

We also set Π the set included in H such that :

$$\Pi = PT(\Sigma)$$

where Σ is the previously defined set of full ρ measure of $L^2_{S^3}$ onto which the flow is globally defined.

Let us now prove the first part of Theorem 2, that is :

Theorem 2.3.29. *Let $p \in]2\alpha, 6[$ and s defined accordingly as $\frac{3}{2} - \frac{4}{p}$. The set Π is of full η measure and for all couple $f_0, f_1 \in \Pi$, we have a solution f of (2.1) with initial data f_0, f_1. This solution is unique in the sense that $f(t,r) - L(t)(f_0, f_1)(r)$ is unique in $C(\mathbb{R}, H^s(\mathbb{R}^3))$, where $L(t)$ is the flow of the linear wave equation, i.e. the flow of $\partial_t^2 - \Delta_{\mathbb{R}^3}$.*

Proof First, Π is of full η-measure since Σ is of full ρ-measure. Then Let u_0 be $(PT)^{-1}(f_0, f_1)$, that is to say :

$$u_0 = \frac{1}{1+\cos R} f_0(\tan(\frac{R}{2})) - iH^{-1}\left(\left(\frac{1}{1+\cos R}\right)^2 f_1(\tan(\frac{R}{2}))\right) .$$

The function u_0 is therefore in Σ. Hence, there exists u such that $u - S(t)u_0$ is unique in X^s_π satisfying (2.10). We then choose f the time dependant Penrose transform of u. The function f satisfies (2.1) with initial condition f_0, f_1, as we have already discussed it in the previous sections.

We also have that $L(t)(f_0, f_1)$ is the reverse Penrose transform of $S(T)u_0$.

Lemma 2.3.30. *Let $f_\infty = f_0, f_1 \in \mathcal{H}$ and f the solution of*

$$\begin{cases} (\partial_t^2 - \Delta)f = 0 \\ f|_{t=0} = f_1 \qquad\qquad \partial_t f|_{t=0} = f_1 \end{cases} \tag{2.13}$$

We set $u_\infty(R) = \frac{f_0(r)}{\Omega(0,R)} - iH^{-1}\frac{f_1(r)}{\Omega^2(0,R)} \in L^2_{S^3}$ and $u = S(T)u_\infty$ the solution of :

$$\begin{cases} i\partial_T u + Hu = 0 \\ u_{T=0} = u_\infty \end{cases}$$

we have $f(t,r) = \Omega Re u(Arctan\,(t+r) + Arctan\,(t-r)\ ,\ Arctan\,(t+r) - Arctan\,(t-r))$.

Proof Indeed, $v(T,R) = \frac{f(t,r)}{\Omega}$ satisfies :

$$\begin{cases} (\partial_T^2 + 1 - \Delta)v = 0 \\ v_{T=0} = \frac{f_0}{\Omega} \qquad\qquad \partial_T v|_{T=0} = \frac{f_1}{\Omega} \end{cases}$$

So, $\widetilde{u} = v - iH^{-1}\partial_T v$ satisfies

$$\begin{cases} i\partial_T\widetilde{u} + H\widetilde{u} = 0 \\ \widetilde{u}_{T=0} = u_\infty \end{cases}$$

which means $\widetilde{u} = u$. Since $f = \Omega v$ and $v = \mathrm{Re}\widetilde{u}$, we have $f = \Omega\mathrm{Re}u$.

◊

Let us now prove the uniqueness in $L(t)(f_0, f_1) + C(\mathbb{R}, H^s)$.

First, the solution f of the non linear wave equation on $\mathbb{R}^3$ lies in $L^p_{t,r}$. Indeed, if f is such a solution and u is the solution on the sphere such that f is its Penrose transform, that is, on $\Omega > 0$:

$$u(T,R) = \Omega^{-1}f(t,r) - i\partial_T H^{-1}\Omega^{-1}f(t,r)\ .$$

Let us compute the $L^p_{t,r}$ norm of f.

First, let us consider a function ϕ in $L^p_{T,R}$ and set ψ its Penrose transform. We have :

$$\int_0^\infty \int_{-\infty}^\infty |\psi(t,r)|^p r^2 drdt = \int\int_{\Omega>0} \frac{1}{\Omega^2}|\Omega\phi(T,R)|^p \frac{\sin^2 R}{\Omega^2} dR$$

since the Jacobean of the Penrose transform is equal to $\frac{1}{\Omega^2}$ and $r = \frac{\sin R}{\Omega}$.

Hence, we get that :

$$\|\psi\|_{L^p_{t,r}} = \|\Omega^{1-4/p}\phi\|_{L^p_{T,R}} \le C\|\phi\|_{L^p_{T,R}}$$

as $p > 2\alpha \ge 4$.

Hence

$$\|f\|_{L^p_{t,r}} \le \|u\|_{L^p_{T,R}} \le \|S(T)(u_0)\|_{L^p_{T,R}} + \|v\|_{X^s}$$

thanks to Sobolev embeddings.

But the same kind of computations as in the proof of lemma 3.18 ensures that $S(T)u_0$ belongs almost surely to any L^q and in particular to L^p (the integral over time is done in $[-\pi, \pi]$). Therefore, f belongs almost surely to L^p.

As Strichartz estimates and Sobolev embeddings also hold on $\mathbb{R}^n$, one can rewrite the local theory for the non linear wave equation on the Euclidean space and prove in this way the uniqueness of f in $L(t)(f_0, f_1) + C([0, T_0], H^s(\mathbb{R}^3)$ for some T_0 depending on the $L^p_{t,r}$ norm of f and then by induction on k in $L(t)(f_0, f_1) + C([kT_0, (k+1)T_0], H^s)$ which ensures the result.

$\diamond$

2.4 Typical properties of the solutions

2.4.1 General considerations

We have seen that the initial data is of the form $u_0 = \sum \frac{\sqrt{2}g_n}{n} e_n$ with g_n complex Gaussian random variables. But remember that $u_0 = v_0 - iH^{-1}v_1$, where v_0, v_1 are the initial data for the problem outside its Hamiltonian form, that is to say : $v|_{T=0} = v_0$, $\partial_T v|_{T=0} = v_1$.

What is more, the initial data for the problem before using the Penrose transform was given by $f_0 = \frac{2}{1+r^2} v_0$ and $f_1 = (\frac{2}{1+r^2})^2 v_1$.

The function $v_0 = \mathrm{Re} u_0$ can be written $\sum \frac{h_n}{n} e_n$ where h_n are independent real Gaussian random variables, and $v_1 = -H\Im u_0$ is $\sum l_n e_n$ where l_n are also independent real Gaussian variables (and independent from the h_n). In other terms, v_1 is in the same spaces as Hv_0. But so, $f_1 = \frac{4}{(1+r^2)^2} v_1$ is in the same spaces as $\frac{4}{(1+r^2)^2} Hv_0 = \frac{4}{(1+r^2)^2} H \frac{1+r^2}{2} f_0$ where H is expressed in terms of t, r instead of T, R.

That is to say, since $H^2 = 1 - \Delta_{S^3} = 1 - \partial_R^2 - \frac{2}{\tan R}\partial_R$, that f_1 lives in the same spaces as

$$(\frac{2}{1+r^2})^2 \sqrt{1 - \left(\frac{1+r^2}{2}\right)^2 \partial_r^2 - \frac{(1+r^2)}{2r}\partial_r} \frac{1+r^2}{2} f_0 \ .$$

We are going to show that f_0 is almost surely in L^p, for $p \in]2, 6[$, and that it is almost surely not in L^p, when p is different. Then, we will use techniques of fractional integration over periodic functions and their Fourier decomposition to both characterize the behaviour of $f_0(r)$ when $r \to \infty$ and show that f_1 belongs to the spaces $W^{-1,p}$ almost surely for all $p \in]2,6[$.

To sum up, we are going to prove in the next subsections the Theorem 1. In the next subsection, we prove that the first component of the initial datum belongs to L^p, $p \in]2,6[$ almost surely, then, in 2.4.3, that this component is almost surely localized, then, in 2.4.4 that it almost surely does not belong to L^p, for $p \notin]2,6[$, and finally in 2.4.5 that the second component is almost surely in $W^{-1,p}$, for $p \in]2,6[$.

2.4.2 Lebesgue spaces the initial data belong to

Definition 2.4.1. Let f_n be the Penrose transform at time $t = 0 \Leftrightarrow T = 0$ of e_n. The functions f_n can be written :

$$f_n(r) = \Omega(t=0)e_n(2\text{Arctan } r) = \sqrt{\frac{2}{\pi}}\frac{2}{1+r^2}\frac{\sin(2n\text{Arctan } r)}{\sin(2\text{Arctan } r)}$$

$$f_n(r) = \sqrt{\frac{2}{\pi}}\frac{\sin(2n\text{Arctan } r)}{r} \ .$$

The initial data f_0 on $\mathbb{R} \times (\mathbb{R}^3)$ is then :

$$f_0(\omega, r) = \sum_n h_n(\omega)\frac{f_n(r)}{n},$$

with h_n independent real Gaussian variables of law $\mathcal{N}(0,1)$.

Proposition 2.4.2. *We have the following inequalities :*

$$\|f_n\|_{L^p} \leq \begin{cases} C_p n^{3/p-1} & \text{if } p < 3 \\ C_p \log n & \text{if } p = 3 \\ C_p n^{1-3/p} & \text{otherwise} \end{cases}$$

Proof By definition,

$$\|f_n\|_{L^p}^p = C\int_0^\infty |\frac{\sin(2n\text{Arctan } r)}{r}|^p r^2 dr$$

By changing r into $R = \tan r$, we get :

$$\|f_n\|_{L^p}^p = \int_0^{\pi/2} |\tan R|^{2-p}|\sin 2nR|^p(1+\tan^2 R)dR$$

This integral has two singularities : at 0 ($r = 0$) and $\pi/2$ ($r = \infty$), we are going to treat them separately by dividing the integral in two parts. We write

$$I = \int_0^{\pi/4} |\tan R|^{2-p}|\sin 2nR|^p(1+\tan^2 R)dR$$

and

$$II = \int_{\pi/4}^{\pi/2} |\tan R|^{2-p} |\sin 2nR|^p (1+\tan^2 R) dR \ .$$

Let us begin with I (i.e. the 0-singularity). Once more, we divide the integral in two, between 0 and $a_n = \text{Arctan}\,\frac{1}{n}$ on the one hand, a_n and $\pi/4$ on the other hand.

$$I.1 = \int_0^{a_n} |\tan R|^{2-p} |\sin 2nR|^p (1+\tan^2 R) dR$$

Since $|\frac{\sin(2nR)}{\tan R}|^p \leq (2n)^p$, we deduce that :

$$I.1 \leq (2n)^p \int_0^{a_n} \tan^2 R (1+\tan^2 R) dR = (2n)^p \frac{1}{3n^3} = C_p n^{p-3} \ .$$

For $I.2$, we use that the sine is less than 1, which gives

$$I.2 \leq \int_{a_n}^{\pi/4} |\tan R|^{2-p} (1+\tan^2 R) dR = \int_{1/n}^1 r^{2-p} dr = \frac{1 - 1/n^{3-p}}{3-p}$$

except when $p = 3$, then the bound is $\log n$. So,

$$I.2 \leq \begin{cases} C_p & \text{if } p < 3 \\ C_p \log n & \text{if } p = 3 \\ C_p n^{p-3} & \text{otherwise} \end{cases}$$

Hence :

$$I \leq \begin{cases} C_p & \text{if } p < 3 \\ C_p \log n & \text{if } p = 3 \\ C_p n^{p-3} & \text{otherwise} \end{cases}$$

Let us look at II. We do the change of variable $R \leftarrow \pi/2 - R$. We have $|\sin(2n(\pi/2 - R))| = |\sin(2nR)|$ and $\tan(\pi/2 - R) = \frac{1}{\tan R}$. The integral II thus is :

$$\begin{aligned} II &= \int_0^{\pi/4} |\sin(2nR)|^p |\tan R|^{p-2} (1+\tan^{-2} R) dR \\ &= \int_0^{\pi/4} |\sin(2nR)|^p |\tan R|^{p-4} (1+\tan^2 R) dR \ . \end{aligned}$$

We suppose that $p > \frac{3}{2}$ or the singularity in 0 (remember it corresponds to $r = \infty$) diverges. We get :

$$II.1 = \int_0^{a_n} |\sin(2nR)|^p |\tan R|^{p-4} (1+\tan^2 R) dR \leq (2n)^p \int_0^{a_n} |\tan R|^{2p-4} (1+\tan^2 R) dR$$

$$II.1 \leq (2n)^p (\frac{1}{n})^{2p-3}/(2p-3) = C_p n^{3-p}$$

Regarding $II.2$, it is given by :

$$II.2 = \int_{a_n}^{\pi/4} |\sin(2nR)|^p |\tan R|^{p-4}(1+\tan^2 R)dR$$

and by bounding the sine by 1, and then doing the change of variable $r = \tan R$,

$$II.2 \leq \int_{a_n}^{\pi/4} |\tan R|^{p-4}(1+\tan^2 R)dR = \int_{1/n}^{1} r^{p-4} dr \ .$$

Computing this integral leads to

$$II.2 \leq \begin{cases} C_p & \text{if } p > 3 \\ C_p \log n & \text{if } p = 3 \\ C_p n^{3-p} & \text{otherwise} \end{cases}$$

Hence, summing $II.1$ and $II.2$,

$$II \leq \begin{cases} C_p & \text{if } p > 3 \\ C_p \log n & \text{if } p = 3 \\ C_p n^{3-p} & \text{otherwise} \end{cases}$$

Finally, combining I and II gives :

$$\|f_n\|_{L^p} \leq \begin{cases} C_p n^{3/p-1} & \text{if } p < 3 \\ C_p (\log n)^{1/3} & \text{if } p = 3 \\ C_p n^{1-3/p} & \text{otherwise} \end{cases} \ .$$

$\diamond$

Proposition 2.4.3. *The initial data $f_0 = \sum \frac{h_n}{n} f_n$ belongs to L^p almost surely as soon as $p \in]2, 6[$.*

Proof The average value of the L^p_r norm to the p of f_0 is :

$$E(\|f_0\|^p_{L^p}) = \int d\omega \int r^2 dr |f_0|^p = \int r^2 dr \int d\omega |f_0|^p = \int r^2 dr \|f_0\|^p_{L^p_\omega} \ .$$

We have seen that $\|\sum a_n h_n\|_{L^p_\omega} \leq C_p \sqrt{\sum |a_n|^2}$, so

$$E(\|f_0\|^p_{L^p}) \leq \int r^2 dr C_p (\sum |\frac{f_n}{n}|^2)^{p/2} = C_p \| \sum |\frac{f_n}{n}|^2 \|^{p/2}_{L^{p/2}}$$

By a triangle inequality,

$$E(\|f_0\|^p_{L^p}) \leq C_p(\sum \frac{\|f_n\|^2_{L^p}}{n^2})^{p/2}$$

$$E(\|f_0\|^p_{L^p}) \leq \begin{cases} C_p(\sum n^{-4+6/p})^{p/2} & \text{if } p \in]3/2,3[\quad \text{which is finite if } p > 2 \\ C_p(\sum n^{-2}\log^2 n)^{p/2} & \text{if } p = 3 \quad < \infty \\ C_p(\sum n^{-6/p})^{p/2} & \text{otherwise} \quad \text{and is finite if } p < 6 \end{cases}$$

Thus, $E(\|f_0\|^p_{L^p})$ is finite when $p \in]2,6[$. We then have that $\|f_0\|_{L^p}$ is almost surely finite for $p \in]2,6[$.

◊

2.4.3 Localization

We are now going to prove the localization of the initial data, that is to say :

Proposition 2.4.4. *The initial data being given by :*

$$f_0(\omega, r) = \sum_{n\geq 1} \frac{h_n(\omega)}{n} \frac{\sin(2nArctan\ (r))}{r} ,$$

almost surely, when $r \to \infty$, $f_0(\omega, r)$ is a $O(\frac{1}{r^{1+\nu}})$ for all $\nu < \frac{1}{2}$.

To prove this localization, we are going to use again a change of variable.

Lemma 2.4.5. *Let $F_0(\omega, R) = \sum_{n\geq 1} \frac{h_n}{n}(-1)^n \sin(2nR)$. If F_0 is bounded in a neighbourhood of $R = 0$ a.s. then $f_0 \to 0$ when $r \to \infty$ a.s. .*

Proof Let $r > 0$. Let us compute $F_0(\omega, \frac{\pi}{2} - \text{Arctan}\ (r))$.

$$F_0(\omega, \frac{\pi}{2} - \text{Arctan}\ (r)) = \sum_{n\geq 1} \frac{h_n}{n}(-1)^n \sin(n\pi - 2n\text{Arctan}\ (r))$$

$$= -\sum_{n\geq 1} \frac{h_n}{n} \sin(2n\text{Arctan}\ (r)) = -rf_0(r) .$$

If $F_0(\omega, .)$ is bounded in a neighbourhood of 0 then there exists $R_0 > 0$ and $M \geq 0$ such that for all $0 \leq R \leq R_0$, $|F_0(\omega, R)| \leq M$. Let $r_0 = \tan(\frac{\pi}{2} - R_0) = \frac{1}{\tan R_0}$. Then for all, $r \geq r_0$, we have $\pi/2 - \text{Arctan}\ (r) \leq R_0$ and so :

$$|f_0(\omega, r)| = \frac{1}{r}|F_0(\omega, \pi/2 - \text{Arctan}\ (r))| \leq \frac{M}{r} \to 0$$

when $r \to \infty$. ◊

We are now going to prove some properties about fractional integration of Fourier series. For more information about trigonometrical series in general and fractional integration in particular, one can refer to [58], chap. IV.

Definition 2.4.6. Let c_n be a sequence of $\mathbb{C}^{\mathbb{Z}}$ such that $c_0 = 0$. We suppose that the distribution f is defined as

$$f(t) = \sum_{n\in\mathbb{Z}} c_n e^{int} \ .$$

We then build f_α, where α is a real number such that $0 < \alpha < 1$ as :

$$f_\alpha(t) = \sum c_n \frac{e^{int}}{(in)^\alpha}$$

where $(in)^\alpha = e^{\text{sign}(n)i\pi\alpha/2}|n|^\alpha$.

We show that f_α can be seen as a convolution product.

Definition 2.4.7. We define $\Psi_\alpha(t)$ for $t \in]0, 2\pi[$ as the limit :

$$\Psi_\alpha(t) = \frac{1}{\Gamma(\alpha)}\left(t^{\alpha-1} + \lim_{N\to\infty}\left(\sum_{n=1}^{N}(t+2n\pi)^{\alpha-1} - \frac{(2\pi)^{\alpha-1}N^\alpha}{\alpha}\right)\right) \ .$$

We also set :

$$r_\alpha^N = \sum_{n=1}^{N}(t+2n\pi)^{\alpha-1} - \frac{(2\pi)^{\alpha-1}N^\alpha}{\alpha}$$

and

$$r_\alpha = \lim r_\alpha^N \ .$$

Proposition 2.4.8. *The sequences r_α^N and $\frac{dr_\alpha^N}{dt}$ are Cauchy sequences for the L^∞ norm. Therefore, r_α is a bounded differentiable function whose derivative is bounded.*

Proof The number $\frac{N^\alpha(2\pi)^{\alpha-1}}{\alpha}$ is equal to :

$$\frac{N^\alpha(2\pi)^{\alpha-1}}{\alpha} = \frac{1}{2\pi}\int_0^{2N\pi} x^{\alpha-1}dx \ .$$

Hence, we get that :

$$
\begin{aligned}
r_\alpha^N(t) = \quad & \textstyle\sum_{n=1}^N \frac{1}{2\pi} \int_{2n\pi}^{2(n+1)\pi} \left((t+2n\pi)^{\alpha-1} - x^{\alpha-1}\right) dx \\
& + \textstyle\frac{1}{2\pi} \int_0^{2\pi} x^{\alpha-1} dx - \frac{1}{2\pi} \int_{2N\pi}^{2(N+1)\pi} x^{\alpha-1} dx \\
r_\alpha^N(t) = \quad & \textstyle\sum_{n=1}^N \frac{1}{2\pi} \int_0^{2\pi} \left((t+2n\pi)^{\alpha-1} - (x+2n\pi)^{\alpha-1}\right) dx \\
& + \textstyle\frac{1}{2\pi} \int_0^{2\pi} x^{\alpha-1} dx - \frac{1}{2\pi} \int_{2N\pi}^{2(N+1)\pi} x^{\alpha-1} dx
\end{aligned}
$$

Since

$$
|\frac{1}{2\pi} \int_0^{2\pi} x^{\alpha-1} dx| = (2\pi)^{\alpha-1} < \infty
$$

and

$$
|\frac{1}{2\pi} \int_{2N\pi}^{2(N+1)\pi} x^{\alpha-1} dx| = \frac{(2\pi)^{\alpha-1}}{\alpha} |(N+1)^\alpha - N^\alpha| \leq C N^{\alpha-1} \to 0
$$

and also

$$
|(t+2n\pi)^{\alpha-1} - (x+2n\pi)^{\alpha-1}| \leq \frac{1}{(2n\pi)^{2-\alpha}} |x-t| \ ,
$$

we have that

$$
|r_N^\alpha(t)| \leq C + \sum_{n=1}^N \frac{1}{(2\pi n)^{2-\alpha}} \int_0^{2\pi} |x-t| dx
$$

and

$$
|r_N^\alpha(t) - r_M^\alpha(t)| \leq C(N^{\alpha-1} + M^{\alpha-1} + \sum_{n=M+1}^N \frac{1}{(2\pi n)^{2-\alpha}}) \ .
$$

as $2-\alpha > 1$ and $|x-t| \leq 4\pi$ we get that $r_N^\alpha(t)$ is bounded by a constant independent from t and from N and that r_N^α is a Cauchy sequence for the L^∞ norm. Then, as the r_N^α are continuous, r_α is also continuous, and it is bounded.

The derivative $\frac{dr_\alpha^N}{dt}$ is

$$
\frac{dr_\alpha^N}{dt} = (\alpha - 1) \sum_{n=1}^N (t+2n\pi)^{\alpha-2}
$$

The L^∞ norm of $(t+2n\pi)^{\alpha-2}$ is $(2n\pi)^{\alpha-2}$ which is the general term of a convergent series. Then $\frac{dr_\alpha}{dt}$ is well defined, continuous and bounded. ◊

Proposition 2.4.9. *The Fourier coefficients of* $\Psi_\alpha(t)$ *are* $c_0 = 0$ *and* $c_n = \frac{1}{(in)^\alpha}$.

Proof As r_α^N converges uniformally and is $\| \,.\, \|_{L^\infty}$-bounded, we can swap the integral and the limit ($t^{\alpha-1}$ is integrable). Then,

$$\begin{aligned}\Gamma(\alpha)c_0 &= \lim \frac{1}{2\pi}\int_0^{2\pi}(t^{\alpha-1} + \sum_{n=1}^{N}(t+2n\pi)^{\alpha-1}) - \frac{1}{2\pi}\int_0^{2N\pi} x^{\alpha-1}dx) \\ &= \lim \frac{1}{2\pi}\int_{2N}^{2(N+1)\pi} t^{\alpha-1}dt \\ &= 0\end{aligned}$$

and for $n \neq 0$,

$$\Gamma(\alpha)c_n = \lim \frac{1}{2\pi}\int_0^{2\pi} e^{-int}(t^{\alpha-1} + \sum_{n=1}^{N}(t+2n\pi)^{\alpha-1}) - \frac{1}{2\pi}\int_0^{2\pi}(2N\pi)^\alpha e^{-int}dt$$

Since $\int_0^{2\pi}(2N\pi)^\alpha e^{-int}dt = 0$ we have :

$$\Gamma(\alpha)c_n = \int_0^\infty t^{\alpha-1}e^{-int}dt\ .$$

The function $z \mapsto z^{\alpha-1}e^{-|n|z}$ is analytic on $\mathbb{C}\setminus\{0\}$. We integrate it over the axis $[\epsilon, R]$ and $[\text{sign}(n)i\epsilon, \text{sign}(n)iR]$ and the arcs of radius ϵ and R. Then, we do $\epsilon \to 0$ and $R \to \infty$. The integral over the arc of radius ϵ behaves like ϵ^α when $\epsilon \to 0$ so it converges to 0. For R, the behaviour is given by $e^{-cR}R^\alpha$. We deduce from that :

$$\Gamma(\alpha)c_n = e^{-i\text{sign}(n)\alpha\pi/2}\int_0^\infty e^{-|n|t}t^{\alpha-1}dt$$

With a change of variable $u = |n|t$, we get

$$\Gamma(\alpha)c_n = e^{-i\text{sign}(n)\alpha/2}|n|^{-\alpha}\int_0^\infty e^{-u}u^{\alpha-1}dt = (in)^{-\alpha}\Gamma(\alpha)\ .$$

◊

A more detailed proof can be found in [58] chap. II.

Proposition 2.4.10. *Let* $\alpha \in]0,1[$ *and r such that* $\alpha r > 1$. *Let* $f = \sum c_n(\omega)e^{int} \in L^r(\Omega\times[0,2\pi])$ *with* $c_0 = 0$ *and suppose that f is the* L^r *limit of* $\sum_{n=-N}^{N} c_n e^{int}$. *Then*

$$\int_0^{2\pi}\frac{dt}{2\pi}f(\omega,t)\Psi_\alpha(x-t)$$

is in L^r_ω, L^∞_x *and is the limit of* $\sum_{n=-N}^{N}\frac{c_n}{(in)^\alpha}e^{inx}$ *that is to say* $f_\alpha(\omega,x)$.

Proof As $\alpha r > 1$, we get that $\alpha - 1 > \frac{1}{r} - 1 = -\frac{1}{r'}$ where r' is the conjugate number of r. Then, as $|\Psi_\alpha(t)| \leq C(1 + t^{\alpha-1})$, we have that Ψ_α is in $L^{r'}$. This way, we have :

$$\|\frac{1}{2\pi}\int_0^{2\pi} f(t)\Psi_\alpha(x-t)dt\|_{L_x^\infty} \leq \|f\|_{L_x^r}\|\Psi_\alpha\|_{L^{r'}}$$

that is to say

$$\|\frac{1}{2\pi}\int_0^{2\pi} f(t)\Psi_\alpha(x-t)dt\|_{L_\omega^r,L_x^\infty} \leq \|\Psi_\alpha\|_{L_x^{r'}}\|f\|_{L_{\omega,x}^r} < \infty\ .$$

What's more,

$$\frac{1}{2\pi}\int_0^{2\pi} c_n e^{int}\Psi_\alpha(x-t)dt = c_n e^{inx}\frac{1}{2\pi}\int_0^{2\pi} e^{-inu}\Psi_\alpha(u)du$$

$$= \frac{c_n}{(in)^\alpha}e^{inx}$$

Then,

$$\frac{1}{2\pi}\int_0^{2\pi} f(t)\Psi_\alpha(x-t)dt - \sum_{n=-N}^{N}\frac{c_n}{(in)^\alpha}e^{inx} = \frac{1}{2\pi}\int_0^{2\pi}(f(t) - \sum_{n=-N}^{N} c_n e^{int})\Psi_\alpha(x-t)dt$$

$$\|\frac{1}{2\pi}\int_0^{2\pi} f(t)\Psi_\alpha(x-t)dt - \sum_{n=-N}^{N}\frac{c_n}{(in)^\alpha}e^{inx}\|_{L_x^\infty} \leq \|f(t) - \sum_{n=-N}^{N} c_n e^{int}\|_{L_t^r}\|\Psi_\alpha\|_{L^{r'}}$$

$$\|\frac{1}{2\pi}\int_0^{2\pi} f(t)\Psi_\alpha(x-t)dt - \sum_{n=-N}^{N}\frac{c_n}{(in)^\alpha}e^{inx}\|_{L_\omega^r,L_x^\infty} \leq \|f(t) - \sum_{n=-N}^{N} c_n e^{int}\|_{L_\omega^r,L_t^r}\|\Psi_\alpha\|_{L^{r'}}$$

which converges towards 0 by hypothesis.

◊

Proposition 2.4.11. *Under the assumptions of Proposition 2.4.10, we have that $f_\alpha(x+h) - f_\alpha(x)$ is a.s. a $O(h^\nu)$ where $\nu = (\alpha - 1/r) > 0$.*

Proof Once more, we can write :

$$|f_\alpha(x+h) - f_\alpha(x)| \leq \|f\|_{L_t^r}\left(\frac{1}{2\pi}\int_0^{2\pi}|\Psi_\alpha(u+h) - \Psi_\alpha(u)|^{r'}du\right)^{1/r'}$$

We the divide the integral over u into two parts : one from 0 to h, the other from h to 2π.

$$\int_0^h |\Psi_\alpha(u+h)-\Psi_\alpha(u)|^{r'}du \le C\int_0^{2h} |\Psi_\alpha(u)|^{r'}du$$

$$\le C\int_0^{2h} |u|^{(\alpha-1)r'}du$$

$$\le Ch^{r'(\alpha-1/r)}$$

$$\int_h^{2\pi} |\Psi_\alpha(u+h)-\Psi_\alpha(u)|^{r'}du \le h^{r'}\int_h^{2\pi} u^{(\alpha-2)r'}$$

$$\le Ch^{r'}h^{r'(\alpha-2)+1} = Ch^{r'(\alpha-1/r)}$$

We have then

$$h^{-\nu}|f_\alpha(x+h)-f_\alpha(x)| \le C\|f\|_{L^r_t}$$

$$\|h^{-\nu}|f_\alpha(x+h)-f_\alpha(x)\|_{L^r_\omega, L^\infty_{x,h}} \le C\|f\|_{L^r_{\omega,t}} < \infty$$

We deduce from that that $\|h^{-\nu}|f_\alpha(x+h)-f_\alpha(x)\|_{L^\infty}$ is a.s. finite, that is to say that $f_\alpha(x+h)-f_\alpha(x)$ is a.s. a $O(h^\nu)$. ◊

Definition 2.4.12. Let $0<\alpha<\frac{1}{2}$, $r>\frac{1}{\alpha}>2$ and $F_{-\alpha}=\sum_{n\neq 0}(-1)^n\frac{h_{|n|}}{2in}(in)^\alpha e^{2inx}$ defined as the limit of $\sum_{n=-N}^N(-1)^n\frac{h_{|n|}}{2in}(in)^\alpha e^{2inx}$ in $L^r_{\omega,x}$.

For this definition to be valid, let us prove that the sequence $\sum_{n=-N}^N(-1)^n\frac{h_{|n|}}{2in}(in)^\alpha e^{2inx}$ converges.

Proof

$$\|\sum_{n=-N}^{N}(-1)^n\frac{h_{|n|}}{2in}(2in)^\alpha e^{2inx} - \sum_{n=-M}^{M}(-1)^n\frac{h_{|n|}}{2in}(in)^\alpha e^{2inx}\|^r_{L^r}$$

$$\le C\int_0^{2\pi}\|\sum_{n=M+1}^{N}(-1)^n\frac{h_n}{n^{1-\alpha}}\frac{e^{inx}i^\alpha - e^{-inx}(-i)^\alpha}{2}\|^r_{L^r_\omega}dx$$

Since $r>2$ we have that $\|\sum c_n h_n\|_{L^r_\omega}\le C_r(\sum|c_n|^2)^{1/2}$, then,

$$\le C_r\int_0^{2\pi}(\sum_{n=M+1}^{N}\frac{1}{n^{2-2\alpha}})^{r/2}$$

As $\alpha < \frac{1}{2}$, $2 - 2\alpha > 1$, so the series of general term $n^{2\alpha-2}$ converges and the sequence $\sum_{n=-N}^{N} \frac{h_{|n|}}{2in}(2in)^{\alpha} e^{2inx}$ is a Cauchy sequence in $L^r_{\omega,x}$, it converges. ◊

Lemma 2.4.13. *Let $\nu < \frac{1}{2}$, there exists $\alpha < \frac{1}{2}$ and $r > \frac{1}{\alpha}$, such that $\nu = \alpha - \frac{1}{r}$ and $(F_{-\alpha})_{\alpha} = F_0$ in $L^r_{\omega}, L^{\infty}_x$ and so $F_0(h)$ is almost surely in Ω a $O(h^{\nu})$.*

Proof Set $\nu < \alpha < \frac{1}{2}$ and let $r = \frac{1}{\alpha-\nu} > \frac{1}{\alpha}$. The function $F_{-\alpha}$ is the limit in $L^r_{\omega,x}$ of the sequence $\sum_{n=-N}^{N}(-1)^n \frac{h_{|n|}}{2in}(in)^{\alpha} e^{inx}$, so $(F_{-\alpha})_{\alpha}$ is the limit in $L^r_{\omega}, L^{\infty}_x$ of

$$\sum_{n=-N}^{N} (-1)^n \frac{h_{|n|}}{2in} \frac{(2in)^{\alpha}}{(2in)^{\alpha}} e^{2inx} = \sum_{n=1}^{N} \frac{h_n}{n} \frac{e^{2inx} - e^{-inx}}{2i} (-1)^n$$

$$= \sum_{n=1}^{N} \frac{h_n}{n} \sin(2nx)(-1)^n \ .$$

That is to say, $F_0 = (F_{-\alpha})_{\alpha}$ in $L^r_{\omega}, L^{\infty}_x$ and so we have that $F_0(x+h) - F_0(x)$ is almost surely in Ω a $O(h^{\nu})$ as $\nu = \alpha - \frac{1}{r}$. We use it for $x = 0$. We have $F_0(\omega, 0) = 0$ so a.s.

$$F_0(\omega, x) = O(x^{\nu}) \ .$$

◊

Proof We have

$$f_0(\omega, r) = \frac{1}{r} F_0(\omega, \pi/2 - \text{Arctan }(r)) \ .$$

Indeed, $\pi/2 - \text{Arctan }(r) = \text{Arctan }(\frac{1}{r}) = O(\frac{1}{r})$. Then, since $F_0(\omega, h) = O(h^{\nu})$ a.s., we have that a.s. :

$$f_0(\omega, r) = \frac{1}{r} O(\text{Arctan }^{\nu}(\frac{1}{r})) = O(\frac{1}{r^{1+\nu}}) \ .$$

◊

2.4.4 Lebesgue spaces the initial data do not belong to

Now, we are going to see that for $p \notin]2, 6[$, $\|f_0\|_{L^p} = \infty$ almost surely.

Proposition 2.4.14. *For almost all $r \in \mathbb{R}$, the function $f_0(., r) = \sum \frac{h_n}{n} f_n(r)$ is a real Gaussian variable of variance $\sigma^2(r) = \sum \frac{|f_n(r)|^2}{n^2}$.*

Proof
Remark that $\sigma(r)$ is a.s. finite since

$$\int r^2 dr\sigma^4(r) = \sum \frac{\|f_n\|_{L^2}^4}{n^4} \leq C \sum n^{-2} < \infty$$

Let's compute $E(e^{ixf_0})$ when $\sigma(r) < \infty$.

$$E(e^{ixf_0}) = E(\prod e^{ix\frac{h_n f_n}{n}}) = \prod E(e^{ix\frac{f_n}{n}h_n})$$

as the h_n are independent. Since h_n is a real centred Gaussian, we have that $E(e^{ix\frac{f_n}{n}h_n}) = e^{-x^2\frac{|f_n(r)|^2}{2n^2}}$ hence :

$$E(e^{ixf_0}) = e^{-x^2\sum\frac{|f_n^2(r)|}{2n^2}} = e^{-x^2\sigma^2(r)/2}$$

so $f_0(r)$ is a real Gaussian variable of variance $\sigma^2(r)$.

◊

Lemma 2.4.15. *For all real centred Gaussian Z,*

$$E(|Z|^p) = C(p)E(|Z|^2)^{p/2}$$

Proof Indeed,

$$E(|Z|^p) = \int |Z|^p e^{-\frac{|Z|^2}{2\sigma^2}} \frac{dZ}{\sqrt{2\pi}\sigma}$$

$$= \int \sigma^p |Y|^p e^{-|Y|^2/2} \frac{dY}{\sqrt{2\pi}} = E(|Z|^2)^{p/2} \int |Y|^p e^{-|Y|^2} \frac{dY}{\sqrt{2\pi}}$$

◊

We deduce from that

$$E(\|f_0\|_{L^p}^p) = \int r^2 dr E(|f_0|^p) = C(p)\int r^2 dr E(|f_0|^2)^{p/2} = C\int r^2 dr (\sum \frac{|f_n|^2}{n^2})^{p/2}$$

Lemma 2.4.16. *Let E be a vector space and $\mathcal{B}(E)$ its borel σ-algebra. We set F a centred Gaussian variable on E and N a pseudo-norm on E that is to say a norm that admits $+\infty$ as a possible value. If the probability $P(N(F) < \infty)$ is strictly positive, then for all $p < \infty$, $E(N^p(F)) < \infty$.*

The proof of this lemma is given by X. Fernique in [30].

Corollary 2.4.17. *Under the previous assumptions, if there exists p such that $E(N^p(F)) = \infty$, then $P(N(F) < \infty) = 0$.*

In particular with $N = \|.\|_{L^p}$, we get that if $E(\|f_0\|_{L^p}^p) = \infty$, then $\|f_0\|_{L^p}$ is a.s. infinite.

We have to study $E(\|f_0\|_{L^p}^p)$, that is $\int r^2 dr (\sum \frac{|f_n|^2}{n^2})^{p/2}$.

Lemma 2.4.18. *Let $R_n = \frac{\pi}{4n}$. For all $R \in [0, R_n]$, $|f_n(\tan R)| \geq n$.*

Proof We have :

$$f_n(\tan R) = \frac{\sin(2nR)}{\tan R}$$

For all $0 \leq R \leq \frac{\pi}{4}$, $|\tan R| \leq \frac{4}{\pi}R$, and for all $0 \leq R \leq \frac{\pi}{2}$, $|\sin R| \geq \frac{2}{\pi}R$ then for all $0 \leq R \leq R_n$,

$$|f_n(\tan R)| \geq \frac{4nR}{\pi}\frac{\pi}{4R} = n$$

◇

We have immediately that

$$\sum \frac{|f_n(\tan R)|^2}{n^2} \geq \sum 1_{R \leq R_n} = \sum 1_{R_{n+1} \leq R \leq R_n} n$$

By changing r into $\tan R$, we get :

$$E(\|f_0\|_{L^p}^p) \geq \int_0^{\pi/2} \tan^2 R(1 + \tan^2 R) dR (\sum \frac{|f_n(\tan R)|^2}{n^2})^{p/2}$$

We divide the integral into its computation I over $[0, \pi/4]$ and II over $[\pi/4, \pi/2]$. For II, we change R into $\pi/2 - R$, we get :

$$II = \int_0^{\pi/4} \tan^{-4} R(1 + \tan^2 R) dR (\sum \frac{|f_n(\tan R)|^2 \tan^4 R}{n^2})^{p/2}$$

$$= \int_0^{\pi/4} \tan^{4(p/2-1)} R(1 + \tan^2 R) dR (\sum \frac{|f_n(\tan R)|^2}{n^2})^{p/2}$$

For $p \geq 6$, we give a lesser bound of I (0 singularity).

$$I \geq \int_0^{\pi/4} \tan^2 R(1 + \tan^2 R) dR \sum n^{p/2} 1_{R_{n+1} \leq R \leq R_n} = \sum \frac{r_n^3 - r_{n+1}^3}{3} n^{p/2}$$

with $r_n = \tan R_n$. We have $r_n^3 - r_{n+1}^3 \sim 3\frac{4}{\pi}\frac{1}{n^4}$, so the general term of the series behaves like $n^{p/2-4}$. For $p \geq 6$, $p/2 - 4 \geq -1$ so the series diverges.

For $p \leq 2$, we give a lesser bound of II (∞ singularity).

$$II \geq \sum n^{p/2} \int_{R_{n+1}}^{R_n} \tan^{2p-4} R(1+\tan^2 R)dR = \sum n^{p/2} \frac{r_n^{2p-3} - r_{n+1}^{2p-3}}{2p-3}$$

$r_n^{2p-3} - r_{n+1}^{2p-3}$ behaves like n^{2-2p} so the general term behaves like $n^{-3p/2+2}$. For $p \leq 2$, $-3p/2+2 \geq -1$ so the series diverges. We get that for $p \leq 2$ or $p \geq 6$, $E(\|f_0\|_{L^p}^p) = \infty$ and so $\|f_0\|_{L^p} = \infty$ almost surely.

2.4.5 Regularity of the second component of the initial datum

We have that $f_1 = \sum l_n \frac{2}{1+r^2} f_n$. Let us do a change of variable $r \mapsto R = \text{Arctan}\,(r)$ and use the result and methods we developed in the localization section. More precisely, we are going to study the behaviour of a "primitive" of f_1 by using the change of variable and use the study of behaviour of periodic functions at two given points $R = 0$ corresponding to $r = 0$ and $R = \frac{\pi}{2}$ corresponding to $r = \infty$.

Definition 2.4.19. Let $R_0 < \frac{\pi}{4}$, $V_1 = [0, \frac{\pi}{2} - R_0]$ and $V_2 = [R_0, \frac{\pi}{2}]$ Set $\psi_n^1(R) = 1_{V_1}\frac{1}{n}(1-\cos(2nR))$ and $\psi_n^2(R) = 1_{V_2}\frac{1}{n}((-1)^n - \cos(2nR))$. Then, let Ψ_α^1 and Ψ_α^2 be the L^r limits for $\alpha < \frac{1}{2}$ and $r > 2$ of

$$\sum_{n>0} (in)^\alpha l_n \psi_n^j$$

with $j = 1, 2$.

This limit exists and hence we can define

$$\Psi^j = \sum_{n>0} l_n \psi_n^j$$

as the limits of the partial sums in L_ω^r, L_R^∞ and have the properties :

$$\Psi^1(R) = O(R^\nu)$$

when $R \to 0$ for all $0 < \nu < \frac{1}{2}$ almost surely in ω and

$$\Psi^2(\frac{\pi}{2} - R) = O(R^\nu)$$

when $R \to 0$ for all $0 < \nu < \frac{1}{2}$ a.s. in ω.

Proposition 2.4.20. *Let $\Phi^j(r) = \frac{\Psi^j(\text{Arctan}\,(r))}{r}$ for $j = 1, 2$. We have that Φ^j is almost surely in L^p for all $p \in]2, 6[$.*

Proof Let $p \in]2, 6[$ and let us compute the L^p norm of Φ^j.

$$\|\Phi^j\|_{L^p}^p = \int_0^\infty |\Phi^j(r)|^p r^2 dr$$

$$= \int_0^{\pi/2} |\Psi^j(R)|^p (1+\tan^2 R) |\tan R|^{2-p} dR\ .$$

Considering the different j, we get :

$$\|\Phi^1\|_{L^p}^p = \int_{V_1} |\Psi^1(R)|^p (1+\tan^2 R) |\tan R|^{2-p} dR$$

and

$$\|\Phi^2\|_{L^p}^p = \int_{V_2} |\Psi^2(R)|^p (1+\tan^2 R) |\tan R|^{2-p} dR$$

that is to say, with the change of variable $R \leftarrow \frac{\pi}{2} - R$:

$$|\Phi^2\|_{L^p}^p = \int_{V_1} |\Psi^2(\pi/2 - R)|^p (1+\tan^2 R) |\tan R|^{p-4} dR\ .$$

Since $1 - \frac{3}{p} < \frac{1}{2}$ as $p < 6$ we can choose $\nu_1 \in]1 - \frac{3}{p}, \frac{1}{2}[$ that is to say $2 - p + p\nu_1 > -1$. Then, given that $\Psi^1(R) = O(R^{\nu_1})$, we get that

$$|\Psi^1(R)|^p |\tan R|^{2-p} = O(R^{\nu_1 p + 2 - p})$$

so the first integral $\|\Phi^1\|_{L^p}^p$ converges almost surely in ω.

Since $\frac{3}{p} - 1 < \frac{1}{2}$ as $p > 2$, we can choose $\nu_2 \in]\frac{3}{p} - 1, \frac{1}{2}[$, that is to say $p - 4 + p\nu_2 > -1$. Then, given that $\Psi^2(\pi/2 - R) = O(R^{\nu_2})$, we get that

$$|\Psi^2(\pi/2 - R)|^p |\tan R|^{p-4} = O(R^{\nu_2 p + p - 4})$$

so the second integral $\|\Phi^2\|_{L^p}^p$ converges a.s.

Hence, Φ belongs to L^p for all $p \in]2, 6[$ almost surely. $\diamond$

Definition 2.4.21. Let χ_1 and χ_2 be two C_c^∞ functions with supports included in V_1 and V_2 respectively such that $\chi_j \in [0,1]$ and $\chi_1 + \chi_2 = 1$ on $[0, \frac{\pi}{2}]$. We set

$$\Phi(r) = \chi_1(\text{Arctan}\,(r))\Phi_1(r) + \chi_2(\text{Arctan}\,(r))\Phi_2(r)$$

and we call F the distribution defined as :

$$F(r) = \Phi'(r) + \frac{\Phi(r)}{r} - \frac{1}{(1+r^2)}\left(\chi_1'(\text{Arctan}\,(r))\Phi^1(r) + \chi_2'(\text{Arctan}\,(r))\Phi^2(r)\right)\ .$$

Proposition 2.4.22. *We deduce from the previous proposition that F belongs to $W^{-1,p}$ for all $p \in]2, 6[$.*

Proof Indeed, Φ' belongs to $W^{-1,p}$ by as it is the derivative of a function in L^p.

Then we use Sobolev embedding theorem. Set q such that $\frac{1}{q} = \frac{1}{p} + \frac{s}{3}$, there exists C such that for all f,

$$\|f\|_{L^p} \leq C\|(1-\Delta_{\mathbb{R}^3})^{s/2} f\|_{L^q} \ .$$

We use Sobolev embedding theorem with $s = 1$ and

$$f = (1-\Delta)^{-1/2}\left(\frac{\Phi}{r} - \frac{1}{(1+r^2)}\left(\chi_1'(\text{Arctan }(r))\Phi^1(r) + \chi_2'(\text{Arctan }(r))\Phi^2(r)\right)\right) \ ,$$

we get that

$$\|\left(\frac{\Phi}{r} - \frac{1}{(1+r^2)}\left(\chi_1'(\text{Arctan }(r))\Phi^1(r) + \chi_2'(\text{Arctan }(r))\Phi^2(r)\right)\right)\|_{W^{-1,p}}$$

$$\leq \|\left(\frac{\Phi}{r} - \frac{1}{(1+r^2)}\left(\chi_1'(\text{Arctan }(r))\Phi^1(r) + \chi_2'(\text{Arctan }(r))\Phi^2(r)\right)\right)\|_{L^q} \ .$$

We then have to prove that $\frac{\Phi}{r}$, $\frac{1}{1+r^2}\chi_j'(\text{Arctan }(r))\Phi^j(r)$ for $j = 1,2$ belong to L^q.

Now, let us remark that $\|\frac{\Phi}{r}\|_{L^q} \leq \|\frac{\Phi^1}{r}\|_{L^q} + \|\frac{\Phi^2}{r}\|_{L^q}$.

Let $p_1 \in]p, 6[$ and $p_2 \in]2, p[$ and p_j' defined as $\frac{1}{p} + \frac{1}{3} = \frac{1}{q} = \frac{1}{p_j} + \frac{1}{p_j'}$. We have :

$$\frac{1}{p_1'} = \frac{1}{3} + \frac{1}{p} - \frac{1}{p_1} \in]\frac{1}{3}, \frac{1}{2} + \frac{1}{p}[$$

So, $p_1' \in]1, 3[$ and by a similar computation $p_2' \in]3, \infty[$. Hence,

$$\|\frac{\Phi^1}{r}\|_{L^q} \leq \|\frac{1_{r<\text{Arctan }(\pi/2-R_0)}}{r}\|_{p_1'}\|\Phi^1\|_{L^{p_1}} < \infty$$

and

$$\|\frac{\chi_1'(\text{Arctan }(r))\Phi^1}{1+r^2}\|_{L^q} \leq \|\frac{\chi'(\text{Arctan }(r))}{1+r^2}\|_{L^{p_1'}}\|\Phi^1\|_{L_1^p}$$

and since $\chi_1'(\text{Arctan }(r))$ is bounded and null outside a bounded set of $\mathbb{R}$ we get that :

$$\|\frac{\chi_1'(\text{Arctan }(r))\Phi^1}{1+r^2}\|_{L^q} < \infty \ .$$

Also,

$$\|\frac{\Phi^2}{r}\|_{L^q} \leq \|\frac{1_{r>\text{Arctan }(R_0)}}{r}\|_{p_2'}\|\Phi^2\|_{L^{p_2}} < \infty$$

and

$$\|\frac{\chi_2'(\text{Arctan }(r))\Phi^2}{1+r^2}\|_{L^q} \leq \|\frac{\chi'(\text{Arctan }(r))}{1+r^2}\|_{L^{p_2'}}\|\Phi^2\|_{L_2^p}$$

since $r \mapsto \chi_2'(\text{Arctan }(r))$ is bounded, we get

$$\|\frac{\chi_2'(\text{Arctan }(r))\Phi^2}{1+r^2}\|_{L^q} < \infty \,.$$

Thus, F belongs to $W^{-1,p}$ for all $p \in]2, 6[$. ◊

Lemma 2.4.23. *The distribution F is equal to the initial data $f_1 = \sum_{n\neq 0} l_n \frac{\sin(2n\mathit{Arctan}\ (r))}{r(1+r^2)}$.*

Proof We have that $F = \sum l_n(\omega)F_n(r)$ with

$$F_n(r) = \left(\frac{d}{dr} + \frac{1}{r}\right)\left(\sum_{j=1,2} \frac{\chi_j(\text{Arctan }(r))((-1)^{n(j+1)} - \cos(2n\text{Arctan }(r)))}{2nr}\right)$$

$$-\frac{1}{1+r^2}\left(\frac{\sum_{j=1,2}\chi_j'(\text{Arctan }(r))((-1)^{n(j+1)} - \cos(2n\text{Arctan }(r)))}{2nr}\right)$$

Let us compute F_n.

$$\left(\frac{d}{dr} + \frac{1}{r}\right)\left(\sum_{j=1,2} \frac{\chi_j(\text{Arctan }(r))((-1)^{n(j+1)} - \cos(2n\text{Arctan }(r)))}{2nr}\right) =$$

$$\frac{d}{2nrdr}\left(\sum_{j=1,2} \chi_j(\text{Arctan }(r))((-1)^{n(j+1)} - \cos(2n\text{Arctan }(r)))\right) \,.$$

So,

$$F_n = \sum_{j=1,2} \frac{\chi_j(\text{Arctan }(r))}{2nr} \frac{d}{dr}\left((-1)^{n(j+1)} - \cos(2n\text{Arctan }(r))\right)$$

$$F_n = (\chi_1(\text{Arctan } r) + \chi_2(\text{Arctan }(r))\frac{1}{1+r^2}\frac{\sin(2n\text{Arctan }(r))}{r} \,.$$

Hence, we get that the initial data

$$f_1(r) = \sum_{n>0} l_n \frac{\sin(2n\text{Arctan }(r))}{r(1+r^2)} = \sum l_n F_n = F \,,$$

and we deduce from that the initial data is almost surely in $W^{-1,p}$. ◊

2.4.6 Consequences regarding the regularity of the solution

We have seen in the definition of the global flow of (2.1) that the solution is of the form $L(t)(f_0, f_1) + g(t,.)$ where $L(t)$ is the flow of the free wave equation, f_0, f_1 is the initial data, and g belongs to $L^p_{(t,r)}$. Thanks to the previous considerations, we will see that $L(t)(f_0, f_1)$, as it highly resembles in spatial structure the initial data f_0, f_1, is localized almost surely. Furthermore, the "controlled" part of the solution g, admits some kind of localization.

Proposition 2.4.24. *Let $f_0, f_1 \in \Pi$ and write the solution of the non linear wave equation* (2.1) *with initial data f_0, f_1*

$$f(t,r) = L(t)(f_0, f_1)(r) + g(t,r) .$$

We have that :

– $L(t)(f_0, f_1)$ is ω - almost surely localized, that is to say, for almost all $f_0, f_1 \in \Pi$, and all $t \in \mathbb{R}$,

$$\lim_{r\to\infty} L(t)(f_0, f_1)(r) = 0 ,$$

– for $p \in]2\alpha, 6[$ such that the solution is defined in $L(t)(f_0, f_1) + L^p_{t,r}$, (see Theorem 2.3.29) we have that $\left(\frac{1+r^2}{2}\right)^{1/2-2/p} g(t,r)$ is in $L^p_{t,r}$, hence, for almost all $t \in \mathbb{R}$, $\left(\frac{1+r^2}{2}\right)^{1/2-2/p} g(t,r)$ is in L^p_r.

Proof Let us prove the second point of the proposition. We have seen in the proof of 2.3.29 that for all radial function ψ, and its global Penrose transform u that :

$$\|\psi\|_{L^p_{t,r}} = \|\Omega^{1-4/p}\mathrm{Re}u\|_{L^p_{T,R}} . \tag{2.14}$$

But if we write u the global Penrose transform of the solution f. We have that $u = S(T)u_0 + v$ with u_0 the Penrose transform at time $T = 0$ of f_0, f_1, $S(T)$ the flow of the linear wave equation on the sphere S^3 and $v \in X^s_{2\pi}$. So, as the global Penrose transform turns $S(T)$ into $L(t)$, and that it is a linear map, we get that g is the global Penrose transform of v. Thus, as

$$\sqrt{\frac{1+r^2}{2}} \leq \begin{cases} \sqrt{\frac{1+(t+r)^2}{2}} & \text{if } t \geq 0 \\ \sqrt{\frac{1+(t-r)^2}{2}} & \text{otherwise} \end{cases} ,$$

we have

$$\sqrt{\frac{1+r^2}{2}} \leq \sqrt{\frac{(1+(r+t)^2)(1+(r-t)^2)}{2}} \leq \sqrt{2}\Omega(t,r)^{-1}$$

and so, by using 2.14 with $\psi = \Omega^{4/p-1}g$ and $\Omega^{4/p-1}\mathrm{Re}v$ its Penrose transform, we get that

$$\|\left(\frac{1+r^2}{2}\right)^{1/2-2/p} g\|_{L^p_{t,r}} \leq C\|\Omega^{4/p-1}g\|_{L^p_{t,r}} \leq C\|v\|_{L^p_{T,R}}$$

and then, using Strichartz estimates :

$$\|\left(\frac{1+r^2}{2}\right)^{1/2-2/p} g\|_{L^p_{t,r}} \leq C\|v\|_{X^s_{2\pi}} .$$

Let us now prove the first part of the proposition. We set $l(t,r) = L(t)(f_0, f_1)(r)$. This function is the global Penrose transform of $S(T)u_0$. Hence, it can be written under the form :

$$l(t,r) = \frac{2\left((1+(t+r)^2)(1+(t-r)^2)\right)^{-1/2}}{\sin(\text{Arctan }(t+r) - \text{Arctan }(t-r))} \times$$

$$\sum_n \frac{h_n}{n} e^{-in(\text{Arctan }(t+r)+\text{Arctan }(t-r))} \sin(2n(\text{Arctan }(t+r) - \text{Arctan }(t-r))) .$$

The factor $2\left(\sqrt{(1+(t+r)^2)(1+(t-r)^2)}\sin(\text{Arctan }(t+r) - \text{Arctan }(t-r))\right)^{-1}$ is equal to $\frac{1}{r}$ as we have seen in the preliminaries, hence it remains to show that the sum is bounded. We will divide the sum in two parts as (consider t as fixed) :

$$l(t,r) = L_1(Y) + L_2(Z)$$

with

$$2Y(r) = \text{Arctan }(t+r) - 3\text{Arctan }(t-r) \ , \ 2Z(r) = 3\text{Arctan }(t+r) - \text{Arctan }(t-r)$$

and

$$L_1(Y) = \sum_{n\geq 1} \frac{h_n}{2in} e^{2inY} \ , \ L_2(Z) = \sum_{n\leq -1} \frac{h_{-n}}{2in} e^{2inZ} .$$

Let $0 < \alpha < \frac{1}{2}$ and set

$$L_1^\alpha(Y) = \sum_{n\geq 1} \frac{h_n}{2in}(2in)^\alpha e^{2inY} \text{ and } L_2^\alpha(Z) = \sum_{n\leq -1} \frac{h_{-n}}{2in}(2in)^\alpha e^{2inZ} .$$

The sum L_1^α (resp. L_2^α) is the limit in $L^r_{\omega,Y}$ (resp. $L^r_{\omega,Z}$) of the partial sum

$$\sum_{n=1}^{N} \frac{h_n}{2in}(2in)^\alpha e^{2inY} \text{ (resp. } \sum_{n=-N}^{-1} \frac{h_{-n}}{2in}(2in)^\alpha e^{2inZ} \text{)}$$

for all $r \geq 2$. Hence the sums L_1 and L_2 are limits of finite sums in L^r_ω, L^∞_Y or L^r_ω, L^∞_Z for all r such that $\alpha r > 1$ and thus are ω-almost surely bounded in Y, Z.

We deduce from that that for all $t \in \mathbb{R}$ fixed, $r \in \mathbb{R}_+^*$ and almost all ω, $L(t)(f_0^\omega, f_1^\omega)$ satisfies :

$$|L(t)(f_0^\omega, f_1^\omega)(r)| \leq \frac{1}{r}\left(\|L_1(\omega)\|_{L_Y^\infty} + \|L_2(\omega)\|_{L_Z^\infty}\right)$$

and so converges toward 0 when r goes to ∞. $\diamond$

2.5 Scattering

2.5.1 Penrose transformed free evolution

We ware going to show that a solution f of the non linear wave equation with initial data in Π tends when $t \to \infty$ towards a solution of the free evolution with different initial data.

For that, we will consider at first the equation on the sphere. However, scattering of f is different from the dynamics of u on $T = \pi$ since $t = \infty$ doesn't correspond to $T = \pi$ but to $\{T, R| \ T = \pi - R\}$, which means that the scattering on f does not come from scattering on u. The decreasing norms of the non linear part in $\mathbb{R}^3$ comes from the fact that the Penrose transform makes terms in negative powers of the time when we compare the norm of the non linear part of f and the one of u.

Definition 2.5.1. Thanks to Lemma 2.3.30, we denote by $L(t)f_\infty = f(t,.)$ the free evolution at time t with initial data f_∞. $L(t)f_\infty(r) = \Omega(T,R)\mathrm{Re}S(T)u_\infty(R)$.

Lemma 2.5.2. *Time $t = \infty$ corresponds by the Penrose transform to $\{T = \pi - R\}$.*

Proof We have that $t = \frac{\sin T}{\Omega}$, so $\sin T$ must be positive that is to say, $T \geq 0$ and $\Omega = 0$, i.e. $\cos T = -\cos R$, $T = \pi - R$, since $0 < R < \pi$. $\diamond$

2.5.2 Scattering result

Definition 2.5.3. For any $u_0 \in \Sigma$ we set :
- $u(T,R)$ the solution of the non linear equation with initial data u_0,
- $u_\infty = u_0 - i\int_0^\pi d\tau S(-\tau)H^{-1}F(\tau,u)$, let us recall that $F(\tau,u) = \Omega^{\alpha-2}(\tau)|\mathrm{Re}u|^\alpha\mathrm{Re}u$,
- $S(T)u_\infty$ the free evolution with initial data u_∞,
- (f_0, f_1) the PT transform of u_0,
- $f = \Omega\mathrm{Re}u$ the solution of the non linear equation with initial data (f_0, f_1),
- f_∞ the PT transform of u_∞,
- $L(t)f_\infty$ the free evolution with initial data f_∞.

Proposition 2.5.4. *Let $p \in]\max(2\alpha, \frac{16}{3}, 6[$, $s = \frac{3}{2} - \frac{4}{p}$, and q such that $\frac{3}{q} = \frac{3}{2} - s$, i.e. $q = \frac{3}{4}p > 4$ and we have the Sobolev embedding $H^s \to L^q$. The function $f - L(t)f_\infty$ is in L^q for all $t > 0$ and its norm converges toward 0 when $t \to \infty$.*

Proof Let's compute $f - L(t)f_\infty$.

$$f - L(t)f_\infty = \Omega\mathrm{Re}(u - S(T)u_\infty)$$

and

$$u = S(T)u_0 - i\int_0^T d\tau S(T-\tau)H^{-1}F(\tau,u)(\tau,R) = S(T)u_\infty + i\int_T^\pi d\tau S(T-\tau)H^{-1}F(\tau,u)$$

so

$$f - L(t)f_\infty = \Omega\mathrm{Re}i\int_T^\pi d\tau S(T-\tau)H^{-1}F(\tau,u)$$

We have now to compute the L^q norm.

$$\|f - L(t)f_\infty\|_{L^q}^q \le \int r^2 dr|\Omega|^q|\int_T^\pi d\tau S(T-\tau)H^{-1}F(\tau,u)|^q$$

Let $R = \text{Arctan}\,(t+r) - \text{Arctan}\,(t-r)$, the change of variable gives

$$\|f - L(t)f_\infty\|_{L^q}^q \le \int \sin^2 R dR \frac{\Omega^{q-4}}{2(t^2+r^2)}|\int_T^\pi d\tau S(T-\tau)H^{-1}F(\tau,u)|^q$$

Note that T depends on t and R but not τ.

$$\|f - L(t)f_\infty\|_{L^q}^q \le \frac{1}{t^2}\int_0^\pi \sin^2 R dR \frac{\Omega^{q-4}}{2}|\int_T^\pi d\tau S(T-\tau)H^{-1}F(\tau,u)|^q$$

$$\|f - L(t)f_\infty\|_{L^q} \le \frac{1}{t^{2/q}}\|\Omega^{(q-4)/q}|\int_T^\pi d\tau S(T-\tau)H^{-1}F(\tau,u)|\ \|_{L_R^q}$$

Since $\Omega \le 2$, we have

$$\|f - L(t)f_\infty\|_{L^q} \le \frac{C}{t^{2/q}}\|\int_T^\pi d\tau S(T-\tau)H^{-1}F(\tau,u)\|_{L_R^q}$$

$$\|f - L(t)f_\infty\|_{L^q} \le \frac{C}{t^{2/q}}\|\int_T^\pi d\tau S(T-\tau)H^{-1}F(\tau,u)\|_{H_R^s}$$

by Sobolev embedding theorem.

$$\|\int_T^\pi d\tau S(T-\tau)H^{-1}F(\tau,u)\|_{H_R^s} = \|H^{s-1}\int_T^\pi d\tau S(-\tau)F(\tau,u)\|_{L_R^2}$$

$$= \|\int_0^\pi 1_{\tau>T}S(-\tau)F(\tau,u)\|_{H^{s-1}}$$

$$\leq C\|1_{\tau>T}F(\tau,u)\|_{Y_\pi^{1-s}}$$

$$\leq C\|F(\tau,u)\|_{Y_\pi^{1-s}}$$

With x' the conjugate number of x such that $\frac{1}{x}+\frac{3}{x}=\frac{3}{2}-1+s$, that is $x'=\frac{4p}{2p+4}$, we get :

$$\|F(\tau,u)\|_{Y_\pi^{1-s}} \leq C\|F(T,u)\|_{L^{x'}_{T,R}} \leq C\|u\|^{\alpha+1}_{L^{(\alpha+1)x'}} \leq C(\|S(T)u_0\|^{\alpha+1}_{L^{(\alpha+1)x'}} + \|u-S(T)u_0\|^{\alpha+1}_{L^{(\alpha+1)x'}}$$

But since $p>2\alpha$, we have $(\alpha+1)x'<p$, so

$$\|f-L(t)f_\infty\|_{L^q} \leq \frac{C}{t^{2/q}}(\|S(T)u_0\|^{\alpha+1}_{L^p_{T,R}} + \|u-S(T)u_0\|^{\alpha+1}_{X_\pi^s}) \leq \frac{C}{t^{2/q}}$$

since for $u_0\in\Sigma$, there exists i such that $\|S(T)u_0\|_{L^p}\leq D\sqrt{i}$ and $\|u-S(T)u_0\|_{X_\pi^s}\leq C\sqrt{i}$.

Hence the result.

◊

Chapitre 3

Consequences of the choice of a particular basis of $L^2(S^3)$ for the cubic wave equation on the sphere and the Euclidean space

Ce chapitre est inspiré de l'article [27].

3.1 Introduction

The first aim of this paper is to extend the result by N. Burq and N. Tzvetkov [14] on the torus to the sphere of dimension 3. In [14], N. Burq and N. Tzvetkov have proved the global well-posedness of the cubic non linear equation when the initial datum is a randomization of some function in the product of Sobolev spaces $H^\sigma(\mathbb{T}^3) \times H^{\sigma-1}(\mathbb{T}^3)$, $\sigma \geq 0$ and $\mathbb{T}^3$ the torus of dimension 3.

The probabilistic estimates they use in order to prove their result are due to the fact that the L^p norms of the canonical basis of $L^2(\mathbb{T}^3)$: $(e^{in.x})_{n\in\mathbb{N}^3}$, are uniformly bounded, whether in n or in p.

Here, a result of N. Burq of G. Lebeau is required to go on in the case of the sphere, as a basis of $L^2(S^3)$ has a priori no reason to be uniformly bounded in L^p. In [12], they proved that there exists a basis of L^2 uniformly bounded in L^p by C_p and formed by eigenfunctions of the Laplace-Beltrami operator on the sphere. Here, the result of [14] is extended to the sphere, despite the dependence of the bound of the norms of the basis on p.

Let us describe the above-mentioned randomization. Let $(e_{n,k})_{n,k}$ be such a basis of $L^2(S^3)$, that is, such that for all n, k

$$||e_{n,k}||_{L^p} \leq C_p \ ,$$

and

$$-\triangle_{S^3} e_{n,k} = n^2 e_{n,k} \ ,$$

let $a_{n,k}$ and $b_{n,k}$ be two sequences of real-valued i.i.d. on a probability space $(\Omega, \mathcal{A}, P)$ and with large Gaussian deviation estimates, which results from the assumption that there exists c such that for all γ, the following mean values satisfy :

$$E(e^{\gamma a_{n,k}}), E(e^{\gamma b_{n,k}}) \leq e^{c\gamma^2} ,$$

and finally, let $\lambda_{n,k}$ and $\mu_{n,k}$ be two sequences of complex numbers such that for some $\sigma \geq 0$:

$$\sum_{n,k} (1+n^2)^{\sigma} |\lambda_{n,k}|^2 < \infty \text{ and } \sum_{n,k} (1+n^2)^{\sigma-1} |\mu_{n,k}|^2 < \infty .$$

Then the equation

$$(\partial_T^2 + 1 - \triangle_{S^3})u + u^3 = 0$$

with initial datum the randomization of $\sum \lambda_{n,k} e_{n,k}, \sum \mu_{n,k} e_{n,k}$ defined as

$$u_{|T=0} = u_0 = \sum_{n,k} \lambda_{n,k} a_{n,k} e_{n,k} \ , \ \partial_T u_{|T=0} = u_1 = \sum_{n,k} \mu_{n,k} b_{n,k} e_{n,k} \ ,$$

is globally well-posed.

The measure μ induced by the couple $(u_0, u_1) \in L^2(\Omega, H^\sigma \times H^{\sigma-1})$ where σ is given in the assumptions on $(\lambda_n)_n, (\mu_n)_n$ is very similar to the one introduced by the randomization in [15].

To phrase it more precisely,

Theorem 3.1.1. *There exists a set E of full μ-measure such that for all $(v_0, v_1) \in E$ the Cauchy problem*

$$\begin{cases} (\partial_T^2 + 1 - \triangle_{S^3})u + u^3 = 0 & \\ u_{|T=0} = v_0 & \partial_T u_{|T=0} = v_1 \end{cases}$$

has a unique solution in $U(T)(v_0, v_1) + C(\mathbb{R}, H^1(S^3))$ where $U(T)$ is the flow of the linear equation $\partial_T^2 + 1 - \triangle_{S^3} = 0$.

Note that the wave operator $\partial_T^2 - \triangle_{S^3}$ has been replaced here by $\partial_T^2 + 1 - \triangle_{S^3}$ for convenience with respect to the second part of this paper, dedicated to the cubic non linear wave equation on $\mathbb{R}^3$. However, the proof for the cubic non linear wave equation on the sphere would be similar to the one with this operator.

What is more, if the $\lambda_{n,k}$ (resp. or the $\mu_{n,k}$) are supposed to be such that

$$\sum_{n,k} (1+n^2)^s |\lambda_{n,k}|^2 = +\infty \text{ (resp. or } \sum_{n,k} (1+n^2)^{s-1} |\mu_{n,k}|^2 = +\infty)$$

for all $s > \sigma$, and the $a_{n,k}$ and $b_{n,k}$ are complex Gaussian of law $\mathcal{N}(0,1)$, then the elements of E are almost surely in $H^\sigma \times H^{\sigma-1}$ and almost surely not in $H^s \times H^{s-1}$.

As it appears, this result recalls one of [24] on the sphere but without the hypothesis of radial symmetry.

The main idea behind the proof is that with large Gaussian deviation estimates, the solution of the linear equation $U(T)(u_0, u_1)$ is made to belong almost surely to L^p for all $p \geq 1$ which ensures local and then global well-posedness. Indeed, it is the gain on integrability on the initial data that helps to gain regularity on the non linear part (namely, the solution minus $U(T)(u_0, u_1)$) of the solution.

A second issue raised on this paper is the properties of the Penrose transform of the solution. The Penrose transform sends solutions of $(\partial_T^2 + 1 - \triangle_{S^3})u + u^3 = 0$ on the sphere to solution of the cubic non linear wave equation on the Euclidean space $\mathbb{R}^3$. Indeed, the change of variable involved in this transform injects $\mathbb{R} \times \mathbb{R}^3$ into $[-\pi, \pi] \times S^3$ and satisfies nice properties with respect to the d'Alembertian $\partial_t^2 - \triangle_{\mathbb{R}^3}$.

Hence, with a solution of $(\partial_T^2 + 1 - \triangle_{S^3})u + u^3 = 0$ on the sphere, the existence of a solution of the cubic NLW on $\mathbb{R}^3$ is expected.

Nevertheless, the use of the Penrose transform raises three problems : first, the space where the solution lives shall be described, then, so does the space where this solution is unique, and finally, the spaces to which the initial data belong or do not belong should be specified.

Unfortunately, the third matter remains unanswered but the author believes that if the work on the sphere is done with $\sigma = 0$, that is with the initial data on the sphere in $L^2 \times H^{-1}$, then the initial data on $\mathbb{R}^3$ should be almost surely in $L^2 \times H^{-1}$ when multiplied by $(\left(\frac{2}{1+r^2}\right)^{1/2}, (\left(\frac{2}{1+r^2}\right)^{-1/2})$, and almost surely not to $H^s \times H^{s-1}$, when $s > 0$.

Nevertheless, the following theorem holds.

Theorem 3.1.2. *There exists a measure ν on $\mathcal{L}^2(\mathbb{R}^3) \times \mathcal{H}^{-1}(\mathbb{R}^3)$ with*

$$\|g\|_{\mathcal{L}^2} = \|\left(\frac{2}{1+r^2}\right)^{1/2} g\|_{L^2} \; , \; \|g\|_{\mathcal{H}^{-1}(\mathbb{R}^3)} = \|\left(\frac{2}{1+r^2}\right)^{-1/2} (1-H_1)^{-1} g\|_{L^2}$$

and

$$H_1 = \left(\frac{1+r^2}{2}\right)^2 \triangle_{\mathbb{R}^3} + 3\frac{1+r^2}{2} r\partial_r + 6\frac{1+r^2}{2}$$

and a set F of full ν-measure such that for all $(g_0, g_1) \in F$, the Cauchy problem :

$$\begin{cases} (\partial_t^2 - \triangle_{\mathbb{R}^3})f + f^3 = 0 & \\ f_{|t=0} = g_0 & \partial_t f_{|t=0} = g_1 \end{cases}$$

has a unique global solution in $L(t)(g_0, g_1) + C(\mathbb{R}, H^1(\mathbb{R}^3))$ where $L(t)$ is the flow of the linear wave equation.

Moreover, the solution f satisfies scattering in the sense that for all $q \in]\frac{18}{5}, 6]$,

$$\|f(t) - L(t)(g_0, g_1)\|_{L^q} = O((1+t^2)^{-1/6})$$

when $t \to \pm\infty$.

The measure ν is the image measure of the measure μ induced by (u_0, u_1) by the Penrose transform.

The main idea of the proof is that for $q \geq 4$, the L^q norm of the solution on $\mathbb{R}^3$ is controlled by the L^q norm of the solution on the sphere, which ensures regularity properties and then uniqueness of the solution.

Plan of the paper The section 2 is dedicated to the proof of the global well-posedness of the equation on the sphere. The first two subsections, which are about local theory and global theory on $H^\sigma \times H^{\sigma-1}$ with $\sigma > 0$, are very similar to [14]. The third one, where the global well-posedness is dealt with the initial datum being almost surely in $L^2 \times H^{-1}$ presents divergences, in particular due to the fact that the bound on the L^p norms of the chosen basis of L^2 depends on p.

The third section is about the Penrose transform and how it acts on the norms of the solution and the norms of the initial data. The transform is presented, along with its trace on the initial data, and then what its trace turns the Laplace-Beltrami operator on the sphere to in order to study the Sobolev norms of the initial data on $\mathbb{R}^3$.

The fourth one is about the uniqueness and scattering properties of the solution of the equation on $\mathbb{R}^3$. It focuses on the integrability of the solution (how a L^p norm of the solution on the sphere is changed by the Penrose transform), then its regularity before stating the uniqueness and scattering results.

Acknowledgements

The author would like to thank Nicolas Burq for suggesting the problem.

3.2 Almost sure existence of global solutions on the sphere

This section deals with the global well-posedness for the cubic wave equation on the sphere with initial data taken as a random variable on $H^\sigma \times H^{\sigma-1}$, $\sigma \geq 0$. The solution has a linear part in $C(\mathbb{R}, H^\sigma)$ and a non linear part in $C(\mathbb{R}, H^1)$.

3.2.1 Definition of the initial data and local theory

Following the techniques of N. Burq and N. Tzvetkov in [14], the random initial datum shall be chosen such that when it is submitted to the linear flow of the wave equation, it has L^p norms in time and space.

Theorem 6 of the third section of [12] provides the existence of a Hilbertian basis of $L^2(S^3)$ composed with spherical harmonics uniformly bounded in L^p. Let us therefore name the different objects that shall be needed to define a suitable initial datum.

First, let us recall that the eigenvalues of $1 - \triangle_{S^3}$ are $n^2, n \geq 1$.

Thanks to the result of N. Burq and G. Lebeau, denote by $(e_{n,k})_{n\geq 1, 1\leq k\leq (n+1)^2}$ a Hilbertian basis of $L^2(S^3)$ uniformly bounded in L^p.

Theorem 3.2.1 (Burq, Lebeau)**.** *There exists a Hilbertian basis $(e_{n,k})_{n,k}$ of $L^2(S^3)$ such that :*

$$(1 - \triangle_{S^3})e_{n,k} = n^2 e_{n,k}$$

for all $n \geq 1, 1 \leq k \leq n^2$, n^2 being the dimension of the subspace of L^2 spanned by the spherical harmonics of degree $n-1$, and such that there exists a constant C_p such that for all n, k

$$\|e_{n,k}\|_{L^p(S^3)} \leq C_p \; .$$

To be more precise, Burq and Lebeau proved the following proposition. Set E_n the vector space spanned by the spherical harmonics of degree $n-1$ and U_n the set of orthonormal basis of E_n.

Proposition 3.2.2 (Burq, Lebeau)**.** *There exist a family of measure ν_n on U_n with $\nu_n(U_n) = 1$ and two constants $c_0, c_1 > 0$ such that for all $p \geq 2$, all $n \in \mathbb{N}^*$, and all $\Lambda \geq 0$, the probability :*

$$\nu_n(\{(e_{n,k})_{1\leq k\leq (n+1)^2} \in U_n \mid \exists k, |\|e_{n,k}\|_{L^p} - M_{n,p}| > \Lambda\})$$

where $M_{n,p}$ is a real number bounded by $C\sqrt{p}$ with C independent from n and p, that is the probability that the difference between $M_{n,p}$ and the norm of at least one of the functions of the basis is bigger than Λ is bounded by :

$$c_0 e^{-c_1 n^{4/p}\Lambda^2} n^2 \; .$$

In the Appendix 3.5, we give a straightforward proof of this Proposition, without using the general framework that Burq and Lebeau used, but largely inspired by their paper, and we deduce from it that there exists a sequence $p_m \to \infty$ and a basis $e_{n,k}$ of spherical harmonics such that

$$\|e_{n,k}\|_{L^{p_m}} \leq C\sqrt{p_m} \; . \tag{3.1}$$

As afore-mentioned, the main difference between this section of this paper and the one by by Burq and Tzvetkov [14] is that in their paper, the basis of L^2 is bounded uniformly in L^p, but uniformly in terms of p too. This property allows them to ask for an almost sure L^p bound (so to speak) on the initial datum and take $p \to \infty$. The difference will appear and be detailed later.

Let $\sigma \in [0, \frac{1}{2}[$ and $(u_0^{n,k})_{n,k}$ and $(u_1^{n,k})_{n,k}$ be two sequences of real numbers such that the series

$$\sum_{n\geq 1} n^{2\sigma} \sum_{1\leq k\leq (n+1)^2} (u_0^{n,k})^2 \text{ and } \sum_{n\geq 1} n^{2(\sigma-1)} \sum_{1\leq k\leq (n+1)^2} (u_1^{n,k})^2$$

converge but at least one of the series

$$\sum_{n\geq 1} n \sum_{1\leq k\leq (n+1)^2} (u_0^{n,k})^2 \text{ or } \sum_{n\geq 1} n^{-1} \sum_{1\leq k\leq (n+1)^2} (u_1^{n,k})^2$$

diverge, that is to say

$$\left(\sum_{n,k} u_0^{n,k} e_{n,k}, \sum_{n,k} u_1^{n,k} e_{n,k}\right)$$

belongs to $H^\sigma \times H^{\sigma-1}$ but is not in the critical space for the cubic NLW $H^{1/2} \times H^{-1/2}$.

Let $(X, \mathcal{A}, P)$ be a probability space large enough such that two sequences $(a_{n,k})$ and $(b_{n,k})$ of random variables can be taken satisfying that the $a_{n,k}$ are independent from each other and from the $b_{n,k}$, the $b_{n,k}$ are independent from each other and that there exists a such that for all n, k and all $\gamma \in \mathbb{R}$:

$$E(e^{\gamma a_{n,k}}), E(e^{\gamma b_{n,k}}) \leq e^{a\gamma^2}$$

where E is the mean value with respect to the probability measure P, which ensures that the random variables have Gaussian-like large deviation estimates.

Proposition 3.2.3. *The sequences of $L^2(X, H^\sigma(S^3))$ and $L^2(X, H^{\sigma-1}(S^3))$ respectively*

$$u_0^N = \sum_{n=1}^{N} \sum_{k=1}^{(n+1)^2} u_0^{n,k} a_{n,k} e_{n,k} \text{ and } u_1^N = \sum_{n=1}^{N} \sum_{k=1}^{(n+1)^2} u_1^{n,k} b_{n,k} e_{n,k}$$

converges. Let u_0 and u_1 their limits.

Proof The proof consists in the fact that the mean values of $a_{n,k}^2$ and $b_{n,k}^2$ are uniformly bounded by $8a$. It ensures that the sequences are Cauchy in their respective spaces and therefore that they converge. ◊

Let $U(T)$ be the flow of the linear equation $(\partial_T^2 + 1 - \triangle_{S^3})u = 0$, that is

$$U(T)\left(\sum_{n,k} v_0^{n,k} e_{n,k}, \sum_{n,k} v_1^{n,k} e_{n,k}\right) = \sum_{n,k} (\cos(nT) v_0^{n,k} + \frac{\sin(nT)}{n} v_1^{n,k}) e_{n,k} .$$

Set

$$S_M^N = \sum_{n=N}^{M} \sum_k n^{2\sigma} (u_0^{n,k})^2 + n^{2(\sigma-1)} (u_1^{n,k})^2 ,$$

$$S_M = S_M^0 , S^N = \lim_{M\to\infty} S_M^{N+1} \text{ and } S = S_N + S^N .$$

Set also $\Pi_N, N \geq 0$ the orthogonal projection on the subspace of L^2 spanned by $\{e_{n,k} \mid n \leq N\}$ with the convention $\Pi_0 = 0$.

The initial data u_0, u_1 satisfy some properties regarding the spaces where they belong. With p_m the sequence that goes to ∞ for which we have the uniform bound (3.1) :

Proposition 3.2.4. *There exists $C, c > 0$ such that for all $\lambda \geq 0$:*

– *for all $N \geq 0$, all p_m and with $\delta_{p_m} = \frac{2}{p_m} > \frac{1}{p_m}$*

$$P(\| \frac{1}{1 + |T|^{\delta_{p_m}}} (1 - \Pi_N) U(T)(u_0, u_1) \|_{L^{p_m}(\mathbb{R}\times S^3)} \geq \lambda) \leq \left(\frac{C p_m \sqrt{S^N}}{\lambda}\right)^{p_m} ,$$

– *with* $\delta_3 = 2/3 > 1/3$,

$$P(\|\frac{1}{1+|T|^{\delta_3}}U(T)(u_0,u_1)\|_{L^3_T,L^6(S^3)} \geq \lambda) \leq Ce^{-c\lambda^2/S}$$

– *for all* $M \geq 1$ *and with* $s = 1$ *if* $\sigma = 0$ *and* $s = 0$ *otherwise*

$$P(\|\frac{1}{1+T^2}\Pi_M U(T)(u_0,u_1)\|_{L^1_T,L^\infty(S^3)} \geq \lambda) \leq Ce^{-c\lambda^2/(M^s S)} .$$

The difference between the choices for σ in the third inequality is due to the fact that if $\sigma > 0$ then by Sobolev embeddings, there exists some p large enough such that $W^{\sigma,p}$ is embedded in L^∞. In the proof, it will appear that $(1+T^2)^{-1}U(T)(u_0,u_1)$ is almost surely and with the same deviation estimates in $L^1(\mathbb{R}, W^{\sigma,p})$, the L^1 norm being taken on the time. Hence, when $\sigma > 0$, the bound does not depend on M, as we can take the left hand-side of the inequality with $M \to \infty$. For $\sigma = 0$, we can not apply Sobolev embedding, thus the bound depends on M, we chose $s = 1$ but we could have chosen any $s > 0$.

In the proof, we will write p instead of p_m.

Remark 3.2.1. *This proposition differs from the similar one in the torus case, [14] where the first inequality corresponded to :*

$$P(\|\frac{1}{1+|T|^{\delta_p}}(1-\Pi_N)U(T)(u_0,u_1)\|_{L^p(\mathbb{R}\times S^3)} \geq \lambda) \leq \left(\frac{C\sqrt{p}\sqrt{S^N}}{\lambda}\right)^p .$$

Proof

Lemma 3.2.5. *There exists C such that for all* $q \geq 1$ *and all couples of* l^2 *sequences* $v_{n,k}, w_{n,k}$ *:*

$$\|\sum_{n,k} a_{n,k}v_{n,k} + b_{n,k}w_{n,k}\|_{L^q_X} \leq C\sqrt{q}\left(\sum |v_{n,k}|^2 + |w_{n,k}|^2\right)^{1/2} .$$

The proof can be found in [15], Lemma 3.1.

Lemma 3.2.6. *There exists C such that for all* $r, p \geq 1$, $s \geq 0$, $M > N \geq 0$ *and* $q \geq r, p$,

$$\|\frac{1}{1+|T|^{2/r}}(1-\Pi_N)(1-\triangle)^{s/2}U(T)(u_0^M,u_1^M)\|_{L^q_X,L^r_T,L^p(S^3)} \leq C\sqrt{p}\sqrt{q}M^{s'}\sqrt{S_M - S_N}$$

with $s' = s - \sigma$ *if* $s \geq \sigma$ *and* $s' = 0$ *otherwise.*

Proof Let

$$\Sigma_N^M(x) = \sum_{n=N+1}^{M}\sum_{k=1}^{1+n^2} n^{2s}\left((u_0^{n,k})^2 + n^{-2}(u_1^{n,k})^2\right)|e_{n,k}(x)|^2 .$$

Using the previous lemma and bounding the sines and cosines by 1 in

$$(1-\triangle)^{s/2}U(T)(u_0^M,u_1^M)\ ,$$

it appears that

$$\|\frac{1}{1+|T|^{2/r}}(1-\Pi_N)(1-\triangle)^{s/2}U(T)(u_0^M,u_1^M)\|_{L_X^q}\leq\frac{C}{1+|T|^{2/r}}\sqrt{q}\sqrt{\Sigma_N^M(x)}\ .$$

Hence, as $q\geq r,p$, and thanks to Minkowski inequality, one can reverse the order of the norms :

$$\|\frac{1}{1+|T|^{2/r}}(1-\Pi_N)(1-\triangle)^{s/2}U(T)(u_0^M,u_1^M)\|_{L_X^q,L_T^r,L^p(S^3)}$$

$$\leq\|\frac{1}{1+|T|^{2/r}}(1-\Pi_N)(1-\triangle)^{s/2}U(T)(u_0^M,u_1^M)\|_{L_T^r,L^p(S^3),L_X^q}$$

$$\leq C\sqrt{q}\|\frac{1}{1+|T|^{2/r}}\|_{L_T^r}\|\Sigma_N^M\|_{L^{p/2}}^{1/2}\ .$$

The map $\frac{1}{1+|T|^{2/r}}$ is in L^r and its norm is less than some constant which does not depend on r and

$$\|\Sigma_N^M\|_{L^{p/2}}\leq\sum_{n=N+1}^{M}\sum_{k=1}^{1+n^2}n^{2s}\left((u_0^{n,k})^2+n^{-2}(u_1^{n,k})^2\right)\|e_{n,k}(x)\|_{L^p}^2$$

as $\|e_{n,k}\|_{L^p}\leq C\sqrt{p}$,

$$\leq Cp\sum_{n=N+1}^{M}\sum_{k=1}^{1+n^2}n^{2s}\left((u_0^{n,k})^2+n^{-2}(u_1^{n,k})^2\right)\leq CpM^{2s'}(S_M-S_N)\ .$$

Therefore,

$$\|\frac{1}{1+|T|^{2/r}}(1-\Pi_N)(1-\triangle)^{s/2}U(T)(u_0^M,u_1^M)\|_{L_X^q,L_T^r,L^p(S^3)}\leq C\sqrt{p}\sqrt{q}M^{s'}\sqrt{S_M-S_N}\ .$$

End of the proof of Lemma 3.2.6 ◊

To prove the first inequality of the proposition, take $M\to\infty$, $s=0$, and $r=q=p$ in the previous lemma to get :

$$\|\frac{1}{1+|T|^{2/p}}(1-\Pi_N)U(T)(u_0,u_1)\|_{L_X^p,L_T^p,L^p(S^3)}\leq Cp\sqrt{S^N}\ .$$

Then,

$$P(\|\frac{1}{1+|T|^{2/p}}(1-\Pi_N)U(T)(u_0,u_1)\|_{L_T^p,L^p(S^3)}\geq\lambda)$$

$$= P(\|\frac{1}{1+|T|^{2/p}}(1-\Pi_N)U(T)(u_0,u_1)\|^p_{L^p_T,L^p(S^3)} \geq \lambda^p)$$

$$\leq \lambda^{-p} E(\|\frac{1}{1+|T|^{2/p}}(1-\Pi_N)U(T)(u_0,u_1)\|^p_{L^p_T,L^p(S^3)})$$

$$= \lambda^{-p}\|\frac{1}{1+|T|^{2/p}}(1-\Pi_N)U(T)(u_0,u_1)\|^p_{L^p_X,L^p_T,L^p(S^3)}$$

$$\leq \left(\frac{Cp\sqrt{S^N}}{\lambda}\right)^p .$$

To prove the second, use the previous lemma with $r = 3$, $p = 6$, $q \geq 6$, $s = 0$, $M \to \infty$, $N = 0$ to get :

$$P(\|\frac{1}{1+|T|^{2/3}}U(T)(u_0,u_1)\|_{L^3_T,L^6(S^3)} \geq \lambda) \leq \left(\frac{C\sqrt{q}\sqrt{S}}{\lambda}\right)^q .$$

For $\lambda \geq \sqrt{6}\sqrt{S}Ce$, choose

$$q = \frac{\lambda^2}{C^2e^2S} \geq 6$$

to get

$$P(\|\frac{1}{1+|T|^{2/3}}U(T)(u_0,u_1)\|_{L^3_T,L^6(S^3)} \geq \lambda) \leq e^{-p} = e^{-c\lambda^2/S}$$

and for small λ use the fact that the probability is bounded by 1 which is less than $e^6e^{-c\lambda^2/S}$ to get for all λ

$$P(\|\frac{1}{1+|T|^{2/3}}U(T)(u_0,u_1)\|_{L^3_T,L^6(S^3)} \geq \lambda) \leq Ce^{-c\lambda^2/S} .$$

For the third inequality with $\sigma = 0$, use the previous lemma with $N = 0$, $r = 1$, $s = \frac{1}{2}$, p some $p_m > 6$, $q \geq p_m$ to get

$$\|\frac{1}{1+T^2}\Pi_M U(T)(u_0,u_1)\|_{L^q,L^1_T,L^\infty(S^3)}$$

$$\leq \|\frac{1}{1+T^2}\Pi_M U(T)(u_0,u_1)\|_{L^q,L^1_T,W^{1/2,p}(S^3)} \leq CM^{1/2}\sqrt{q}\sqrt{S_M} \leq CM^{1/2}\sqrt{q}\sqrt{S}$$

thanks in particular to the Sobolev embedding $W^{1/2,7} \to L^\infty$ and then

$$P(\|\frac{1}{1+|T|^2}\Pi_M U(T)(u_0,u_1)\|_{L^1_T,L^\infty(S^3)} \geq \lambda) \leq \left(\frac{C\sqrt{q}M^{1/2}\sqrt{S}}{\lambda}\right)^q$$

and finally

$$P(\|\frac{1}{1+|T|^2}\Pi_M U(T)(u_0,u_1)\|_{L^1_T,L^\infty(S^3)} \geq \lambda) \leq Ce^{-c\lambda^2/(MS)} \ .$$

For the third inequality with $\sigma > 0$, use the previous lemma with $N=0$, $r=1$, $s=\sigma$, p some p_m larger than $\frac{4}{\sigma}$, $q \geq p$ to get

$$\|\frac{1}{1+T^2}\Pi_M U(T)(u_0,u_1)\|_{L^q,L^1_T,L^\infty(S^3)}$$

$$\leq \|\frac{1}{1+T^2}\Pi_M U(T)(u_0,u_1)\|_{L^q,L^1_T,W^{1/2,p}(S^3)} \leq C\sqrt{q}\sqrt{S_M} \leq C\sqrt{q}\sqrt{S} \ ,$$

then

$$P(\|\frac{1}{1+|T|^2}\Pi_M U(T)(u_0,u_1)\|_{L^1_T,L^\infty(S^3)} \geq \lambda) \leq \left(\frac{C\sqrt{q}\sqrt{S}}{\lambda}\right)^q \ ,$$

and finally, with an appropriate choice for q,

$$P(\|\frac{1}{1+|T|^2}\Pi_M U(T)(u_0,u_1)\|_{L^1_T,L^\infty(S^3)} \geq \lambda) \leq Ce^{-c\lambda^2/S} \ .$$

◇

Thanks to previous proposition, it is known now that $\frac{1}{1+|T|^{2/3}}U(T)(u_0,u_1)$ belongs almost surely to $L^3_T, L^6(S^3)$. Let us use this property in the local theory.

First, rewrite the equation on the sphere in a more convenient way.

The map u solves

$$\left\{\begin{array}{ll} \partial_T^2 u + (1-\triangle)u + u^3 = 0 & \\ u_{|T=0} = v_0 & \partial_T u_{|T=0} = v_1 \end{array}\right. \tag{3.2}$$

if and only if $v = u - U(T)(v_0,v_1)$ solves, with $g(T) = U(T)(v_0,v_1)$:

$$\left\{\begin{array}{ll} \partial_T^2 v + (1-\triangle)v + (g+v)^3 = 0 & \\ v_{|T=0} = 0 & \partial_T v_{|T=0} = 0 \end{array}\right. . \tag{3.3}$$

Proposition 3.2.7. *There exists C such that for all $\Lambda > 0$, all $T_0 \in \mathbb{R}$ and all g, v_0, v_1 such that*

$$\|\frac{1}{1+|T|^{2/3}}g\|^3_{L^3_T,L^6(S^3)} \leq \Lambda \ , \ \|v_0\|_{H^1} \leq \Lambda \ , \ \|v_1\|_{L^2} \leq \Lambda \ ,$$

the equation

$$\begin{cases} \partial_T^2 v + (1-\triangle)v + (g+v)^3 = 0 \\ v_{|T=T_0} = v_0 \qquad\qquad \partial_T v_{|T=T_0} = v_1 \end{cases} \tag{3.4}$$

has a unique solution in $C([T_0 - T_1, T_0 + T_1], H^1)$ *with* $T_1 = \min(1, \frac{1}{C\Lambda^2(1+T_0^2)^3})$.

Proof Let

$$\phi_{g,v_0,v_1}(v)(T) = S(T-T_0)(v_0,v_1) - \int_{T_0}^{T} \frac{\sin((T-\tau)\sqrt{1-\triangle})}{\sqrt{1-\triangle}} \left((g+v)^3(\tau)\right) d\tau \ .$$

The equation (3.4) can be rewritten as the fixed point problem $\phi_{g,v_0,v_1}(v) = v$. The map ϕ_{g,v_0,v_1} satisfies :

$$\|\phi_{g,v_0,v_1}(v)(T)\|_{H^1} \le C\Lambda + \int_{T_0}^{T} \|(g+v)^3\|_{L^2}$$

$$\|(g+v)^3\|_{L^2} = \|g+v\|_{L^6}^3 \le C(\|g\|_{L^6}^3 + \|v\|_{L^6}^3) \le C(\|g\|_{L^6}^3 + \|v\|_{H^1}^3)$$

$$\|\phi_{g,v_0,v_1}(v)(T)\|_{H^1} \le \ C\Big(\Lambda + (1+|T-T_0|^2 + |T_0|^2)\|\tfrac{1}{1+|\tau|^{2/3}} g\|_{L^3_\tau, L^6(S^3)} + \int_{T_0}^{T} \|v(\tau)\|_{H^1}^3 d\tau\Big) \ .$$

With $T \in [T_0 - T_1, T_0 + T_1]$,

$$\|\phi_{g,v_0,v_1}(v)\|_{L^\infty_T, H^1} \le C\left((2+T_0^2+T_1^2)\Lambda + |T_1|\, \|v\|_{L^\infty_T, H^1}\right) \ .$$

If $T_1 \le \min(1, \frac{1}{C^3\Lambda^2(4+T_0^2)^3}$ and $\|v\|_{L^\infty, H^1} \le C\Lambda(4+T_0^2)$, then

$$\|\phi_{g,v_0,v_1}(v)\|_{L^\infty_T, H^1} \le C(4+T_0^2)\Lambda$$

so the ball of radius $C(4+T_0^2)\Lambda$ in $C([T_0-T_1, T_0+T_1], H^1(S^3))$ is stable under ϕ_{g,v_0,v_1}.

What is more, in this ball

$$\|\phi_{g,v_0,v_1}(v) - \phi_{g,v_0,v_1}(w)\|_{L^\infty_T, H^1}$$

$$\le C\|v-w\|_{L^\infty_T, H^1}\Big((2+T_0^2)\|\frac{1}{1+|T|^{2/3}} g\|^2_{L^3_T, L^6} \|1_{[T_0-T_1, T_0+T_1]}\|_{L^3_T} + T_1(\|v\|^2_{L^\infty, H^1} + \|w\|^2_{L^\infty_T, H^1})\Big)$$

$$\le C\|v-w\|_{L^\infty_T, H^1}\left(T_1^{1/3}(2+T_0^2)\Lambda^{2/3} + T_1\Lambda^2(4+T_0^2)^2\right) \ .$$

Therefore with C large enough and $T_1 \leq \frac{1}{C\Lambda^2(1+T_0^2)}$, the fixed point theorem applies which concludes the proof. $\diamond$

Thanks to the local Cauchy theory, one can see that the solution of (3.3) can be extended for bigger times as long as the energy :

$$\mathcal{E}(T) = \int v(1-\triangle)v + \int (\partial_T v)^2 + \frac{1}{2}\int v^4$$

is finite.

To bound this quantity, different arguments are used depending on whether the initial data have been built with $\sigma = 0$ or $\sigma > 0$.

3.2.2 Global solutions on the sphere : case 1

Theorem 3.2.8. *Suppose that $\sigma > 0$. There exists a set $E_\sigma \subseteq H^\sigma \times H^{\sigma-1}$ such that the probability*

$$P((u_0, u_1) \in E_\sigma) = 1$$

and that for all $v_0, v_1 \in E_\sigma$, the Cauchy problem (3.2) *with initial datum v_0, v_1 is globally well-posed in $U(T)(v_0, v_1) + C(\mathbb{R}, H^1)$.*

Proof The third inequality of Proposition 3.2.4 ensures that, when $\sigma > 0$, $\frac{1}{1+T^2}U(T)(u_0, u_1)$ belongs almost surely to $L^1_T, L^\infty(S^3)$ and $\frac{1}{1+|T|^{2/3}}U(T)(u_0, u_1)$ belongs almost surely to $L^3_T, L^6(S^3)$. Therefore, take for E_σ the set of initial data which satisfy

$$\|\frac{1}{1+T^2}U(T)(v_0, v_1)\|_{L^1_T, L^\infty(S^3)} < \infty\ ,$$

$$\|\frac{1}{1+|T|^{2/3}}U(T)(v_0, v_1)\|_{L^3_T, L^6(S^3)} < \infty\ .$$

For $v_0, v_1 \in E_\sigma$, call $g(T) = U(T)(v_0, v_1)$ and let v be the local solution of

$$\partial_T^2 v + (1-\triangle)v + (g+v)^3 = 0$$

with initial datum $0, 0$.

According to the local Cauchy theory, the solution v exists as long as

$$\int (\partial_T v)^2 + \int v(1-\triangle)v$$

is finite.

Take

$$\mathcal{E}^2(T) = \int (\partial_T v)^2 + \int v(1-\triangle)v + \frac{1}{2}\int v^4$$

and differentiate this quantity with respect to T.

$$(\partial_T \mathcal{E})\,\mathcal{E} = \int \partial_T v \partial_T^2 v + \int \partial_T v(1-\triangle)v + \int \partial_T v v^3$$
$$= \int (\partial_T v)\left(v^3 - (g+v)^3\right)\ .$$

Hence,

$$\partial_T \mathcal{E} \leq \|v^3 - (g+v)^3\|_{L^2} \leq C\left(\|g(T)^3\|_{L^2} + \|g^2 v(T)\|_{L^2} + \|gv^2\|_{L^2}\right)$$

$$|\partial_T \mathcal{E}| \leq C\left(\|g(T)\|_{L^6}^3 + \|g\|_{L^6}^2\|v\|_{L^6} + \|g\|_{L^\infty}\|v\|_{L^4}^2\right)$$

$$|\partial_T \mathcal{E}| \leq C\left(\|g(T)\|_{L^6}^3 + \|g\|_{L^6}^2\mathcal{E} + \|g\|_{L^\infty}\mathcal{E}\right)$$

thanks to Sobolev embedding $H^1 \to L^6$, and applying Gronwall lemma :

$$\mathcal{E}(T) \leq C\int_0^T \|g(\tau)\|_{L^6}^3 d\tau e^{c\int_0^T (\|g(\tau)\|_{L^6}^2 + \|g(\tau)\|_{L^\infty})d\tau} < \infty\ ,$$

the energy is bounded, which concludes the proof of Theorem 3.2.8. ◊

3.2.3 Global solutions on the sphere : case 2

Theorem 3.2.9. *Suppose that $\sigma = 0$. There exists a set $E \subseteq L^2 \times H^{-1}$ such that the probability*

$$P((u_0, u_1) \in E) = 1$$

and that for all $v_0, v_1 \in E$, the Cauchy problem (3.2) with initial datum v_0, v_1 is globally well-posed in $U(T)(v_0, v_1) + C(\mathbb{R}, H^1)$.

Proposition 3.2.10. *Let $T_0 > 0$. There exists $C(T_0)$ such that for all $\theta > 0$ and $p = \frac{6}{\theta}$, supposing that $g = U(T)(v_0, v_1)$ can be written $g = g_1 + g_2$ with*

$$C(T_0)\|\frac{1}{1+T^{2/3}}g\|_{L_T^3, L_x^6}^3 \leq e^{p/18}$$

and

$$C(T_0)(\|\frac{1}{1+T^{2/3}}g\|_{L_T^3, L_x^6}^2 + \|\frac{1}{1+T^2}g_1\|_{L_T^2, L_x^\infty} + \|\frac{1}{1+T^{2/p}}g_2\|_{L_{T,x}^p}) \leq \frac{p}{18}$$

then the equation (3.4) has a unique solution onto $C([-T_0, T_0], H^1)$. The constant C depends on T_0 but is independent of θ.

Proof We consider the energy given in the previous subsection :

$$\mathcal{E}(v)^2 = \int v(1-\triangle)v + \int (\partial_T v)^2 + \frac{1}{2}\int v^4 \ .$$

If v is the local solution of (3.4), on its interval of definition, it comes :

$$\partial_T \mathcal{E}(v)\mathcal{E}(v) = \int (\partial_T v)\left(g^3 + 3g^2 v + 3(g_1+g_2)v^2\right)$$

and by using Hölder inequalities, in particular on the last term, with $1/p' + 1/p = 1/2$,

$$\Big|\int (\partial_T v) g_2 v^2 \leq \|\partial_T v\|_{L^2}\|g_2(T)\|_{L^p}\|v^2\|_{L^{p'}}$$

we get

$$\partial_T \mathcal{E}(v) \leq \|g(T)\|_{L^6_x}^3 + 3\|g(T)\|_{L^6_x}^2\|v\|_{L^6} + 3\|g_1(T)\|_{L^\infty}\|v^2\|_{L^2} + 3\|g_2(T)\|_{L^p}\|v^2\|_{L^{p'}} \ .$$

Then, by using Sobolev embedding $H^1 \subset L^6$ and because the H^1 norm of v is controlled by the energy

$$\|v\|_{L^6} \leq C\|v\|_{H^1} \leq C\mathcal{E}$$

and because the L^4 norm to the square is controlled by the energy :

$$\|v^2\|_{L^2} = \|v\|_{L^4}^2 \leq C\mathcal{E}$$

finally, as $\theta = \frac{6}{p}$,

$$\frac{1}{2p'} = \frac{1}{4} - \frac{1}{2p} = \frac{1}{4} - \frac{\theta}{12} = \frac{1-\theta}{4} + \frac{\theta}{6}$$

we get

$$\|v^2\|_{L^{p'}} = \|v\|_{L^{2p'}}^2 \leq (\|v\|_{L^4}^{1-\theta}\|v\|_{L^6}^{\theta})^2 \leq C\mathcal{E}^{1-\theta}\|v\|_{H^1}^{2\theta} \leq C\mathcal{E}(v)^{1+\theta} \ .$$

Thus,

$$\partial_T \mathcal{E}(v) \leq \|g(T)\|_{L^6_x}^3 + C\left((\|g(T)\|_{L^6_x}^2 + \|g_1(T)\|_{L^\infty})\mathcal{E} + \|g_2(T)\|_{L^p}\mathcal{E}^{1+\theta}\right) \ .$$

As $\mathcal{E}$ is continuous and initially 0, suppose that until time T_1 it is less than $e^{p/6} = e^{1/\theta}$, then until time T_1, it appears that :

$$\partial_T \mathcal{E}(v) \leq \|g(T)\|_{L^6_x}^3 + C\left((\|g(T)\|_{L^6_x}^2 + \|g_1(T)\|_{L^\infty}) + \|g_2(T)\|_{L^p}\right)\mathcal{E} \ .$$

Using Gronwall lemma,

$$\begin{aligned}
\mathcal{E}(v) \quad & \leq C(1+T_0^2)\|g\|_{L^3_T,L^6_x}^3 e^{C((1+T_0^2)^{2/3}\|\frac{1}{1+T^{2/3}}g\|_{L^3_T,L^6_x}^2 + (1+T_0^2)\|\frac{1}{1+T^2}g_1\|_{L^1_T,L^\infty} + (1+T_0^2)^{(1+p)/2p}\|\frac{1}{1+T^{2/p}}g_2\|_{L^p_{T,x}})} \\
& \leq C(1+T_0^2)\|g\|_{L^3_T,L^6_x}^3 e^{C(1+T_0^2)\left(\|\frac{1}{1+T^{2/3}}g\|_{L^3_T,L^6_x}^2 + \|\frac{1}{1+T^2}g_1\|_{L^1_T,L^\infty} + \|\frac{1}{1+T^{2/p}}g_2\|_{L^p_{T,x}}\right)} \ .
\end{aligned}$$

Choosing $C(T_0) = C(1+T_0^2)$, by hypothesis :

$$\mathcal{E}(v) \leq e^{p/9} < e^{p/6} \ .$$

Suppose that the solution v is not well posed on $[-T_0, T_0]$, then as $\mathcal{E}(v)$ controls the H^1 norm of v and the L^2 norm of $\partial_T v$, there exists a time T_1 such that for all time T smaller than T_1, the energy $\mathcal{E}(v)$ is smaller than $e^{p/6}$ and a ϵ such that for all $T \in]T_1, T_1+\epsilon[$, $\mathcal{E}(v) > e^{p/6}$. Then, thanks to the previous computation, until T_1, the energy is strictly less than $e^{p/6}$ and as is it continuous, there exists ϵ' such that the energy remains smaller than $e^{p/6}$ until $T_1+\epsilon'$ with contradicts the hypothesis.

Hence, the equation (3.3) has a unique solution in $\mathcal{C}([-T_0, T_0], H^1)$ provided that g satisfies the right properties. ◊

Definition 3.2.11. Let $\theta \in \{\frac{6}{p_m}, m \in \mathbb{N}\}$, $p = \frac{6}{\theta}$. As

$$S^N = \sum_{n>N} (u_0^{n,k})^2 + n^{-1}(u_1^{n,k})^2$$

converges toward 0 when N goes to ∞, there exists $N(T_0)$ such that $\sqrt{S^{N(T_0)}}$ is smaller than $\frac{1}{54eC(T_0)C_1}$, where C_1 is the constant involved in the first inequality of Proposition 3.2.4 and $C(T_0)$ is the one involved in Proposition 3.2.10, let

$$F_\theta(T_0) = \{v_0, v_1 \mid C(T_0)\|U(T)(v_0,v_1)\|^3_{L^3_T, L^6_x} \leq e^{p/18}\} \ ,$$

$$G_\theta(T_0) = \{v_0, v_1 \mid C(T_0)\|U(T)(v_0,v_1)\|^2_{L^2_T, L^6_x} \leq \frac{p}{54}\} \ ,$$

$$H_\theta(T_0) = \{v_0, v_1 \mid C(T_0)\|U(T)\Pi_N(v_0,v_1)\|_{L^1_T, L^\infty_x} \leq \frac{p}{54}\} \ ,$$

$$I_\theta(T_0) = \{v_0, v_1 \mid C(T_0)\|U(T)(1-\Pi_N)(v_0,v_1)\|_{L^p_{T,x}} \leq \frac{p}{54}\} \ ,$$

$$J_\theta(T_0) = F_\theta \cap G_\theta \cap H_\theta \cap I_\theta \ .$$

Call then

$$E(T_0) = \bigcup_{\theta \in \{\frac{6}{p_m}, m\in\mathbb{N}\}} J_\theta \ .$$

Remark 3.2.2. *The separation between the high and low frequencies is useful there, as S^N can be taken as small as one wants and ensures that the measure of I_θ^c is small enough.*

Proposition 3.2.12. *The set $E(T_0)$ is of full μ-measure.*

Proof The measures of the complementary of the different sets defined satisfy :

$$\mu(F_\theta^c) = \mu\left(\{v_0, v_1 \mid \|U(T)(v_0, v_1)\|_{L_T^3, L_x^6} > e^{p/54}\}\right) \leq Ce^{-c(T_0)e^{p/27}}$$

$$\mu(G_\theta^c) \leq \mu\left(\{v_0, v_1 \mid \|U(T)(v_0, v_1)\|_{L_T^3, L_c^6} > \sqrt{\frac{p}{54C}}\}\right) \leq Ce^{-c(T_0)p}$$

$$\mu(H_\theta^c) = \mu\left(\{v_0, v_1 \mid \|U(T)\Pi_N(v_0, v_1)\|_{L_T^1, L_x^\infty} > \frac{p}{54C}\}\right) \leq Ce^{-c(T_0)p^2/N}$$

$$\mu(I_\theta^c) = \mu\left(\{v_0, v_1 \mid \|U(T)(1 - \Pi_N)(v_0, v_1)\|_{L_{T,x}^p} > \frac{p}{54C}\}\right) \leq \left(\frac{C_1 p 54C(T_0)\sqrt{S^N}}{p}\right)^p$$

$$\mu(I_\theta^c) \leq e^{-p} \ .$$

It comes :

$$\mu(J_\theta^c) \leq Ce^{-c(T_0)p} \ .$$

Thus, for all θ, $E(T_0)$ satisfies

$$\mu(E^c(T_0)) \leq \mu(J_\theta^c) \leq Ce^{-c6/\theta} \ .$$

Taking the limit when θ goes to 0 (as when $m \to \infty$, $p_m \to \infty$ and then $\frac{6}{p_m} \to 0$) :

$$\mu(E^c(T_0)) = 0 \ , \ \mu(E(T_0)) = 1 \ .$$

◊

Proposition 3.2.13. *For all $(v_0, v_1) \in E(T_0)$, the cubic non linear wave equation on the sphere* (3.2) *with initial datum v_0, v_1 has a unique solution in $U(T)(v_0, v_1) + C([-T_0, T_0], H^1)$.*

Proof The equation (3.3) with $g = U(T)(v_0, v_1) = g_1 + g_2$, $g_1 = \Pi_N g$, $g_2 = (1 - \Pi_N)g$ is equivalent to (3.2) and satisfies the hypothesis of Proposition 3.2.10 for some $\theta \in]0, 1[$, hence it is well posed in $C([-T_0, T_0], H^1)$. Thus, (3.3) is well-posed in $U(T)(v_0, v_1) + C([-T_0, T_0], H^1)$. ◊

Definition 3.2.14. Let T_N be an increasing sequence of $\mathbb{R}$ going to $+\infty$ and let

$$E = \limsup E(T_N) \ .$$

Proposition 3.2.15. *The set E is of full μ-measure.*

Proof Indeed, using Fatou's lemma,

$$\mu(E^c) = \mu(\liminf E(T_N)^c) \leq \liminf \mu(E(T_N)^c) = 0 \ .$$

◇

Proof of Theorem 3.2.9. Let $T \geq 0$. As the sequence T_N is increasing toward ∞ there exists N_0 such that for all $N \geq N_0$,

$$T_N \geq T_{N_0} \geq T \ .$$

Since $E = \limsup E(T_N)$, for all N_1 there exists $N \geq N_1$ such that $v_0, v_1 \in E(T_N)$. With $N_1 = N_0$ there exists $N \geq N_0$ such that

$$T_N \geq T \text{ and } v_0, v_1 \in E(T_N) \ .$$

Hence the equation has a unique solution on $U(\tau)(v_0, v_1) + C([-T_N, T_N], H^1)$ and thus in $U(\tau)(v_0, v_1) + C([-T, T], H^1)$. Therefore, this property holding for all time T, the equation has a unique solution in $U(T)(v_0, v_1) + C(\mathbb{R}, H^1)$. ◇

3.3 Reduction to the sphere and almost sure solutions on the Euclidean space

In this section, the problem on the Euclidean space is reduced thanks to the Penrose transform to the problem on the sphere. The existence of solution for the Cauchy problem with initial data on a suitable space is derived in this way. Note that for all the sequel $\sigma = 0$.

3.3.1 Penrose transform and reduction to the sphere

As a basis of L^2 uniformly bounded in L^p is required to use the techniques developed by N. Burq and N. Tzvetkov in [14] and according to [12], the problem needs to be reduced to the sphere. For that, the Penrose transform seems appropriate, since it turns the d'Alembertian of $\mathbb{R}^3$ into the d'Alembertian of S^3 added to the identity on distributions.

The Penrose transform has been introduced in Chapter 2.

Definition 3.3.1 (Penrose Transform on the variables)**.** For all $t \in \mathbb{R}$ and $r \in \mathbb{R}^+$, define $T(t, r)$, $R(t, r)$, $R_0(r)$,

$\Omega(t,r)$ and $\Omega_0(r)$ as :

$$T = \text{Arctan}\,(t+r) + \text{Arctan}\,(t-r)\ ,\ R = \text{Arctan}\,(t+r) - \text{Arctan}\,(t-r)\ ,$$
$$R_0(r) = R(0,r) = 2\text{Arctan}\,(r)\ ,$$
$$\Omega(t,r) = \cos T + \cos R = \frac{2}{\sqrt{(1+(t+r)^2)(1+(t-r)^2)}}$$
$$\Omega_0(r) = \Omega(0,r) = \frac{2}{1+r^2}\ .$$

Proposition 3.3.2. *The map*

$$t,r,\omega \in \mathbb{R}\times\mathbb{R}^+\times S^2 \mapsto T(t,r), R(t,r), \omega$$

is a bijection from $\mathbb{R}\times\mathbb{R}^3$ *to* $S = \{(T,R,\omega)\ |\ \cos T + \cos R > 0\}$ *and its inverse is given by*

$$T,R,\omega \mapsto t = \frac{\sin T}{\cos T + \cos R}\ ,\ r = \frac{\sin R}{\cos T + \cos R}\ ,\ \omega\ .$$

See [55, 20] for the proof.

Remark 3.3.1. *The map* $r,\omega \mapsto 2Arctan\,(r),\omega$ *is a bijection from* $\mathbb{R}^3$ *to* S^3 *deprived of one of its poles,* $R_0(r) = 2Arctan\,(r) \in [0,\pi[$ *being the third angle describing a point in* S^3*.*

Definition 3.3.3 (Penrose Transform on distributions)**.** Let f be a distribution on $\mathbb{R}\times\mathbb{R}^3$ and (f_0,f_1) be a pair of distributions on $\mathbb{R}^3$. Define then $v = \text{PT}(f)$ the distribution on S and $(v_0,v_1) = \text{PT}_0(f_0,f_1)$ the pair of distributions on S^3 deprived of one of its poles such that

$$v(T,R,\omega) = f(\frac{\sin T}{\cos T + \cos R}, \frac{\sin R}{\cos T + \cos R}, \omega)(\cos T + \cos R)^{-1}$$

and

$$v_0(R,\omega) = \frac{f_0(\tan(R/2),\omega)}{1+\cos R}\ ,\ v_1(R,\omega) = \frac{f_1(\tan(R/2),\omega)}{(1+\cos R)^2}\ .$$

Remark 3.3.2. *The definition of* PT_0 *may appear a little awkward but the idea hidden behind the notations is that* f *solves the cubic non linear wave equation with initial datum* (g_0,g_1) *if an extension of* $PT(f)$ *solves the equation of the first section with initial datum an extension to* S^3 *of* $PT_0(g_0,g_1)$*.*

Definition 3.3.4. Let u be a distribution on $\mathbb{R}\times S^3$ and v_0,v_1 two distributions on S^3, the inverse Penrose transform is given by :

$$\text{PT}^{-1}u(t,r,\omega) = \Omega(t,r)u(\text{Arctan}\,(t+r) + \text{Arctan}\,(t-r), \text{Arctan}\,(t+r) - \text{Arctan}\,(t-r), \omega)\ ,$$

which depends only on the restriction of u to S and the inverse Penrose transform at time $t = 0 \Leftrightarrow t = 0$ by

$$\mathrm{PT}_0^{-1}(r,\omega)(v_0,v_1) = \Big(\Omega_0(r)v_0(2\mathrm{Arctan}\ (r),\omega)\ ,\ ,\Omega_0^2(r)v_1(2\mathrm{Arctan}\ (r),\omega)\Big)\ ,$$

witch depends only on the restriction on S^3 deprived of one of its poles of v_0, v_1.

Lemma 3.3.5. *If u solves the problem*

$$\begin{cases} (\partial_T^2 + 1 - \triangle_{S^3})u + u^3 = 0 \\ (u_{|T=0}, \partial_T u_{|T=0}) = v_0, v_1 \end{cases} \tag{3.5}$$

then the map f defined as the inverse Penrose transform of u restricted to S, that is

$$f = PT^{-1}(u)$$

solves the problem :

$$\begin{cases} (\partial_t^2 - \triangle_{\mathbb{R}^3})f + f^3 = 0 \\ f|_{t=0} = g_0 \qquad\qquad \partial_t f|_{t=0} = g_1 \end{cases} \tag{3.6}$$

where

$$(g_0, g_1) = PT_0^{-1}(v_0, v_1)\ .$$

Proof The fact that the Penrose transform sends the action of $\partial_t^2 - \triangle_{\mathbb{R}^3}$ on $\mathbb{R}\times\mathbb{R}^3$ onto the action of $\Omega^3(\partial_T^2 + 1 - \triangle_{S^3})$ on S is known and the proof can be found in [55]. Thus, on S

$$\Big((\partial_t^2 - \triangle_{\mathbb{R}^3})f + f^3\Big)(t,r,\omega) = \Omega^3(\partial_T^2 + 1 - \triangle_{S^3})\mathrm{PT}(f) + \Omega^3\mathrm{PT}(f)^3(T(t,r),R(t,r),\omega) = 0\ .$$

What is more, $T = 0 \Leftrightarrow t = 0$,

$$g_0 = f(t=0) = \Omega_0 u(T=0) = \Omega_0 u(R_0(r))$$

and

$$\begin{aligned} g_1 &= \partial_t f(t=0) = (\partial_t\Omega)(t=0)u(0,R_0(r)) + \Omega_0\partial_t T(t=0)\partial_T u + \Omega_0\partial_t R(t=0)\partial_R u \\ &= \Omega_0(r)^2\partial_T u = \Omega_0^2 v_1(R_0(r))\ . \end{aligned}$$

◊

3.3.2 Properties of the change of variable

In this subsection, the properties of the change of variables involved in the Penrose transform is studied, in particular what it implies on operators and norms.

Definition 3.3.6. Let Ψ be the change of variable corresponding to the Penrose transform at time $T = 0$, that is to say :

$$\Psi(v)(r, \omega) = v(2\text{Arctan}\,(r), \omega) \ .$$

Proposition 3.3.7. *This change of variable satisfies :*

- *for all* v, w, $\int v(R, \omega) w(R, \omega) \sin^2 R d\omega dR = \int \Psi(v)(r, \omega)\Psi(w)(r, \omega) r^2 \left(\frac{2}{1+r^2}\right)^3 dr$,
- *for all* v, $\int |v|^p \sin^2 R dR = \int |\Psi(v)|^p r^2 \left(\frac{2}{1+r^2}\right)^3 dr$,
- $\Psi(\triangle_{S^3} v) = \left(\frac{1+r^2}{2}\right)^2 \triangle_{\mathbb{R}^3} \Psi(v) - \frac{1+r^2}{2} r\partial_r \Psi(v)$.

Proof The proof comes from the facts that :
- $dR = \frac{2}{1+r^2} dr$,
- $\sin R = \frac{2r}{1+r^2}$ and
- $\tan R = \frac{2r}{1-r^2}$.

Hence, to do the change of variable in the integrals, one can use :

$$\sin^2 R dR = \left(\frac{2}{1+r^2}\right)^3 r^2 dr \ .$$

The computation of the change of variable on the Laplace-Beltrami operator is quite similar :

$$\begin{aligned}
\Psi(\partial_R v) &= (\partial_r R)^{-1} \partial_r \Psi(v) \\
\Psi(\sin^2(R)\partial_R v) &= \frac{2r^2}{1+r^2} \partial_r \Psi(v) \\
\Psi(\partial_R \sin^2(R)\partial_R v) &= (\partial_r R)^{-1} \partial_r \Psi(\sin^2 R \partial_R v) \\
\Psi(\partial_R \sin^2(R)\partial_R v) &= -\frac{2r^3}{1+r^2} \partial_r \Psi(v) + \partial_r \left(r^2 \partial_r \Psi(v)\right) \\
\Psi(\frac{1}{\sin^2 R} \partial_R \sin^2(R)\partial_R v) &= -\frac{1+r^2}{2} r\partial_r \Psi(v) + \left(\frac{1+r^2}{2}\right)^2 \frac{1}{r^2} \partial_r (r^2 \partial_r \Psi(v)) \\
\Psi(\triangle_{S^3} v) &= \left(\frac{1+r^2}{2}\right)^2 \triangle_{\mathbb{R}^3} \Psi(v) - \frac{1+r^2}{2} r\partial_r \Psi(v).
\end{aligned}$$

◇

Definition 3.3.8. Let f_0, f_1 be the inverse Penrose transform at time $T = 0$ of (u_0, u_1) and $g_{n,k}, h_{n,k}$ the inverse Penrose transform at time $T = 0$ of $e_{n,k}, e_{n,k}$, that is to say :

$$f_0 = \sum_{n,k} u_0^{n,k} a_{n,k} g_{n,k} \ , \ f_1 = \sum_{n,k} u_1^{n,k} b_{n,k} h_{n,k} \ .$$

Proposition 3.3.9. *The $g_{n,k}$ are the eigenfunctions of $H_0 = \left(\frac{1+r^2}{2}\right)^2 \triangle_{\mathbb{R}^3} + \frac{1+r^2}{2} r\partial_r + \frac{3+r^2}{2}$ with eigenvalues $1 - n^2$ and the $h_{n,k}$ are the eigenfunctions of $H_1 = \left(\frac{1+r^2}{2}\right)^2 \triangle_{\mathbb{R}^3} + 3\frac{1+r^2}{2} r\partial_r + 6\frac{1+r^2}{2}$ with eigenvalues $1 - n^2$, $n \geq 1$.*

Proof As

$$g_{n,k} = \frac{2}{1+r^2}\Psi(e_{n,k})$$

they are the eigenfunctions of the operator H_0 such that

$$H_0(g) = \frac{2}{1+r^2}\Psi\left(\triangle_{S^3}\Psi^{-1}(\frac{1+r^2}{2}g)\right) \ .$$

It remains to compute H_0.

$$H_0 g = \left(\frac{1+r^2}{2}\right)^2 \triangle_{\mathbb{R}^3} g + \frac{1+r^2}{2} r\partial_r g + \frac{3+r^2}{2} g$$

As the $h_{n,k}$ are given by

$$h_{n,k} = \left(\frac{2}{1+r^2}\right)^2 e_{n,k} \ ,$$

they are the eigenfunctions of the operator H_1 defined by

$$H_1 h = \left(\frac{2}{1+r^2}\right)^2 \Psi\left(\triangle_{S^3}\Psi^{-1}\left(\frac{1+r^2}{2}\right)^2 h\right) \ .$$

By manipulating the expression of H_1, we get that

$$H_1 h = \left(\frac{1+r^2}{2}\right)^2 \triangle_{\mathbb{R}^3} h + 3\frac{1+r^2}{2} r\partial_r h + 6\frac{1+r^2}{2} h \ .$$

◊

Corollary 3.3.10. *Let f_0 and f_1 given by $(f_0, f_1) = PT_0^{-1}(u_0, u_1)$. We have that :*

$$\|u_0\|_{W^{s,p}(S^3)} = \|\left(\frac{2}{1+r^2}\right)^{3/p-1} (1-H_0)^{s/2} f_0\|_{L^p(\mathbb{R}^3)}$$

$$\|u_1\|_{W^{s,p}(S^3)} = \|\left(\frac{2}{1+r^2}\right)^{3/p-2} (1-H_1)^{s/2} f_1\|_{L^p(\mathbb{R}^3)} .$$

Proof First, do the change of variable in the L^p-norm :

$$\|u_0\|_{W^{s,p}} = \|\left(\frac{2}{1+r^2}\right)^{3/p} \Psi((1-\triangle_{S^3})^{s/2} u_0)\|_{L^p} .$$

Then, compute $\Psi((1-\triangle_{S^3})^{s/2} u_0)$:

$$\Psi((1-\triangle_{S^3})^{s/2} u_0) = \Psi\left(\sum_{n,k} n^s u_0^{n,k} a_{n,k} e_{n,k}\right) = \frac{1+r^2}{2} \sum_{n,k} n^s u_0^{n,k} a_{n,k} g_{n,k}$$

$$= \frac{1+r^2}{2} \sum_{n,k} (1-H_0)^{s/2} u_0^{n,k} a_{n,k} g_{n,k} = \frac{1+r^2}{2} (1-H_0)^{s/2} f_0 .$$

In the end, it comes :

$$\|u_0\|_{W^{s,p}} = \|\left(\frac{2}{1+r^2}\right)^{3/p-1} (1-H_0)^{s/2} f_0\|_{L^p} .$$

The second equality is proved the same way. ◊

3.3.3 Spaces of definition of the initial data

Considering the results of the previous subsection, the choice of the random variable $a_{n,k}$ and $b_{n,k}$ will be made such that the initial datum u_0, u_1 of the equation reduced to the sphere is a Gaussian vector, in order to state which norms of u_0 and u_1 and then of the initial datum of the wave equation on the Euclidean space are almost surely finite or infinite.

In this subsection, suppose that $a_{n,k}$ and $b_{n,k}$ not only satisfy the Gaussian large deviation estimate, but that they are Gaussian. To ensure that the initial datum is almost surely not into some spaces, Fernique's theorem should be used :

Theorem 3.3.11 (Fernique, 1974)**.** *Let X be a Gaussian vector with value into a Banach space B and N a pseudo-norm on B (a pseudo-norm has the same properties as a norm except that ∞ is one of its possible value), then for all $p \geq 1$ if the mean value of $N(X)^p$ is infinite, then $N(X)$ is almost surely ∞ :*

$$E(N(X)^p) = \infty \Rightarrow P(N(X) = \infty) = 1 .$$

For the proof, see [30].

Proposition 3.3.12. *The initial datum u_0, u_1 is almost surely not in $H^s \times H^{s-1}$ for all $s > 0$.*

Proof Use Fernique's theorem with $B = L^2$, $p = 2$ and N being the H^s norm and X either u_0 or u_1. As

$$E(||u_0||^2_{H^s}) = \sum_{n,k} (u_0^{n,k})^2 n^{2s} \ ,$$

$$E(||u_1||^2_{H^{s-1}}) = \sum_{n,k} (u_1^{n,k})^2 n^{2(s-1)} \ ,$$

either one of this series diverges and u_0 and u_1 are pseudo Gaussian vectors, it comes that almost surely

$$||u_0||_{H^s} = \infty$$

or almost surely

$$||u_1||_{H^{s-1}} = \infty \ .$$

◇

Considering the remarks on the norms of the initial datum (f_0, f_1) with respect to the ones of (u_0, u_1) in the previous subsection, the author believes that the initial datum f_0, f_1 of the cubic non linear wave equation belongs almost surely to $L^2 \times H^{-1}$ with weight $\frac{1}{\sqrt{1+r^2}}$ but is almost surely not in $H^s \times H^{s-1}$ for all $s > 0$.

Nevertheless, the proof would require the equivalence between the norms

$$|| \left(\frac{2}{1+r^2}\right)^{1/2} (1 - H_0)^{s/2} . ||_{L^2(\mathbb{R}^3)} \text{ and } || \left(\frac{2}{1+r^2}\right)^{1/2-s} (1 - \triangle_{\mathbb{R}^3})^{s/2} . ||_{L^2(\mathbb{R}^3)}$$

in the one hand and

$$|| \left(\frac{2}{1+r^2}\right)^{-1/2} (1 - H_1)^{s/2} . ||_{L^2(\mathbb{R}^3)} \text{ and } || \left(\frac{2}{1+r^2}\right)^{-1/2-s} (1 - \triangle_{\mathbb{R}^3})^{s/2} . ||_{L^2(\mathbb{R}^3)}$$

on the other hand.

Definition 3.3.13. Let $\mathcal{H}_0^s(\mathbb{R}^3)$ and $\mathcal{H}_1^s(\mathbb{R}^3)$ be the topological spaces defined by the norms

$$||g||_{\mathcal{H}_0^s(\mathbb{R}^3)} = || \left(\frac{2}{1+r^2}\right)^{1/2} (1 - H_0)^s g ||_{L^2}$$

and

$$||g||_{\mathcal{H}_1^s(\mathbb{R}^3)} = || \left(\frac{2}{1+r^2}\right)^{-1/2} (1 - H_1)^s g ||_{L^2} \ .$$

Proposition 3.3.14. *The set $F = PT_0^{-1}(E)$ is almost surely included in $\mathcal{H}_0^0 \times \mathcal{H}_1^{-1}$ and almost surely disjoint from $\mathcal{H}_0^s \times \mathcal{H}_1^{s-1}$ for all $s > 0$.*

Setting $f_0, f_1 = PT_0^{-1}(u_0, u_1)$, the random variable which is used as initial datum of the cubic NLW on $\mathbb{R}^3$ and ν the image measure of μ under PT_0, the set F satisfies $\nu(F^c) = 0$, which means that there exists ν almost surely a solution of the cubic NLW.

3.4 Uniqueness of the solution and scattering

In this section, the uniqueness of the solution, alongside with some scattering properties is proved.

3.4.1 Uniqueness

Theorem 3.4.1. *Let $g_0, g_1 \in PT_0^{-1}(E)$. The Cauchy problem*

$$\begin{cases} \partial_t^2 f - \triangle f + f^3 = 0 \\ f|_{t=0} = g_0, \qquad\qquad \partial_t f_{|t=0} = g_1 \end{cases}$$

has a unique solution in $L(t)(g_0, g_1) + C(\mathbb{R}, H^1(\mathbb{R}^3))$ where $L(t)$ is the flow of $\partial_t^2 - \triangle = 0$.

Proof Let $v_0, v_1 \in E$ such that (g_0, g_1) is the inverse Penrose transform of v_0, v_1, let u be the solution of the equation on the sphere with initial datum v_0, v_1. Let f be the Penrose transform of u restricted to S, this map f satisfies the Cauchy problem with initial datum g_0, g_1, which gives the existence of the solution. Prove now that this solution is unique.

Lemma 3.4.2. *The flow of the linearised around* 0 *wave equation is the inverse Penrose transform of the linear flow $U(T)$, that is :*

$$PT^{-1}U(T)(v_0, v_1) = (L(t)(g_0, g_1)) \ .$$

Proof The map $w = U(T)(v_0, v_1)$ satisfies

$$\partial_T^2 w + (1 - \triangle_{S^3})w = 0$$

with initial datum v_0, v_1. Hence its inverse Penrose transform h satisfies

$$\partial_t^2 h - \triangle_{\mathbb{R}^3} h = \Omega^3 \Psi \left(\partial_T^2 h + (1 - \triangle_{S^3})h \right) = 0$$

with initial datum g_0, g_1, that is, $h = L(t)(g_0, g_1)$. ◊

Then, let $g = f - L(t)(g_0, g_1)$, g is the reverse Penrose transform of $v = u - U(T)(v_0, v_1)$ and is the solution of

$$\partial_t^2 g - \triangle g + (L(t)(g_0, g_1) + g)^3 = 0$$

with initial datum 0, 0, hence it is a fixed point of

$$\phi(g) = \int_0^t \frac{\sin(\sqrt{-\triangle}(t-s))}{\sqrt{-\triangle}}(L(t)(g_0, g_1) + g)ds \ .$$

Lemma 3.4.3. *Let $w \in L^q([-\pi, \pi] \times S^3)$, $q \geq 4$ and h its reverse Penrose transform, then*

$$\|h\|_{L^q(\mathbb{R}\times\mathbb{R}^3)} \leq C\|w\|_{L^q([-\pi,\pi]\times S^3)} \ .$$

Proof Computing the change of variable $(T, R) = (\text{Arctan }(t+r) + \text{Arctan }(t-r), \text{Arctan }(t+r) - \text{Arctan }(t-r))$ leads to :

$$\int_{\mathbb{R}\times\mathbb{R}^3} |h(t,r,\omega)|^q r^2 drdtd\omega = \int_{\Omega>0} |\Omega w(R,T,\omega)|^q \Omega^{-4} \sin^2 R dRdTd\omega \ .$$

With $q \geq 4$ and $\Omega = \cos T + \cos R$ being bounded by 2,

$$\|h\|_{L^q(\mathbb{R}\times\mathbb{R}^3)} \leq C\|w\|_{L^q} \ .$$

◊

Therefore, $L(t)(g_0, g_1)$, g and so f belong to $L^6(\mathbb{R}\times\mathbb{R}^3)$. Indeed,

$$\|L(t)(g_0, g_1)\|_{L^6(\mathbb{R}\times\mathbb{R}^3)} \leq C\|U(T)(v_0, v_1)\|_{L^6}$$

$$\leq C\left(\int_{-\pi}^{\pi}(1+T^{\delta_3})^6\right)^{1/6} \|(1+T^{\delta_3})^{-1}U(T)(v_0,v_1)\|_{L^3_T, L^6(S^3)} < \infty \ ,$$

$$\|g\|_{L^6} \leq C\|v\|_{L^6} \leq C(2\pi)^{1/6}\|v\|_{L^\infty_T, L^6} \leq C\|v\|_{L^\infty_T, H^1} < \infty \ .$$

Lemma 3.4.4. *The map g belongs to $C(\mathbb{R}, H^1(\mathbb{R}^3))$ and $\partial_t g \in C(\mathbb{R}, L^2(\mathbb{R}^3))$.*

Proof The map g satisfies

$$g(\tau) = -\int_0^\tau \frac{\sin(\tau - s)\sqrt{-\triangle}}{\sqrt{-\triangle}}(L(s)(g_0, g_1) + g)^3 ds \ .$$

Hence, by differentiating this expression :

$$\partial_t g = -\int_0^\tau \cos((\tau - s)\sqrt{-\triangle})(L(s)(g_0, g_1) + g)^3 ds \ ,$$

and using a Hölder inequality on the integral over time :

$$\|\partial_t g\|_{L^2(\mathbb{R}^3)} \leq \int_0^\tau \|L(s)(g_0, g_1) + g\|^3_{L^6(\mathbb{R}^3)} \leq \sqrt{|\tau|}\|L(s)(g_0, g_1) + g\|^3_{L^6(\mathbb{R}\times\mathbb{R}^3)}$$

then, we use Minkowski inequality on the L^2 norm of $\partial_t g$,

$$\|\partial_t g\|_{L^\infty([0,t],L^2(\mathbb{R}^3))} \leq \sqrt{|t|}\|L(s)(g_0,g_1)+g\|^3_{L^6(\mathbb{R}\times\mathbb{R}^3)} \ .$$

Therefore, as $L(s)(g_0,g_1)$ and g belong to $L^6(\mathbb{R}\times\mathbb{R}^3)$, g belongs to $C(\mathbb{R},L^2)$.

Besides,

$$\|g\|_{\dot H^1} \leq \int_0^\tau \|L(s)(g_0,g_1)+g\|^3_{L^6(\mathbb{R}^3)} \ .$$

Hence,

$$\|g\|_{\dot H^1} \leq \sqrt{|t|}\|L(s)(g_0,g_1)+g\|^3_{L^6(\mathbb{R}\times\mathbb{R}^3)} \ .$$

Therefore, g belongs to $C([0,t],H^1(\mathbb{R}^3))$. ◊

Prove now the uniqueness of the solution in $L(t)(g_0,g_1)+C([0,t],H^1(\mathbb{R}^3))$. Let f_2, f_3 be two solutions of the cubic wave equation with initial datum g_0,g_1, let $h = f_2 - f_3$. The map h satisfies :

$$\partial_t^2 h - \triangle h + f_2^3 - f_3^3 = 0 \ .$$

Remark that h is in H^1 and $\partial_t h$ is in L^2. Let

$$H(t)^2 = \int (\partial_t h)^2 + \int h(1-\triangle)h \ .$$

$$2H'(t)H(t) = -2\int(\partial_t h)\left(-h+f_2^3-f_3^3\right) = -2\int(\partial_t h)\left(-h+h(f_2^2+f_2f_3+f_3^2\right)$$

$$H'(t) \leq C\|h\|_{L^2} + C\|h\|_{L^6}\left(\|f_2\|^2_{L^6}+\|f_3\|^2_{L^6}\right)$$

As $H(0)=0$ and

$$\int_0^t \left(\|f_2\|^2_{L^6}+\|f_3\|^2_{L^6}\right) \leq |t|^{2/3}\left(\|f_2\|^2_{L^6}+\|f_3\|^2_{L^6_{t,x}}\right) \ ,$$

by Gronwall lemma, $H(t)=0$ for all time t, which proves the uniqueness. ◊

3.4.2 Scattering property

Finally, with those particular initial data for the wave equation, it satisfies a scattering property. More precisely, when t goes to $\pm\infty$ the solution tends to behave like the solution of the linearised around 0 solution of the equation with same initial datum. This property does not result from a scattering property of the wave equation on the sphere. Indeed, it is the fact that the Penrose transform divides the solution by something that behaves like $\frac{1}{t^2}$ that ensures scattering.

Theorem 3.4.5. *Let $q \in]\frac{18}{5}, 6]$, $(g_0, g_1) \in PT_0^{-1}(E)$, $f(t)$ the solution of the cubic wave equation with initial datum g_0, g_1.*

There exists a constant C depending on the initial datum such that

$$\|f(t) - L(t)(g_0, g_1)\|_{L^q} \leq \frac{C}{(1+t^2)^{1/6}} \ .$$

Proof Let $v_0, v_1 \in E$ such that $(g_0, g_1) = \mathrm{PT}_0^{-1}(v_0, v_1)$ and u the solution of (3.2) with initial datum v_0, v_1. The map u satisfies :

$$u(T) - U(T)(v_0, v_1) = -\int_0^T \frac{\sin((T-\tau)\sqrt{1-\triangle})}{\sqrt{1-\triangle}}(u^3(\tau))d\tau \ .$$

Taking the inverse of the Penrose transform of this equality leads to :

$$f(t) - L(t)(g_0, g_1) = -\Omega(t,r)\left(\int_0^{T(t,r)} \frac{\sin(T-\tau)\sqrt{1-\triangle})}{\sqrt{1-\triangle}}(u^3(\tau))d\tau\right)(R(t,r)) \ .$$

$$\|f(t) - L(t)(g_0, g_1)\|_{L^q} \leq \|\frac{\Omega}{2\Omega^{2/3}(1+t^2+r^2)^{1/6}}\|_{L^p}$$

$$\|\left(\int_0^{T(t,r)} \frac{\sin(T-\tau)\sqrt{1-\triangle})}{\sqrt{1-\triangle}}(u^3(\tau))d\tau\right)(R(t,r))2\Omega^{2/3}(1+t^2+r^2)^{1/6}\|_{L^6}$$

with $\frac{1}{q} = \frac{1}{p} + \frac{1}{6}$ $(q \leq 6)$.

Let

$$A = \|\frac{\Omega}{2\Omega^{2/3}(1+t^2+r^2)^{1/6}}\|_{L^p}$$

and

$$B = \|\left(\int_0^{T(t,r)} \frac{\sin(T-\tau)\sqrt{1-\triangle})}{\sqrt{1-\triangle}}(u^3(\tau))d\tau\right)(R(t,r))2\Omega^{2/3}(1+t^2+r^2)^{1/6}\|_{L^6} \ .$$

Apply the change of variable $R = \text{Arctan}\,(t+r) - \text{Arctan}\,(t-r)$ (t is fixed) in B. We have :

$$dR = \Omega^2 2(1 + t^2 + r^2)dr$$

$$\sin^2 R dR = \Omega^4 2(1 + t^2 + r^2)r^2 dr \ .$$

the quantity B can be rewritten as :

$$B = \| \int_{-\pi}^{\pi} 1_{\tau \le T(t,R)} \frac{\sin(T-\tau)\sqrt{1-\triangle})}{\sqrt{1-\triangle}}(u^3(\tau))d\tau\|_{L^6}$$

thus

$$B \le \int_{-\pi}^{\pi} \|1_{\tau \le T(t,R)} \frac{\sin(T-\tau)\sqrt{1-\triangle})}{\sqrt{1-\triangle}}(u^3(\tau))\|_{L^6} d\tau$$

$$B \le \int_{-\pi}^{\pi} \| \frac{\sin(T-\tau)\sqrt{1-\triangle})}{\sqrt{1-\triangle}}(u^3(\tau))\|_{L^6} d\tau \ .$$

Then , we use that thanks to Sobolev embedding $H^1 \subset L^6$:

$$\begin{aligned} \| \frac{\sin(T-\tau)\sqrt{1-\triangle})}{\sqrt{1-\triangle}}(u^3(\tau))\|_{L^6} &\le \| \tfrac{\sin(T-\tau)\sqrt{1-\triangle})}{\sqrt{1-\triangle}}(u^3(\tau))\|_{H^1} \\ &\le \|u^3\|_{L^2} = \|u\|^3_{L^6} \end{aligned}$$

Hence, we have :

$$B \le C\|u\|^3_{L^3_{T\in[-\pi,\pi]}, L^6} \ .$$

To bound A remark that $\Omega \le \frac{2}{\sqrt{1+r^2}}$.

$$A \le \frac{1}{(1+t^2)^{1/6}} \|\Omega^{1/3}\|_{L^p} = \frac{1}{(1+t^2)^{1/6}} \|\Omega\|^{1/3}_{L^{p/3}} \ .$$

As $q > \frac{18}{5}$,

$$\frac{p}{3} = \frac{2q}{6-q} > 3$$

which ensures that $\Omega \in L^{p/3}$ and bounded uniformly in t.

Finally,

$$\|f(t) - L(t)(g_0, g_1)\|_{L^q} \le AB \le \frac{C}{(1+t^2)^{1/6}} \ .$$

$\diamond$

3.5 Appendix : Uniformly bounded basis

In this appendix, we will build a measure ν_n on the set of orthogonal basis of the space E_n spanned by spherical harmonics of degree $n-1$ (in dimension 3) that satisfies the property required to take an orthonormal basis of $L^2(S^3)$ that is uniformly bounded in L^p.

Let us begin with giving some notations.

For $n \geq 1$, let $(f_{n,k})_{1\leq k\leq N_n}$ be a fixed orthonormal basis of E_n, that is, for all k

$$-\triangle_{S^3} f_{n,k} = \lambda_n^2 f_{n,k}$$

with $\lambda_n = \sqrt{n^2-1}$ and N_n is the dimension of E_n, that is $N_n = n^2$.

We identify the functions of E_n whose L^2 norm is equal to 1 with $S_n = S^{N_n-1}$ the unit sphere of $\mathbb{R}^{N_n}$. Call p_n the uniform measure on S_n.

We should admit one theorem before we can go on :

Theorem 3.5.1 (see, [40]). *If $F : S_n \to \mathbb{R}$ is Lipschitz continuous on S_n (for the distance $d(u,v) = \|u-v\|_2$) and $M(F)$ is its median defined as :*

$$p_n(F \geq M(F)) \geq \frac{1}{2} \text{ and } p_n(F \leq M(F)) \geq \frac{1}{2})$$

then, we have that

$$p_n(|F - M(F)| > r) \leq 2e^{-(N_n-1)\frac{r^2}{2\|F\|_{\mathrm{Lip}}^2}} .$$

We then give other definitions.

Definition 3.5.2. We identify the set of all orthonormal basis $(b_k)_{1\leq k\leq N_n}$ of E_n with the group of all orthogonal operators of $\mathbb{R}^{N_n}$, that is $O(N_n)$, and we call ν_n the Haar measure on $O(N_n)$ and $\Pi_{n,k}$ the map that takes a matrix in $O(N_n)$ and gives its k-th column.

Proposition 3.5.3. *The image measure of ν_n by $\Pi_{n,k}$ is equal to p_n.*

Proof For all $R \in O(N_n)$ and for all measurable set A, we have by definition :

$$\Pi_{n,k}^* \nu_n(RA) = \nu_n(\Pi_{n,k}^{-1}(RA))$$

but then the fact that U is in $\Pi_{n,k}^{-1}(RA)$ is equivalent to the fact that $R^{-1}\Pi_{n,k}U$ belongs to A. We also have that R and $\Pi_{n,k}$ commute :

$$R^{-1}\Pi_{n,k}U = \Pi_{n,k}(R^{-1}U)$$

hence $\Pi_{n,k}^{-1}(RA) = R\Pi_{n,k}^{-1}(A)$ and thus

$$\Pi_{n,k}^* \nu_n(RA) = \nu_n(R\Pi_{n,k}^{-1}(A)) .$$

Then, as ν_n is the Haar measure of $O(N_n)$ its is invariant through multiplication to the left, so

$$\Pi_{n,k}^* \nu_n(RA) = \Pi_{n,k}^* \nu_n(A) \ .$$

Hence $\Pi_{n,k}^* \nu_n$ is invariant through every isometry of $\mathbb{R}^{N_n}$, it is the uniform measure on S_n, which is p_n. ◊

To apply Theorem 3.5.1, the L^q norm has to be Lipschitz continuous on S_n. The next Lemma results in this property.

Lemma 3.5.4. *There exists C such that for all $u \in E_n$ and $q \in [2, \infty]$, we have that :*

$$\|u\|_{L^q} \le C n^{1-2/q} \|u\|_{L^2} \ .$$

Proof Let π_n be the orthogonal projection on E_n and K its kernel, that is, for all $f \in L^2$ and all x :

$$\int K(x,y) f(y) dy = \pi_n f(x) \ .$$

This kernel is given by

$$K(x,y) = \sum_k g_k(x) g_k(y)$$

for all orthonormal basis $(g_k)_{1 \le k \le n^2}$ of E_n. Hence, as for any rotation R of S_n, $(f_{n,k})_k$ and $(f_{n,k} \circ R)_k$ are orthonormal basis of E_n, we have :

$$K(x,y) = \sum_k f_{n,k}(x) f_{n,k}(y) = \sum_k f_{n,k}(Rx) f_{n,k}(Ry) = K(Rx, Rx) \ .$$

Therefore $K_n(x)$ defined as $\sqrt{K(x,x)}$ is a constant on S_n. Let us compute its value.

$$K_n(x)^2 = \frac{1}{vol(S^3)} \int K_n(t)^2 dt = \frac{1}{vol(S^3)} \int \sum_k |f_{n,k}(t)|^2 dt = C N_n = C n^2 \ .$$

where $C = \frac{1}{vol(S^3)}$ does not depend on n.

If u is in E_n, with a Cauchy-Schwartz inequality, we get that :

$$|u(x)| \le \|u\|_{L^2} K_n(x) \le C n \|u\|_{L^2} \ .$$

Therefore, we have :

$$\|u\|_{L^\infty} \le C n \|u\|_{L^2}$$

and by interpolation,

$$\|u\|_{L^q} \le C \|u\|_{L^2}^{\theta} \|u\|_{L^\infty}^{1-\theta}$$

with $\theta = \frac{2}{q}$, hence

$$\|u\|_{L^q} \le C n^{1-2/q} \|u\|_{L^2} \ .$$

◊

Proposition 3.5.5. *With $M_{n,q}$ the median of $\|.\|_{L^q}$ on S_n, we have that there exists $c_1 > 0$ such that for all r, all n and all $q \geq 2$,*

$$\nu_n\left(\{(b_k)_{1\leq k\leq N_n} \mid \exists k_0\ ,\ \left|\|b_{k_0}\|_{L^q} - M_{n,q}\right| > r\}\right) \leq 2N_n e^{-c_1 n^{4/q} r^2}\ .$$

Proof First, we apply the previous Lemma to prove that $\|.\|_{L^q}$ is Lipschitz continuous with Lipschitz constant equal to $Cn^{1-2/q}$ as

$$\left|\|u\|_{L^q} - \|v\|_{L^q}\right| \leq \|u-v\|_{L^q} \leq Cn^{1-2/q}\|u-v\|_{L^2}\ .$$

Then, we apply Proposition 3.5.3 to get, with k_0 fixed,

$$\nu_n\left(\left|\|b_{k_0}\| - M_{n,q}\right| > r\right) = p_n\left(\left|\|b\|_{L^q} - M_{n,q}\right| > r\right)\ .$$

Finally, we use Theorem 3.5.1 to get

$$\nu_n\left(\left|\|b_{k_0}\| - M_{n,q}\right| > r\right) \leq 2e^{-(N_n-1)\frac{r^2}{2Cn^{2-4/q}}}\ .$$

Since $\frac{N_n-1}{n^{2-4/q}} \geq Cn^{4/q}$ and by summing over k_0, we get the result, that is

$$\nu_n\left(\{(b_k)_{1\leq k\leq N_n} \mid \exists k_0\ ,\ \left|\|b_{k_0}\|_{L^q} - M_{n,q}\right| > r\}\right) \leq 2N_n e^{-c_1 n^{4/q} r^2}\ .$$

◇

Let us now estimate $M_{n,q}$.

Lemma 3.5.6. *For all $t \in \mathbb{R}_+$, we have*

$$p_n(|x_1| > t) \leq 2e^{-(N_n-1)\frac{t^2}{2}}\ .$$

Proof We use the fact that the projection on the first coefficient of one vector is Lipschitz continuous on the sphere S_n with Lipschitz constant equal to 1, and that as p_n is uniformly distributed on S_n, the median of x_1 is 0, hence, applying Theorem 3.5.1 :

$$p_n(|x_1| > t) \leq 2e^{-(N_n-1)\frac{t^2}{2}}\ .$$

◇

Proposition 3.5.7. *There exists C such that for all n, q,*

$$M_{n,q} \leq C\sqrt{q}\ .$$

Proof Let us compute the mean value with respect to p_n of $\|.\|_{L^q}^q$. Let

$$A_{n,q}^q = E(\|.\|_{L^q}^q) = \int_{S_n} \Big(\int_{S^3} |u(x)|^q dx \Big) dp_n(u) \ .$$

We can reverse the order of the integrals :

$$A_{n,q}^q = \int_{S^3} \Big(\int_{S_n} |u(x)|^p dp_n(u) \Big) dx = \int_{S^3} \int_{R_+} q\lambda^{q-1} p_n(|u(x)| > \lambda) d\lambda dx \ .$$

With our particular basis $(f_{n,k})_k$, we get that u is written

$$u = \sum_k a_k f_{n,k}$$

hence with $K_n(x) = \sqrt{\sum_k |f_{n,k}(x)|^2}$ and $\epsilon(x)$ the unit vector :

$$\epsilon_k(x) = \frac{f_{n,k}(x)}{K_n(x)} \ ,$$

we get

$$p_n(|u(x)| > \lambda) = p_n\Big(|\langle a, \epsilon(x) \rangle| > \frac{\lambda}{K_n(x)} \Big)$$

and since p_n is invariant by the action of $O(N_n)$, $\epsilon(x)$ can be replaced by $(1, 0, \ldots, 0)$:

$$p_n(|u(x)| > \lambda) = p_n(|a_1| > \frac{\lambda}{K_n(x)}) \leq 2e^{-(N_n - 1)\frac{\lambda^2}{2K_n(x)^2}} \ .$$

We already proved that $K_n(x) = Cn \leq C\sqrt{N_n - 1}$, hence

$$A_{n,q}^q \leq \int_{S^3} \Big(\int_{\mathbb{R}_+} q\lambda^{q-1} 2e^{-\frac{\lambda^2}{2}} d\lambda \Big) dx$$

As we have by induction on q

$$\int_{\mathbb{R}^+} q\lambda^{q-1} e^{-\lambda^2/2} \leq Cq^{q/2} \ ,$$

we get that

$$A_{n,q} \leq C\sqrt{q}$$

with C independent from n and q.

To bound $M_{n,q}$, we use the definition of the median :

$$\frac{1}{2} \leq p_n(\|u\|_{L^q} \geq M_{n,q}) = p_n(\|u\|_{L^q}^q \geq M_{n,q}^q)$$

and then using Markov's inequality,

$$\frac{1}{2} \le M_{n,q}^{-q} A_{n,q}^{q} \ .$$

We deduce from that that

$$M_{n,q} \le 2^{1/q} A_{n,q} \le C \sqrt{q} \ ,$$

which concludes the proof. ◊

We know prove the existence of a sequence p_m that goes to ∞ such that there exists an orthonormal basis $(e_{n,k})_{n,k}$ such that $e_{n,k}$ belongs to E_n and

$$||e_{n,k}||_{L^{p_m}} \le C \sqrt{p_m}$$

where C is independent from n, k and m.

We have that for some constant C independent from n and p, the set

$$B_{n,p} = \{(e_{n,k})_{1 \le k \le (n+1)^2} \in U_n \mid \forall k ||e_{n,k}||_{L^p} \le C \sqrt{p}\}$$

satisfies that

$$\nu_n(B_{n,p}^c) \le c_0 n^2 e^{-c_1' C^2 n^{4/p} p}$$

where $B_{n,p}^c$ is the complementary set of $B_{n,p}$ by taking $\Lambda = C \sqrt{p}$ in the Proposition 3.5.5. By taking the product measure ν of the ν_n and with

$$B_p = \prod_n B_{n,p}$$

we get

$$\nu(B_p^c) \le c_0 \sum_{n \ge 1} n^2 e^{-c_1' C^2 n^{4/p} p} \le c_0 e^{-c_1' C^2 p} \sum_{n \ge 1} n^{2 - 4 c_1' C^2} \ .$$

Hence, for C large enough, we have for all p :

$$\nu(B_p^c) \le \frac{1}{2} \ .$$

By Fatou lemma,

$$\nu(\limsup_{p \to \infty} B_p) \ge \frac{1}{2}$$

Therefore, the set $\limsup_{p \to \infty} B_p$ is not empty. This is equivalent to the existence of a sequence $p_m \to \infty$ and a basis $e_{n,k}$ of spherical harmonics such that

$$||e_{n,k}||_{L^{p_m}} \le C \sqrt{p_m} \ . \tag{3.7}$$

Chapitre 4

Invariant measure for the cubic wave equation on the unit ball of R^3

Ce chapitre est issu de l'article [26].

4.1 Introduction

The goal here is to prove the invariance of the measure ρ constructed in [16], under the flow of the cubic non linear wave equation on the unit ball of $\mathbb{R}^3$, hence answering the remark 6.3 of the paper by Nicolas Burq and Nikolay Tzvetkov.

The equation studied is :

$$\left\{ \begin{array}{ll} \partial_t^2 f - \Delta_{B^3} f + f^3 = 0 & (t,x) \in \mathbb{R} \times B^3 \\ f|_{t=0} = f_0 & \partial_t f|_{t=0} = f_1 \end{array} \right. , \tag{4.1}$$

where f is real, radial, B^3 is the unit ball in $\mathbb{R}^3$, and Δ_{B^3} is the Laplace-Beltrami operator on B^3 with Dirichlet boundary conditions. Though, it is soon to be changed into its complex form, that is, writing $H = \sqrt{-\Delta_{B^3}}$ and $u = f - iH^{-1}\partial_t f$,

$$\left\{ \begin{array}{l} i\partial_t u + Hu + H^{-1}(\mathrm{Re} u)^3 = 0 \\ u|_{t=0} = u_0 = f_0 - iH^{-1} f_1 \end{array} \right. . \tag{4.2}$$

In order to define the measure invariant under the flow of (4.2), a sequence of independent complex centred and normalised (law $\mathcal{N}(0,1)$) $(g_n)_n$ is introduced, along with the measure μ, which is the image measure of the (well-defined) map from a probability space to the Sobolev space H^σ, $\sigma < \frac{1}{2}$:

$$\varphi(\omega, r) = \sum_{n=1}^{\infty} \frac{g_n}{\pi n} e_n$$

where e_n are the radial eigenfunctions of Δ_{B^3} with eigenvalues $\pi^2 n^2$. It makes μ a sort of limit of Gaussian on $\mathbb{R}^N$ when N goes to infinity.

The measure ρ is then defined as :

$$d\rho(u) = e^{-\frac{1}{4}\|u\|^4_{L^4(B^3)}} d\mu(u)$$

absolutely continuous with respect to μ. It has been proved that ρ is genuine, the norm $\| \ . \ \|_{L^4(B^3)}$ being μ - almost surely finite.

In [16], Burq and Tzvetkov prove the following theorem :

Theorem 3 (Burq,Tzvetkov). *Let $\sigma < \frac{1}{2}$. There exists a set $\Sigma \subseteq H^\sigma$ of full μ or ρ (which is equivalent) measure such that for any initial data u_0, the flow is globally well-defined and what is more the solution of* (4.2) *is unique in $S(t)u_0 + C(\mathbb{R}, H^s)$ where $S(t)$ is the flow of the linear equation $i\partial_t u + Hu = 0$ and s is some real number $s > \frac{1}{2}$.*

Using the ideas of the proof of this theorem and the local property of the solution, the following theorem will be proved.

Theorem 4. *There exists a set $\Pi \subseteq H^\sigma$ of full ρ measure such that the solution of* (4.2) *is strongly globally well-defined for any initial data taken in Π and that all measurable sets A included in Π satisfies at all times t :*

$$\rho(\psi(t)A) = \rho(A) \ .$$

Before going further, it has to be understood that ρ is built to be invariant under the flow of the non linear wave equation. Actually, by applying a cut-off on the frequency of the Lapace-Beltrami operator with radial symmetry and Dirichlet boundary conditions of the unit ball in $\mathbb{R}^3$, the NLW is approached by PDE in finite dimension, susceptible to finite dimension theory, like Cauchy-Lipschitz theorem. Indeed, call E_N the space linearly spanned by N first eigenfunctions of the radial Laplace-Beltrami operator on the unit ball of $\mathbb{R}^3$ with Dirichlet boundary conditions, by using projectors on E_N, or, better to say, operators that send H^s into E_N which are more regular than mere orthogonal projectors, the non linear wave equation can be reduced onto a problem on E_N, which admits a unique maximal condition thanks to Cauchy-Lipschitz theorem that can be proved to be a global one thanks to the existence of a conserved positive energy. The reduction is chosen such that the solutions converges in the space of distributions towards a solution of the non linear wave equation.

Then, finding a measure ρ_N on E_N which is invariant under the flow of this equation relies mostly on the existence of a conserved energy and Liouville theorem. It happens that the sequence ρ_N extended to H^σ converges towards a non trivial measure ρ on H^σ. The measure ρ being a limit of ρ_N, it is expected to be invariant under the flow of NLW.

In previous works, like [8, 7, 9, 54], the strategy applied to prove the invariance of the measure in infinite dimension used the fact that the initial datum was taken in spaces $\mathcal{H}$ such that the orthogonal projectors Π_N on E_N were uniformly bounded, that is to say, there exists C independent from N, such that $\|\Pi_N\|_{\mathcal{H}\to\mathcal{H}} \leq C$. So the projections $\Pi_N u$ converged toward u in $\mathcal{H}$ uniformly in every compact subset of $\mathcal{H}$. In these cases, by approaching the the initial data in $\mathcal{H}$ by its projections, the flow with initial data u_0 could be approached by the finite dimensional flow of $\Pi_N u_0$, the convergence being uniform on any compact set of initial data in $\mathcal{H}$. However, here, the control that ensures the existence of global strong solution in [16] is the norm :

$$\|S(t)u_0\|_{L^p_{t\in[0,2],x\in B^3}} ,$$

and thus the convergence of $\Pi_N u_0$ in H^σ does not ensure the convergence of this norm applied to $\Pi_N u_0$ and thus not the uniform (regarding the initial data) convergence of finite dimensional solutions towards the global solution.

This problem can be solved though by introducing slightly different "finite" measures and dimensional equations. Instead of entirely reducing the problem to a problem on E_N, only its non linear part will, that is to say, the initial data will be taken in H^σ but the non linear part will be projected on E_N. Therefore, the reduced problem will present two parts : a linear and of infinite dimension one and a finite dimensional though non linear one. Then, the reduced measure ρ_N, instead of being defined on E_N will be defined on all H^σ (and still invariant under the flow). The initial data, thus, will not have to be approached, only the flow will, like in [13]. Unlike in [13] though, considerations on the reversibility of the flow will be confined to the linear treatment. Indeed, the flow of the linear equation is defined on all H^σ which makes it easier to manipulate. The afore-mentioned strategy using the uniform bound of the projectors will be used to prove the invariance of μ under the linear flow. Nevertheless, the flow of the NLW being defined only on a subset Σ of H^s, this subset has to be invariant under the flow if the reversibility of the flow must be used.

To sum up, [15, 16] will provide the topological framework and the local results of existence for the non linear wave equation, [13] the descriptions of the "new" partly finite dimensional measures, [54] a guideline to prove the invariance of the measure μ under the linear flow, [2] the main ideas and properties about random Gaussian series, and thanks to all these results, the invariance of ρ shall be proved.

Plan of the paper. The first part is a reminder of the results of [15, 16, 54, 13] rewritten in a slightly different form in order to fit with the framework. The results of [15, 16] are stated at the beginning to display the theorems that compose the starting point. Then, the approximation of the non linear wave equation by finite dimensional problems is detailed. Finally, the first part of the proof of Theorem 4 is given, that is, the construction of the measures μ and ρ and the invariance of μ under the linear flow.

The second part is mainly analytical, it deals with the local properties of the flow. First, the local existence of the flow is derived from [15, 16], then a result of local (in time) uniform (for the initial data) convergence of the approached flow towards the local flow of NLW is given, which leads to a result of local invariance of ρ under the local flow.

The last one is dedicated to the extension of the local solution to a global one when the initial data is taken in Π and then of the extension of the local invariance result to a global one.

4.2 Existence of solution for the cubic NLW

4.2.1 Statement of the main results

In [15, 16], Nicolas Burq and Nikolay Tzvetkov have proved that there existed a large subset of H^σ with $\sigma < \frac{1}{2}$ that could be taken as initial data for the 3D-non linear wave equation :

$$(\partial_t^2 - \Delta)u + u^3 = 0 \tag{4.3}$$

with u a radial function and Δ the Laplace-Beltrami operator on the unit ball of $\mathbb{R}^3$.

The first paper shows the existence of local solution using a randomization of the initial data.

The randomization is given by :

Definition 4.2.1. Let $s \geq \frac{8}{21}$, $f = (f_1, f_2) \in H^s \times H^{s-1}$ and α_n, β_n the sequences defined as :

$$f_1 = \sum_n \alpha_n e_n \ , \ f_2 = \sum_n \beta_n e_n$$

with e_n the eigenfunctions of Δ on the unit ball with Dirichlet or Neumann conditions.

Then, let h_n, l_n be sequences of real centred Gaussian variables, independent from each other on a probability space Ω, P. Set :

$$f^\omega = (f_1^\omega, f_2^\omega)$$

with

$$f_1^\omega = \sum_n h_n(\omega)\alpha_n e_n \ , \ f_2^\omega = \sum_n l_n(\omega)\beta_n e_n \ .$$

Then, a local solution exists :

Theorem 4.2.2. *Assume $s \geq \frac{8}{21}$ and $f \in H^s \times H^{s-1}$. Set f^ω defined according to the previous randomization. There exists a regularity parameter $\sigma \geq \frac{1}{2}$ such that for almost all $\omega \in \Omega$, there is a time $T_\omega > 0$ such that there is a unique solution to* (4.3) *in*

$$\cos(\sqrt{-\Delta}t)f_1^\omega + \frac{\sin(\sqrt{-\Delta}t)}{\sqrt{-\Delta}} f_2^\omega + C([-T_\omega, T_\omega], H^\sigma) \ .$$

The second one is dedicated to the global extension of these solutions (with Dirichlet boundary conditions). It states :

Theorem 4.2.3. *Fix $p \in]4, 6[$. Let f_1^ω and f_2^ω be :*

$$f_1^\omega = \sum_n \frac{h_n(\omega)}{n\pi} e_n \ , \ f_2^\omega = \sum_n l_n(\omega) e_n \ .$$

Then, for all $s < \frac{1}{2}$ and almost all $\omega \in \Omega$, the problem (4.3) *has a unique global solution in*

$$C(\mathbb{R}_t, H^s) \cap L^p_{loc}(\mathbb{R}_t, L^p) .$$

To prove this theorem, a cut-off on the frequencies of the Laplace-Beltrami operator is used. The idea is to solve the equation on finite dimensional functional spaces, spanned by the N first eigenfunctions of the Laplacian. Then, by taking limits of the finite dimensional solutions, a subset of Ω of full measure appears into which the norms of the local solutions with initial data of the form $f_0(\omega, \ . \ ), f_1(\omega, \ . \ )$ are controlled as the limits of finite dimensional solutions whose norms are themselves controlled. Therefore, this set of full measure provides a set of functions such that the local solution can be extended. The way these sets of control at finite times are built will inspire the construction of other sets onto which not the flow is strongly globally defined but also onto which the measure that we will define is invariant under this flow.

Before going further, the way the problem is reduced to a finite dimensional one will be described, as the definitions involved shall prove themselves useful for the sequel.

Definition 4.2.4. Let χ be a C^∞_c function with support included in $[-1, 1]$ and satisfying

$$\chi \equiv 1$$

on $[\frac{-1}{2}, \frac{1}{2}]$. Then, for all $N \in \mathbb{N}$ we call S_N the operator $\chi(\frac{-\Delta}{N^2})$ that is to say the operator that maps

$$\sum_n c_n e_n$$

to

$$\sum_n c_n \chi(\frac{n^2}{N^2}) e_n \ .$$

The set linearly spanned by $\{e_n \mid n \leq N\}$ is now called E_N and Π_N is the orthogonal projection on E_N.

Proposition 4.2.5. *The operators S_N are uniformly continuous from L^p to E_N normed by L^p, that is to say that there exists a constant C independent from N such that for all $f \in L^p$,*

$$\|S_N f\|_{L^p} \leq C\|f\|_{L^p} \ .$$

Also, for all $f \in L^p$, the sequence $(S_N f)_N$ converges towards f in L^p.

The proof of this proposition can be found in [11].

The reduced problem in finite dimension becomes :

$$\left\{ \begin{array}{l} i\partial_t u + (-\Delta)^{-1/2} u + S_N((S_N \mathrm{Re} u)^3) = 0 \\ u|_{t=0} = u_0 = f_1 + i(-\Delta)^{-1/2} f_1 \end{array} \right. \ . \tag{4.4}$$

This should be explained in the next subsection.

4.2.2 Approximation of the flow by finite dimensional problems

First, one should see how the equation (4.4) is derived from the non linear wave equation on the unit ball.

Conserved quantities The initial equation is :

$$\begin{cases} \partial_t^2 f - \Delta_{B^3} f + f^3 = 0 & t, r \in \mathbb{R} \times B^3 \\ f|_{t=0} = f_0 & (\partial_t f)|_{t=0} = f_1 \end{cases} \tag{4.5}$$

where B^3 is the unit ball of $\mathbb{R}^3$ and Δ_{B^3} is the Laplace-Beltrami operator on B^3 with Dirichlet boundary conditions.

Now, by setting $H = \sqrt{-\Delta_{B^3}}$, $u_0 = f_0 - iH^{-1} f_1$ and $u = f - iH^{-1}\partial_t f$, u satisfies :

$$\begin{cases} i\partial_t u + Hu + H^{-1}(\mathrm{Re} u)^3 = 0 \\ u|_{t=0} = u_0 \end{cases} \tag{4.6}$$

Proposition 4.2.6. *The equation* (4.6) *is a Hamiltonian equation with energy :*

$$\mathcal{E}(u) = \frac{1}{2}\int_{B^3} |Hu|^2(r) r^2 dr + \frac{1}{4}\int_{B^3} |Reu|^4 r^2 dr \ .$$

The operator S_N is then introduced in order to reduce the problem into an almost finite dimensional one.

Definition 4.2.7. Set $\mathcal{E}_N$ the quantity :

$$\mathcal{E}_N(u) = \frac{1}{2}\int_{B^3} |Hu|^2 r^2 dr + \frac{1}{4}|S_N \mathrm{Re} u|^4 r^2 dr \ .$$

This quantity is the hamiltonian of the equation

$$i\partial_t u + Hu + S_N(H^{-1}(S_N \mathrm{Re} u)^3) = 0 \ .$$

Proposition 4.2.8. *Set $u_0 \in H^\sigma$ with $\sigma < \frac{1}{2}$ and $S(t) = e^{iHt}$ the flow of the linear equation $i\partial_t u + Hu = 0$ and consider the equation :*

$$\begin{cases} i\partial_t v + Hv + S_N H^{-1}((S_N(S(t)u_0 + v))^3) \\ v|_{t=0} = 0 \end{cases} \ . \tag{4.7}$$

There exists a global strong solution called v_N.
Furthermore, $u_N = S(t)u_0 + v_N$ satisfies

$$i\partial_t u_N + Hu_N + S_N(H^{-1}(S_N Reu_N)^3) = 0 \tag{4.8}$$

with initial data u_0. The flow of this equation is written $u_N(t) = \psi_N(t)(u_0)$.

The equation (on v) $i\partial_t v + Hv + S_N H^{-1}((S_N(S(t)u_0 + v))^3)$ is on E_N and thus the Cauchy-Lipschitz theorem holds and shows that it admits a local unique solution for any initial condition v_0, and with u_0 fixed. Then, the quantity $\mathcal{E}_N(\Pi_N S(t)u_0 + v)$ does not depend on time and controls v, which implies that the local solution does not explode and therefore the solution v_N is global.

4.2.3 Building invariant measures

Now, invariant measures under the flows ψ_N are built. First, call $e_n(r) = \frac{\sin n\pi r}{\sqrt{\pi} r}$ the eigenfunctions of the Laplacian with Dirichlet boundary conditions. Then, let Ω, P be a probability space and $(g_n)_n$ a sequence of independent centred and normalized Gaussian variables. Set :

$$\varphi_N(\omega, r) = \sum_{n=1}^{N} \frac{g_n(\omega)}{n\pi} e_n(r) \; .$$

The image measure μ_N of φ_N on E_N is absolutely continuous with respect to the Lebesgue measure on E_N and :

$$d\mu_N(\sum_{n=1}^{N}(a_n + ib_n e_n)) = d_N \prod_{n=1}^{N} e^{-(n\pi)^2(a_n^2+b_n^2)/2} da_n db_n$$

$$= d_N e^{-\int_{B^3} |H\sum(a_n+ib_n)e_n|^2} \prod_{n=1}^{N} da_n db_n$$

where d_N is a normalization factor.

Thanks to this point of view, it appears that μ_N is invariant under the flow $S(t)$ on E_N. Indeed, by Liouville theorem, the Lebesgue measure on E_N is invariant under the flow and the quantity $\frac{1}{2}\int |Hu|^2$ is invariant under $S(t)$.

Furthermore, the sequence φ_N converges in L^2_Ω, H^σ_r for all $\sigma < \frac{1}{2}$. Denote its limit by φ and call μ the image measure on H^σ of $\omega \mapsto \varphi(\omega, .)$.

Also, considering the measure $\mu_N^\perp$ on $E_N^\perp$ (the orthogonal being taken in H^σ) such that

$$\mu = \mu_N \otimes \mu_N^\perp \; ,$$

it comes that on $E_N^\perp$, $\mu_N^\perp$ is the image measure of

$$\varphi^N : \omega \mapsto \sum_{n=N+1}^{\infty} \frac{g_n(\omega)}{n\pi} e_n \; .$$

Lemma 4.2.9. *Let U be an open (for the trace topology of H^σ on $E_M^\perp$) set of $E_M^\perp$ and call μ_N^M the image measure of*

$$\varphi_N^M : \omega \mapsto \sum_{n=M+1}^{N} \frac{g_n(\omega)}{n\pi} e_n$$

on E_N^M the space linearly spanned by $\{e_{M+1}, \dots, e_N\}$, such that $\mu_M^\perp = \mu_M^N \otimes \mu_N^\perp$. It comes,

$$\mu_M^\perp(U) \leq \liminf_{N\to\infty} \mu_N^M(U \cap E_N^M) \ .$$

In particular, for $M = 0$, this leads to, for all open set U of H^σ,

$$\mu(U) \leq \liminf_{N\to\infty} \mu_N(U \cap E_N) \ .$$

Proof Let $\sigma < \sigma_1 < \frac{1}{2}$. Let A be the set of Ω, $A = (\varphi^M)^{-1}(U)$ and $A_N = (\varphi_N^M)^{-1}(U \cap E_N^M)$.

If A is empty, then $\mu_M^\perp(U) = 0 = \mu_N^M(U \cap E_N^M)$.

If not, let $\omega \in A$. Since U is an open set, there exists a ball of radius $\epsilon > 0$ such that $\varphi^M(\omega) + B_\epsilon \cap E_M^\perp \subseteq U$. Also,

$$\|\varphi_N^M(\omega) - \varphi^M(\omega)\|_{H^\sigma} \leq N^{\sigma-\sigma_1} \|\varphi(\omega)\|_{H^{\sigma_1}} .$$

The norm $\|\varphi\|_{L^2_\omega, H^{\sigma_1}}$ being finite, for almost all ω, the $\|\varphi(\omega)\|_{H^{\sigma_1}}$ is finite. So, for almost all $\omega \in A$, there exists $N_0 \geq 0$ such that for all $N \geq N_0$, $\varphi_N^M(\omega) \in \varphi^M(\omega) + B_\epsilon \cap E_M^\perp \subseteq U$, as $\varphi_N^M - \varphi^M(\omega) \in E_M^\perp$, that is there exists N_0 such that for all $N \geq N_0$

$$\omega \in A_N \ \text{ that is to say } \omega \in \liminf A_N \ .$$

So, $A \subseteq \liminf A_N$ with the possible exception of a negligible set. As $\mu_M^\perp$ is the image measure of φ^M,

$$\mu_M^\perp(U) = P((\varphi^M)^{-1}(U)) = P(A)$$

and as A is almost surely included in $\liminf A_n$

$$\mu_M^\perp(U) \leq P(\liminf A_N) \ .$$

Thanks to Fatou's lemma

$$\mu_M^\perp(U) \leq \liminf P(A_N) = \liminf \mu_N^M(U \cap E_N) \ .$$

◊

Remark 4.2.1. *For all closed sets F of $E_M^\perp$,*

$$\mu_M^\perp(F) \geq \limsup \mu_N^M(F \cap E_N^M) \ .$$

Proposition 4.2.10. *The measures $\mu_M^\perp$ are invariant under the flow $S(t)|_{E_M^\perp}$. Therefore, with $M = 0$, μ is invariant under $S(t)$.*

Proof Let F be a closed set of $E_M^\perp$ and for all $\epsilon > 0$, call $B_\epsilon^M = B_\epsilon \cap E_M^\perp$ with B_ϵ the open ball of H^σ of radius ϵ. For all $t \in \mathbb{R}$, $S(t)$ is a linear isometry of H^σ and $E_M^\perp$ is invariant under $S(t)$. Thus, as $F + \overline{B_\epsilon^M}$ is a closed set of $E_M^\perp$, $S(t)F + \overline{B_\epsilon^M} = S(t)(F + \overline{B_\epsilon^M})$ is also closed :

$$\mu_M^\perp(S(t)F + \overline{B_\epsilon^M}) = \mu(S(t)(F + \overline{B_\epsilon^M})) \geq \limsup \mu_N^M(S(t)(F + \overline{B_\epsilon}) \cap E_N^M)$$

and as $S(t)A \cap E_N^M = S(t)(A \cap E_N^M)$,

$$\mu_M^\perp(S(t)F + \overline{B_\epsilon^M}) \geq \limsup \mu_N^M(S(t)(F + \overline{B_\epsilon^M} \cap E_N^M)) .$$

Then, μ_N^M is invariant under the flow $S(t)|_{E_N^M}$ for the same reasons as μ_N, so

$$\mu_M^\perp(S(t)F + \overline{B_\epsilon^M}) \geq \limsup \mu_N^M(F + \overline{B_\epsilon^M} \cap E_N^M) \geq \liminf \mu_N^M(F + \overline{B_\epsilon^M} \cap E_N^M)$$

$$\mu_M^\perp(S(t)F + \overline{B_\epsilon^M}) \geq \liminf \mu_N^M(F + B_\epsilon^M \cap E_N^M) .$$

As $F + B_\epsilon^M$ is open in $E_M^\perp$,

$$\mu_M^\perp(S(t)F + \overline{B_\epsilon^M}) \geq \mu_M^\perp(F + B_\epsilon^M) \geq \mu_M^\perp(F)$$

and by the dominated convergence theorem when $\epsilon \to 0$,

$$\mu_M^\perp(S(t)F) \geq \mu_M^\perp(F) .$$

The linear equation is reversible on all $E_M^\perp$ and $S(t)F$ is closed so,

$$\mu_M^\perp(F) = \mu_M^\perp(S(-t)S(t)F) \geq \mu_M^\perp(S(t)F)$$

which gives

$$\mu_M^\perp(F) = \mu_M^\perp(S(t)F)$$

for all time t and all closed set F.

Then, again because $S(t)$ is an isometry on $E_M^\perp$ and thus preserves the topology, this equality is stable under the passage to the complementary and to denumerable union. Therefore, this property is true for all measurable sets A and all time t. ◊

As the quantity $\frac{1}{4}\int_{B^3} |S_N \mathrm{Re} u|^4$ is μ almost surely finite (see [15, 16]) the measure

$$d\rho_N(u) = e^{-\frac{1}{4}\int_{B^3} |S_N \mathrm{Re} u|^4} d\mu(u)$$

is well-defined on all H^σ.

Proposition 4.2.11. *The measure ρ_N is invariant under the flow $\psi_N(t) : H^\sigma \to H^\sigma$ of the equation* (4.8).

Proof Consider a measurable set A of initial data u_0. For each u_0 in A, we can write :

$$u_0 = \Pi_N u_0 + \Pi_N^\perp u_0$$

where $\Pi_N^\perp$ is the orthonormal projector (in H^σ) on $E_N^\perp$. It suffices to consider A of product type, that is of the type :

$$A = \{u_0 \mid \Pi_N u_0 \in B\ ,\ \Pi_N^\perp u_0 \in C\}$$

with B and C measurable sets of respectively E_N and $E_N^\perp$ since the topology (and so the measurable sets) of H^σ is the same as the one of the Cartesian product $E_N \times E_N^\perp$.

Therefore,

$$\psi_N(t)u_0 = S(t)\Pi_N u_0 + S(t)\Pi_N^\perp u_0 + v(t) = \psi_N|_{E_N}(t)(\Pi_N u_0) + S(t)|_{E_N^\perp}\Pi_N^\perp u_0$$

and thus

$$\psi_N(t)(A) = \psi_N|_{E_N}(t)(B) \times S(t)_{E_N^\perp}(C)\ .$$

So, the invariance of ρ_N under ψ_N is reduced to the invariance of $\mu_N^\perp$ under $S(t)$ and the invariance of $e^{-\frac{1}{4}\int_{B^3}|S_N \mathrm{Re} u|^4} d\mu_N(u)$ (on E_N) under $\psi_N|_{E_N}$. The first invariance has already been dealt with. For the second one, all $B \subseteq E_N$ measurable satisfies :

$$\begin{aligned}\int_{\psi_N|_{E_N}(t)(B)} e^{-\frac{1}{4}\int_{B^3}|S_N \mathrm{Re} u|^4} d\mu_N(u) &= \int_{\psi_N|_{E_N}(t)(B)} e^{-\frac{1}{2}\int_{B^3}|Hu|^2 - \frac{1}{4}\int_{B^3}|S_N \mathrm{Re} u|^4} dL_N(u) \\ &= \int_{\psi_N|_{E_N}(t)(B)} e^{-\mathcal{E}_N(u)} dL_N(u)\end{aligned}$$

where L_N is the Lebesgue measure on E_N. By Liouville theorem, L_N is invariant under $\psi_N|_{E_N}$, therefore the following change of variable $u = \psi_N|_{E_N}(t)(w)$ holds :

$$\int_{\psi_N|_{E_N}(t)(B)} e^{-\frac{1}{4}\int_{B^3}|S_N \mathrm{Re} u|^4} d\mu_N(u) = \int_B e^{\mathcal{E}_N(\psi_N|_{E_N}(t)(w))} dL_N(w)\ .$$

Then, remarking that on E_N, $\mathcal{E}_N(\psi_N|_{E_N}(t)(w))$ can be derived over t and is equal to $\mathcal{E}_N(w)$, the measure is invariant and so, ρ_N is invariant under ψ_N.

◊

Definition 4.2.12. Let f_N and f be the application on H^σ defined as :

$$f_N(u) = e^{-\frac{1}{4}\int_{B^3}|S_N \mathrm{Re} u|^4} \text{ and } f(u) = e^{-\frac{1}{4}\int_{B^3}|\mathrm{Re} u|^4}\ .$$

The following statement comes from the analysis of [13].

Proposition 4.2.13. *The quantity*

$$\frac{1}{4}\int_{B^3} |Reu|^4$$

is finite for μ-almost all $u \in H^\sigma$.

Besides, f_N converges towards f in L^1_μ norm.

Therefore, the measure ρ can be introduced as :

Proposition 4.2.14. *The measure ρ such that :*

$$d\rho(u) = f(u)d\mu(u)$$

is well defined and non trivial. And for all A measurable,

$$\rho(A) = \lim_{N\to\infty} \rho_N(A) \ .$$

The proof of the convergence is very similar to the one in the case of the defocusing NLS, as can be found in [13], and then :

$$\rho(A) = \int_A f(u)d\mu(u)$$

$$|\rho(A) - \rho_N(A)| \leq \int_A |f(u) - f_N(u)|d\mu(u) \leq \|f - f_N\|_{L^1_\omega} \ .$$

The fact that there exists a set of full ρ measure onto which the flow of (4.6) is well-defined has been proved in [16].

Now, the fact that the measure ρ is invariant under the flow shall be seen.

4.3 Uniform convergence of the approached flows

4.3.1 Toolbox

Sobolev embedding For a start, here is the fundamental Sobolev embedding theorem on $\mathbb{R}^n$.

Theorem 4.3.1. *Let $n \in \mathbb{N}$ and $s \in \mathbb{R}$. Set $p \in [2,\infty[$ such that $\frac{1}{2} = \frac{1}{p} + \frac{s}{n}$. The functional space $H^s(\mathbb{R}^n)$ is continuously embedded into $L^p(\mathbb{R}^n)$. That is to say, there exists a constant $C(s)$ such that for all $f \in H^s(\mathbb{R}^n)$,*

$$\|f\|_{L^p} \leq C\|f\|_{H^s} \ .$$

Remark 4.3.1. *By considering f radial and with compact support on B^3 the unit ball in dimension 3, as a particular case of the previous theorem for all f radial and with compact support on B^3 and in $H^s(\mathbb{R}^3)$, that is to say for all $f \in \mathcal{H}^s$, it comes :*

$$\|f\|_{L^p(B^3)} \leq C\|f\|_{H^s(B^3)}$$

as long as $\frac{1}{2} = \frac{1}{p} + \frac{s}{3}$.

The proof of Sobolev embedding theorem can be found in [1].

Deep into the local existence of solution for the cubic NLW The goal here is to show that on certain sets, the flows ψ_N converges uniformly towards ψ. So, first, deterministic Strichartz estimates and needed properties of the flow are described.

Definition 4.3.2. Let $p > 2$, q such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$, $T > 0$ and $s = \frac{2}{p}$. Call

$$X_T^s = C^0([-T,T], H^s(B^3)) \cap L^p((-T,T), L^q(B^3))$$

where B^3 is the unit ball in $\mathbb{R}^3$ and

$$Y_T^s = L^1([-T,T], H^{-s}(B^3)) + L^{p'}((-T,T), L^{q'}(B^3))$$

its dual where p' and q' are the conjugate numbers of p and q.

Proposition 4.3.3. *Let $p \in]4,6[$ and $s = \frac{3}{2} - \frac{4}{p}$. There exists a constant C such that for all $T \in]0,1]$ and all f,*

$$\|f\|_{L^p([0,T],L^p(B^3))} \leq C\|f\|_{X_T^s} \text{ and } \|f\|_{Y_T^s} \leq C\|f\|_{L^{p'}([-T,T]\times B^3)} .$$

The proof comes from a particular case of interpolation between the two functional spaces described in the definition of X_T^s.

Thanks to a combination of Sobolev embedding theorem and Strichartz inequality (see [33] for further details), the following property holds :

Proposition 4.3.4. *Let $p \in]4,6[$ and s defined as $s = \frac{3}{2} - \frac{4}{p}$, there exists $C \leq 0$ such that for all $T \in [0,1]$ and all $f \in H^s$:*

$$\|S(t)f\|_{L^p([-T,T]\times B^3)} \leq C\|f\|_{H^s} .$$

Proposition 4.3.5. *Let $p \in]4,6[$, $p_1 \in]4,6[$ such that $p_1 > p$, $s = \frac{3}{2} - \frac{4}{p}$ and $s_1 = \frac{3}{2} - \frac{4}{p_1} > s$. There exists C such that for all $T \in]0,1]$ and all f,*

$$\int_0^t S(t-u)H^{-1}f(u)du\|_{X_T^s} \leq C\|f\|_{Y_T^{1-s}}$$

$$\|(1-S_N)\int_0^t H^{-1}S(t-u)f(u)du\|_{X_T^s} \leq CN^{s-s_1}\|f\|_{Y_T^{1-s_1}} .$$

The proof can be found in [16]. The last crucial result needed from this article is the local existence theorem, and its implication regarding the X^s_T norms of the function $v(t) = \psi(t)u_0 - S(t)u_0$ where $\psi(t)$ would be defined as the flow of

$$i\partial_t u + Hu + H^{-1}(\mathrm{Re}u)^3 = 0$$

that is to say v is the solution of

$$\begin{cases} i\partial_t v + Hv + H^{-1}((\mathrm{Re}(S(t)u_0 + v(t)))^3) = 0 \\ v|_{t=0} = 0 \end{cases} . \tag{4.9}$$

Theorem 4.3.6. *Choose a real number $p \in]4,6[$ and define s as $s = \frac{3}{2} - \frac{4}{p}$. There exists $C > 0$, $c > 0$, $\gamma = 1 - \frac{4}{p}$ such that for any arbitrary large number A, there exists a time of existence $\tau \in]0,1]$ depending on A as $\tau = c(1+A)^{-\gamma}$ such that for all initial data u_0 satisfying $\|S(t)u_0\|_{L^p_{t,x}\in[0,2]\times B^3} \leq A$, there exist unique solutions of the equations* (4.7) *and* (4.9)*, v_N and v, belonging to X^S_τ and satisfying :*

$$\|v\|_{X^s_\tau}, \|v_N\|_{X^s_\tau} \leq CA\ .$$

Also, as $S(t)$ is 2 periodic (the eigenvalues of the Laplacian on B^3 with Dirichlet boundary conditions are of the form $(n\pi)^2$, $n \in \mathbb{N}^*$) and thanks to the Proposition 4.3.4, there exists another constant C' such that for each $t \in [-\tau,\tau]$:

$$\|S(t')(u(t))\|_{L^p_{t',x}}, \|S(t')(u_N(t))\|_{L^p_{t',x}} \leq \|S(2t')u_0\|_{L^p_{t',x}} + \|S(t')v(t)\|_{L^p_{t',x}} \leq C'A$$

and if $u_0 \in H^\sigma$, the solutions satisfies :

$$\|u(t)\|_{H^\sigma}, \|u_N(t)\|_{H^\sigma} \leq C'\|u_0\|_{H^\sigma}$$

with $u(t) = S(t)u_0 + v$ and $u_N(t) = S(t)u_0 + v_N$.

Remarks on sets' measurements The local results of existence will provide local properties of uniform convergence of the sequence of flows ψ_N toward the flow ψ and then induce properties on the invariance of the flow that will remain local. In order to extend those next to appear local properties into a global invariance of the flow, we will have to control the quantity denoted as A in the previous theorem. But this control has to satisfy certain properties, such as the set that describes the initial data that lead to a controlled solution must be of full measure.

To this purpose, consider the following proposition.

Proposition 4.3.7. *Let $\sigma < \frac{1}{2}$, let $p \in]4,6[$, let $D \geq 0$ and consider the sets :*

$$B(D)^c = \{u_0 \in H^\sigma \mid \|S(t)u_0\|_{L^p_{t,x}} > D\ \}$$

and

$$E(D)^c = \{u_0 \in H^\sigma \mid \|u_0\|_{H^\sigma} > D\} .$$

There exists $c > 0$ independent from D such that :

$$\rho(B(D)^c), \rho_N(B(D)^c) \leq \mu(B(D)^c) \leq e^{-cD^2}$$

and

$$\rho(E(D)^c), \rho_N(E(D)^c) \leq \mu(E(D)^c) \leq e^{-cD^2} .$$

The proof depends on the lemma 3.3 that can be found in [13].

Remark 4.3.2. *It will appear that the time $\tau_1 < \tau$ such that there is local convergence on $X^s_{\tau_1}$ depends on D. It will then be necessary to prove that τ_1 is big enough to control $u(t)$ at some finite times t_k with $k \in \mathbb{Z}$ cover all times, and still have a set of initial data of full ρ measure.*

4.3.2 Local uniform convergence

We now want to prove that the flows ψ and ψ_N are such that $\psi(t)u_0 - \psi_N(t)u_0$ converges in $X^s_{\tau_1}$ for some τ_1 uniformly in u_0.

Lemma 4.3.8. *Let $\sigma \in]0, \frac{1}{2}[$, $p \in]4, 6[$ and s defined as $s = \frac{3}{2} - \frac{4}{p}$. Fix $D \geq 0$ and consider $A(D)$ the set*

$$A(D) = \{u_0 \in H^\sigma \mid \|S(t)u_0\|_{L^p} \leq D \text{ and } \|u_0\|_{H^\sigma} \leq D\}.$$

There exists $c_1 > 0$ and $\gamma_1 > 0$ such that by fixing $\tau_1 = \min(c_1(1 + D)^{-\gamma_1}, \tau)$, where τ is the time provided by the Theorem 4.3.6 for all $\epsilon > 0$, there exists $N_0 \geq 0$ such that for all $u_0 \in A(D)$ and all $N \geq N_0$,

$$\|\psi(t)u_0 - \psi_N(t)u_0\|_{X^s_{\tau_1}} < \epsilon ,$$

that is to say that $\psi_N(t)u_0$ converges uniformly in $u_0 \in A(D)$ in $X^s_{\tau_1}$.

Proof Let $u_0 \in A(D)$ and v and v_N be such that

$$\psi(t)u_0 = S(t)u_0 + v(t) \text{ and } \psi_N(t)u_0 = S(t)u_0 + v_N(t) .$$

The functions v and v_N are both in X^s_τ so the norm

$$\|\psi(t)u_0 - \psi_N(t)u_0\|_{X^s_{\tau_1}} = \|v - v_N\|_{X^s_{\tau_1}}$$

is finite for all $\tau_1 \leq \tau$.

For all $t \leq \tau$:

$$v(t) - v_N(t) = \int_0^t S(t-s)H^{-1}\left((\mathrm{Re}\psi(s)u_0)^3 - S_N((S_N\mathrm{Re}\psi_N(s)u_0)^3)\right)ds$$

that is to say $v - v_N = I_N + II_N$ with

$$I_N = (1 - S_N)\int_0^t S(t-s)H^{-1}\left((\mathrm{Re}\psi(s)u_0)^3\right)ds$$

and

$$II_N = \int_0^t S(t-s)S_N H^{-1}\left((\mathrm{Re}\psi(s)u_0)^3 - (\mathrm{Re}S_N\psi_N(s)u_0)^3\right)ds\ .$$

Thanks to Proposition 4.3.5, for $T \le \tau$ and $s_1 > s$

$$\|I_N\|_{X_T^s} \le CN^{s-s_1}\|(\mathrm{Re}\psi u_0)^3\|_{Y_T^{1-s_1}}$$

with C independent from D, T, and N. Let p_1 be such that $1 - s_1 = \frac{3}{2} - \frac{4}{p_1}$. The condition $s_1 > s$ is equivalent to $p_1 < \frac{2p}{p-2}$. Hence :

$$\|I_N\|_{X_T^s} \le CN^{s-s_1}\|(\mathrm{Re}\psi u_0)^3\|_{L_{t,x}^{p_1'}}$$

$$\|I_N\|_{X_T^s} \le CN^{s-s_1}\|(\mathrm{Re}\psi u_0)\|^3_{L^{3p_1'}}\ .$$

To bound this norm, the condition $3p_1' \le p$ is wanted. This condition is equivalent to $p_1 \ge \frac{p}{p-3}$, which means p_1 has to be chosen in the interval $[\frac{p}{p-3}, \frac{2p}{p-2}[$. But since $p > 4$, $\frac{p}{p-3} < \frac{2p}{p-2}$, and so, such a choice is possible, and in particular, by choosing $p_1 = \frac{p}{p-3}$, or $3p_1' = p$, it comes :

$$\|I_N\|_{X_T^s} \le CN^{s-s_1}\left(\|S(t)u_0\|_{L^p} + \|v\|_{L^p}\right)^3$$

$$\|I_N\|_{X_T^s} \le CN^{s-s_1}\left(\|S(t)u_0\|_{L^p} + \|v\|_{X_T^s}\right)^3$$

and thanks to Theorem 4.3.6, $\|v\|_{X_T^s} \le CD$ with C independent from D and T as long as $T \le \tau$ so

$$\|I_N\|_{X_T^s} \le CN^{s-s_1}D^3\ .$$

Therefore, for all $\epsilon > 0$, there exists N_0 such that for all $u_0 \in A(D)$, all $T \le \tau$ and all $N \ge N_0$,

$$\|I_N\|_{X_T^s} \le \epsilon\ .$$

Once more, thanks to (4.3.5),

$$\|II_N\|_{X_T^s} \leq \| \left((\mathrm{Re}\psi(s)u_0)^3 - (\mathrm{Re}S_N\psi_N(s)u_0)^3 \right) \|_{Y_T^{1-s}}$$

and with p_2 such that $1 - s = \frac{3}{2} - \frac{4}{p_2}$ that is $p_2 = \frac{2p}{p-2}$,

$$\|II_N\|_{X_T^s} \leq \| \left((\mathrm{Re}\psi(s)u_0)^3 - (\mathrm{Re}S_N\psi_N(s)u_0)^3 \right) \|_{L^{p_2'}} .$$

Since

$$|(\mathrm{Re}\psi(s)u_0)^3 - (\mathrm{Re}S_N\psi_N(s)u_0)^3| \leq \frac{3}{2}|\mathrm{Re}\psi(s)u_0 - \mathrm{Re}S_N\psi_N(s)u_0| \left((\mathrm{Re}\psi(s)u_0)^2 + (\mathrm{Re}S_N\psi_N(s)u_0)^2 \right) ,$$

by Hölder inequality with $\frac{1}{p_2'} = \frac{1}{3p_2'} + \frac{2}{3p_2'}$,

$$\|II_N\|_{X_T^s} \leq \|\mathrm{Re}\psi(s)u_0 - \mathrm{Re}S_N\psi_N(s)u_0\|_{L^{3p_2'}} \|(\mathrm{Re}\psi(s)u_0)^2 + (\mathrm{Re}S_N\psi_N(s)u_0)^2\|_{L^{3p_2'/2}}$$

$$\leq \|\mathrm{Re}\psi(s)u_0 - \mathrm{Re}S_N\psi_N(s)u_0\|_{L^{3p_2'}} \times$$
$$\left(\|(\mathrm{Re}\psi(s)u_0)^2\|_{L^{3p_2'/2}} + \|(\mathrm{Re}S_N\psi_N(s)u_0)^2\|_{L^{3p_2'/2}} \right)$$

As $3p_2' = \frac{6p}{p+2} < p$ and

$$\|(\mathrm{Re}\psi(s)u_0)^2\|_{L^{3p_2'/2}} = \|\mathrm{Re}\psi(s)u_0\|_{L^{3p'2}}^2 \leq T^{2\gamma_2}\|\psi(s)u_0\|_{L^p}^2$$

with $\gamma_2 = \frac{p-4}{6p}$, it comes that :

$$\|II_N\|_{X_T^s} \leq CD^2T^{2\gamma_2}\|\mathrm{Re}\psi(s)u_0 - \mathrm{Re}S_N\psi_N(s)u_0\|_{L^{3p_2'}}$$

with C independent from D and T as long as $T \leq \tau$.

Now, the quantity $\|\mathrm{Re}\psi(s)u_0 - \mathrm{Re}S_N\psi_N(s)u_0\|_{L^{3p_2'}} \leq \alpha_N + \beta_N$ remains to be considered, with $\alpha_N = \|(1 - S_N)\psi(s)u_0\|$ and $\beta_N = \|S_N(\psi(s)u_0 - \psi_N(s)u_0)\|$.

By the same convex inequalities as previously,

$$\beta_N \leq CT^{\gamma_2}\|\psi(s)u_0 - \psi_N(s)u_0\|_{L^p} \leq CT^{\gamma_2}\|\psi(s)u_0 - \psi_N(s)u_0\|_{X_T^s} .$$

Choose $2 < p_3 < \frac{3}{3/2-\sigma} \in]2,3[$ and call $\sigma_3 = 3\left(\frac{1}{2} - \frac{1}{p_3}\right) < \sigma$. As $p_3 < 3 < 3p_2' < p$ so, there exists $\theta \in]0,1[$ such that $\frac{1}{3p_2'} = \frac{\theta}{p_3} + \frac{1-\theta}{p}$, thus

$$\alpha_N \leq \|(1 - S_N)\psi(s)u_0\|_{L^{p_3}}^{\theta}\|(1 - S_N)\psi(s)u_0\|_{L^p}^{1-\theta}$$

$$\|(1 - S_N)\psi(s)u_0\|_{L^p} \leq C(\|S(t)u_0\|_{L^p} + \|v\|_{X_T^s}) \leq CD$$

and by Sobolev embedding theorem :

$$\|(1-S_N)\psi(s)u_0\|_{L^{p_3}} \leq \|(1-S_N)\psi(s)u_0\|_{L_t^{p_3},H_x^{\sigma_3}}$$

Now, for all s,

$$\|(1-S_N)\psi(s)u_0\|_{H^{\sigma_3}} \leq CN^{\sigma_3-\sigma}\|\psi(s)u_0\|_{H^\sigma} \leq CDN^{\sigma_3-\sigma}$$

and so

$$\alpha_N \leq CN^{\theta(\sigma_3-\sigma)}D^\theta D^{1-\theta} = CN^{\theta(\sigma_3-\sigma)}D\ .$$

Therefore, for all $\epsilon > 0$, there exists N_0 such that for all $u_0 \in A(D)$, all $T \leq \tau$, and all $N \geq N_0$,

$$\alpha_N \leq \epsilon\ .$$

Now, let us sum up these inequalities.

$$\begin{aligned}\|\psi(t)u_0 - \psi_N(t)u_0\|_{X_T^s} &\leq I_N + CD^2T^{2\gamma_2}(\alpha_N+\beta_N) \\ &\leq I_N + C\alpha_N + CD^2T^{3\gamma_2}\|\psi(t)u_0-\psi_N(t)u_0\|_{X_T^s}\ .\end{aligned}$$

Set $\gamma_1 = \max(\gamma, \frac{2}{3\gamma_2})$ and $\tau_1 = \min(\tau, C^{-1/(3\gamma_2)}(1+D)^{-\gamma_1})$, such that $CD^2\tau_1^{3\gamma_2} < CD^2\tau_1^{2/\gamma_1} < 1$, hence

$$\|\psi(t)u_0 - \psi_N(t)u_0\|_{X_{\tau_1}^s} \leq C(I_N+\alpha_N)$$

so for all $\epsilon > 0$ there exists N_0 such that for all $u_0 \in A(D)$ and all $N \geq N_0$

$$\|\psi(t)u_0 - \psi_N(t)u_0\|_{X_{\tau_1}^s} \leq \epsilon\ .$$

◇

Remark 4.3.3. *Note that the construction of γ_1 ensures that $\gamma_1 \geq \gamma$ but that τ_1 is still a power of D.*

4.3.3 Local invariance

Let us show that the measure is invariant under the flow locally in time. That is, as long as the sequence $\psi_N(t)u_0$ converges uniformly on some sets, it will appear that $\rho(\psi(t)A) \geq \rho(A)$. This is the first step in order to reach a global invariance result for the measure.

Lemma 4.3.9. *Let $\sigma \in]0, \frac{1}{2}[$, $p \in]4, 6[$, $s = \frac{3}{2} - \frac{4}{p}$, and $D > 0$. Set $A(D)$ the set described in* (4.3.8), $\tau = c(1+D)^{-\gamma}$ *the local existence time coming from theorem* (4.3.6) *and $\tau_1 = \min(\tau, c_1(1+D)^{-\gamma_1})$ the local time of uniform convergence, all three depending only on D and p. Then, for all $A \subseteq A(D)$ measurable, and all $t \in [-\tau_1, \tau_1]$, the set $\psi(t)A$ is measurable and :*

$$\rho(\psi(t)A) = \rho(A) .$$

Proof First, for all A measurable, $\psi(t)A$ is also measurable thanks to the local continuity of the flow. Assume now that A is a closed set of H^σ included in $A(D)$ and set $\epsilon > 0$. By lemma (4.3.8), there exists N_0 such that for all $u_0 \in A(D)$ and all $N \geq N_0$

$$\|\psi(t)u_0 - \psi_N(t)u_0\|_{X^s_{\tau_1}} \leq \epsilon .$$

But by definition, $\| \, . \, \|_{X^s_{\tau_1}} \geq \| \, . \, \|_{C^0([-\tau_1,\tau_1],H^s(B^3))}$. So for all $t \in [-\tau_1, \tau_1]$,

$$\|\psi(t)u_0 - \psi_N(t)u_0\|_{H^s} \leq \epsilon .$$

Let B_ϵ be the ball in H^s of centre 0 and radius ϵ, as $A \subseteq A(D)$, for all $N \geq N_0$, and all $t \in [-\tau_1, \tau_1]$,

$$\psi_N(t)(A) \subseteq \psi(t)(A) + B_\epsilon$$

therefore

$$\rho_N(\psi_N(t)(A)) \leq \rho_N(\psi(t)(A) + B_\epsilon) .$$

Then, since the measure ρ_N is invariant under the flow ψ_N, as it is stated in Proposition 4.2.11,

$$\rho_N(\psi_N(t)A) = \rho_N(A)$$

and thus

$$\rho_N(A) \leq \rho_N(\psi(t)A + B_\epsilon) .$$

Then, using the fact that (Proposition 4.2.13), for all μ measurable sets B,

$$\rho(B) = \lim_{N\to\infty} \rho_N(B) ,$$

and as the property $\rho_N(A) \leq \rho_N(\psi(t)A + B_\epsilon)$ is true for all $N \geq N_0$, by taking the limit :

$$\rho(A) \leq \rho(\psi(t)A + B_\epsilon)$$

and then by making ϵ go to 0, thanks to the dominated convergence theorem,

$$\rho(A) \leq \rho(\psi(t)A)$$

and that for all $t \in [-\tau_1, \tau_1]$. Indeed, $\psi(-t) = \psi(t)^{-1}$ is continuous in H^σ, so $\psi(t)A$ is closed in H^σ.

For the reverse inequality, use the fact that indeed, $\psi(t)A \subseteq \psi_N(t)A + B_\epsilon$ for all $u_0 \in A(D)$ and $n \geq N_0$. It is also true that the ball $\widetilde{B}_\epsilon$ of radius ϵ in H^σ contains B_ϵ as $\sigma < \frac{1}{2} < s$, so :

$$\psi(t)A \subseteq \psi_N(t)A + \widetilde{B}_\epsilon \ .$$

Then, the fact that the equation is reversible, and so $\psi_N(t)^{-1} = \psi_N(-t)$ is used. Also, thanks to the continuity of the local flow on H^σ, there exists a constant C depending on the time t but not on ϵ or N such that :

$$\psi_N(-t)(\psi_N(t)A + \widetilde{B}_\epsilon) \subseteq A + \widetilde{B}_{C\epsilon}$$

so

$$\psi(t)A \subseteq \psi_N(t)A + \widetilde{B}_\epsilon \subseteq \psi_N(t)(A + \widetilde{B}_{C\epsilon})$$

and

$$\rho_N(\psi(t)A) \leq \rho_N\left(\psi_N(t)(A + \widetilde{B}_{C\epsilon})\right) = \rho_N(A + \widetilde{B}_{C\epsilon})$$

thanks to the invariance of ρ_N under ψ_N.

By passing to le limit $N \geq N_0 \to \infty$,

$$\rho(\psi(t)A) \leq \rho(A + \widetilde{B}_{C\epsilon})$$

and then $\epsilon \to 0$,

$$\rho(\psi(t)A) \leq \rho(A)$$

so for all closed set A of H^σ included in $A(D)$,

$$\rho(\psi(t)A) = \rho(A) \ .$$

Then, remark that $A(D)$ is a closed set of H^σ, so

$$\rho(\psi(t)A(D)) = \rho(A(D))$$

and thus the property of invariance under the flow passes to the complementary and the denumerable unions. It holds on every measurable set. $\diamond$

4.4 Measure invariance

4.4.1 Building sets of full measure with global existence

We now need to build a set of full ρ measure such that in this set not only the local invariance result holds but also can be extended to a global one, that is to say that in this set the flow must be globally well defined.

Definition 4.4.1. Let

$$D_{i,j} = (i + j^{1/\gamma_1})^{1/2}$$

with $i, j \in \mathbb{N}$ and set $T_{i,j} = \sum_{l=1}^{j} \tau_1(D_{i,l})$. Let

$$\Pi_{N,i} = \{u_0 \mid \forall j \in \mathbb{N} \ , \ \psi_N(\pm T_{i,j}) \in A(D_{i,j+1})\}$$

and

$$\Pi_i = \limsup_{N\to\infty} \Pi_{N,i}$$

and finally

$$\Pi = \bigcup_{i\in\mathbb{N}} \Pi_i \ .$$

Proposition 4.4.2. *The set* Π *is of full* ρ *measure.*

Proof Let us compute the measure of the complementary set of Π. For all μ (and thus ρ or ρ_N) measurable sets A, we have that

$$|\rho(A) - \rho_N(A)| \leq \int (f(u) - f_N(u))1_A(u)d\mu(u) \leq \|f - f_N\|_{L^1_\mu}$$

hence

$$\rho(\Pi^c_{N,i}) \leq \|f - f_N\|_{L^1_\mu} + \rho_N(\Pi^c_{N,i}) \ .$$

Then, as $\Pi_{N,i}$ is an intersection :

$$\Pi_{N,i} = \bigcap_{j\in\mathbb{N}} \psi_N(\pm T_{i,j})^{-1}(A(D_{i,j+1})) \ ,$$

we have

$$\Pi^c_{N,i} = \bigcup_{i\in\mathbb{N}} \psi_N(\pm T_{i,j})^{-1}(A(D_{i,j+1}))^c \ ,$$

$$\rho_N(\Pi^c_{N,i}) \leq \sum_{j=0}^{\infty} \rho_N\Big(\Big(\psi_N(\pm T_{i,j})^{-1}(A(D_{i,j+1})\Big)^c\Big) \ .$$

Then, using the invariance of ρ_N under the flow ψ_N, we get that the measure of $\left(\psi_N(\pm T_{i,j})^{-1}(A(D_{i,j+1})\right)^c = \psi_N(\pm T_{i,j})^{-1}(A(D_{i,j})^c)$ is equal to the measure of $A(D_{i,j})^c$,

$$\rho_N(\Pi_{N,i}^c) \leq 2\sum_j \rho_N(A(D_{i,j})^c) \leq 2\sum_j \mu(A(D_{i,j})^c)$$

But, $A(D)^c = B(D)^c \cup E(D)^c$ with B and E the sets defined in (4.3.7) so

$$\mu(A(D)^c) \leq \mu(B(D)^c) + \mu(E(D)^c) \leq 2e^{-cD^2}$$

so

$$\rho_N(\Pi_{N,i}) \leq C\sum_j e^{-cD_{i,j}^2}$$

and so

$$\rho(\Pi_{N,i}^c) \leq Ce^{-ci}\sum_j e^{-cj^{1/\gamma_1}} \leq Ce^{-ci}$$

as $e^{-cj^{1/\gamma_1}}$ is the general term of a convergent series.

Therefore :

$$\rho(\Pi_i^c) = \rho(\liminf \Pi_{N,i}^c) \leq \liminf \rho(\Pi_{N,i}^c) = e^{-cD^2 i} + \liminf \|f - f_N\|_{L^1_\mu} = Ce^{-ci}$$

and then

$$\rho(\Pi^c) = \rho(\bigcap_i \Pi_i^c) \leq \lim \rho(\Pi_i^c) = 0$$

that is to say that Π is of full measure. ◇

Lemma 4.4.3. *Let $u_0 \in \Pi_i$, the flow $\psi(t)u_0$ is strongly globally defined and for all $j \in \mathbb{N}$ and $\psi(\pm T_{i,j})u_0$ belongs to $A(D_{i,j+1})$.*

Proof As $u_0 \in \Pi_i = \limsup \Pi_{N,i}$, there exists a sequence $N_k \to \infty$ such that for all k, $u_0 \in \Pi_{N_k,i}$, which is equivalent to $\psi_{N_k}(\pm T_{i,j})u_0 \in A(D_{i,j})$ for all $j \in \mathbb{N}$.

Then, by recurrence over j, it can be proved that $\psi(t)u_0$ is defined on $[-T_{i,j}, T_{i,j}]$ and that $\psi(\pm T_{i,j})u_0 - \psi_{N_k}(\pm T_{i,j})u_0$ converges toward 0 in $\mathcal{H}^s$ when $k \to \infty$.

For $j = 0$, $T_{i,0} = 0$ and so $u_0 = \psi_{N_k}(\pm T_{i,0})u_0 \in A(D_{i,1}$ and $\psi(T_{i,0})u_0 - \psi_{N_k}(T_{i,0})u_0 = 0$ converges toward 0 in H^s.

Suppose that at rank j, $\psi(t)u_0$ strongly exists on $[-T_{i,j}, T_{i,j}]$ and $\psi(\pm T_{i,j})(u_0) - \psi_{N_k}(\pm T_{i,j})(u_0)$ converges toward 0 in H^s. Let us show that the property holds at rank $j+1$. As

$$\|\psi(\pm T_{i,j})(u_0) - \psi_{N_k}(\pm T_{i,j})(u_0)\|_{H^\sigma} \leq \|\psi(\pm T_{i,j})(u_0) - \psi_{N_k}(\pm T_{i,j})(u_0)\|_{H^s}$$

and

$$\|S(t)\left(\psi(\pm T_{i,j})(u_0) - \psi_{N_k}(\pm T_{i,j})(u_0)\right)\|_{L^p_{t,x}} \leq \|\psi(\pm T_{i,j})(u_0) - \psi_{N_k}(\pm T_{i,j})(u_0)\|_{H^s}$$

and for all k,

$$\|\psi_{N_k}(\pm T_{i,j})(u_0)\|_{H^\sigma} \leq D_{i,j+1}$$

and

$$\|S(t)\left(\psi_{N_k}(\pm T_{i,j})(u_0)\right)\|_{L^p_{t,x}} \leq D_{i,j+1} \ ,$$

by taking the limits when $k \to \infty$, it comes that $u_\pm := \psi(\pm T_{i,j})u_0 \in A(D_{i,j+1})$.

So, thanks to Theorem 4.3.6 $\psi(t)u_\pm$ is strongly defined on $[-\tau_1, \tau_1] \subseteq [-\tau, \tau]$ and thanks to Lemma 4.3.8,

$$\psi(t)u_\pm - \psi_{N_k}(t)u_\pm$$

converges toward 0 in $X^s_{\tau_1(D_{i,j+1})}$. In particular,

$$\psi(\pm\tau_1(D_{i,j+1})u_\pm - \psi_{N_k}(\pm\tau_1(D_{i,j+1}))u_\pm$$

converges toward 0 in H^s.

Then, as

$$\psi(\pm T_{i,j+1})u_0 - \psi_{N_k}(\pm T_{i,j+1})u_0 = \psi(\pm\tau_1(D_{i,j+1}))u_\pm - \psi_{N_k}(\pm\tau_1(D_{i,j+1}))u_\pm +$$

$$\psi_{N_k}(\pm\tau_1(D_{i,j+1}))u_\pm - \psi_{N_k}(\pm\tau_1(D_{i,j+1}))(\psi_{N_k}(\pm T_{i,j})u_0$$

and since $v \mapsto \psi_N(\pm\tau_1(D_{i,j+1}))v$ is uniformly in N continuous from $H^s \cap A(D_{i,j+1})$ to H^s, it implies that

$$\psi_{N_k}(\pm\tau_1(D_{i,j+1}))u_\pm - \psi_{N_k}(\pm\tau_1(D_{i,j+1}))(\psi_{N_k}(\pm T_{i,j})u_0$$

also converges toward 0 in H^s. Therefore,

$$\psi(\pm T_{i,j+1})u_0 - \psi_{N_k}(\pm T_{i,j+1})u_0$$

converges toward 0 in H^s and as it has previously been seen, it implies that $\psi(\pm T_{i,j+1}) \in A(D_{i,j+2})$. $\diamond$

4.4.2 Global invariance

Now, a first result of global invariance can be proved.

Proposition 4.4.4. *Let A be a measurable set included in Π_i. Then for all $t \in \mathbb{R}$, we have*

$$\rho(\psi(t)(A)) = \rho(A) .$$

Proof

In order to prove such a fact, it is required that the sequence $T_{i,j}$ where i is fixed diverges.

Indeed,

$$\tau_1(D_{i,j}) = \min(\tau(D_{i,j}, c_1(1 + D_{i,j})^{-\gamma_1}) = \min(c(1 + D_{i,j})^{-\gamma}, c_1(1 + D_{i,j})^{-\gamma_1})$$

and $D_{i,j} = \sqrt{i + j^{1/\gamma_1}}$ diverges. Therefore, as $\gamma_1 \geq \gamma$, above a certain rank $\tau_1(D_{i,j}) = c_2(1+D_{i,j})^{-\gamma_1}$ with $c_2 = c_1$ if $\gamma < \gamma_1$ or $c_2 = \min(c, c_1)$ otherwise.

So, $\tau_1(D_{i,j})$ behaves like $j^{-1/2}$ when $j \to \infty$ and so the sequence $T_{i,j}$ diverges.

Let $t \in \mathbb{R}$, there exists j such that $t \in [T_{i,j}, T_{i,j+1}]$ if $t \geq 0$ or $t \in [-T_{i,j+1}, -T_{i,j}]$. Let us show by recurrence over j that for all $t \in [T_{i,j}, T_{i,j+1}] \cup [-T_{i,j+1}, -T_{i,j}]$,

$$\rho(\psi(t)A) = \rho(A)$$

.

For $j = 0$, we have $T_{i,0} = 0$, $T_{i,1} = \tau_1(D_{i,1})$ and $A = \psi(T_{i,0})(A) \subseteq \psi(T_{i,0})(\Pi_i) \subseteq A(D_{i,1})$ thanks to Lemma 4.4.3. So, the local invariance Lemma 4.3.9 holds : for all $t \in [-\tau_1(D_{i,1}), \tau_1(D_{i,1})]$,

$$\rho(\psi(t)(A)) = \rho(A) .$$

For $j - 1 \Rightarrow j$, $\psi(\pm T_{i,j})(A) \subseteq \psi(T_{i,j})(\Pi_i) \subseteq A(D_{i,j+1})$ (Lemma 4.4.3). So, by using Lemma 4.3.9, for all $t \in [0, \tau_1(D_{i,j+1})]$,

$$\rho\left(\psi(\pm t)\left(\psi(\pm T_{i,j})(A)\right)\right) = \rho\left(\psi(\pm T_{i,j})(A)\right) .$$

Then, by using the recurrence hypothesis,

$$\rho\left(\psi(\pm T_{i,j})(A)\right) = \rho(A) .$$

And so, for all

$$t \in [T_{i,j}, T_{i,j} + \tau_1(D_{i,j+1})] \cup [-T_{i,j} - \tau_1(D_{i,j+1}), -T_{i,j}]$$

$$= [T_{i,j}, T_{i,j+1}] \cup [-T_{i,j+1}, -T_{i,j}] ,$$

it comes

$$\rho(\psi(t)(A)) = \rho(\psi(t)(A)) \ .$$

◊

Theorem 4.4.5. *For all ρ measurable sets included in Π, we have :*

$$\rho(\psi(t)(A)) = \rho(A) \ .$$

Proof As $\Pi = \bigcup_i \Pi_i$, and $A \subseteq \Pi$, A can be written :

$$A = \bigsqcup_{i \in \mathbb{N}} A_i$$

with $A_i \subseteq \Pi_i$, and the A_i disjoint. So,

$$\psi(t)(A) = \bigsqcup_{i \in \mathbb{N}} \psi(t) A_i$$

since the flow is strongly defined in Π.

$$\rho(\psi(t)A) = \sum_{i \in \mathbb{N}} \rho(\psi(t)A_i) = \sum_{i \in \mathbb{N}} \rho(A_i) = \rho(\bigsqcup_{i \in \mathbb{N}} A_i) = \rho(A) \ .$$

◊

Chapitre 5

Wave turbulence for the BBM equation : Stability of a Gaussian statistics under the flow of BBM

Ce chapitre est issu de l'article [25].

5.1 Introduction

Wave turbulence studies the evolution of some particular statistics under the flow of non linear equations with a weak non linearity. An early reference on the subject is the one by Peierls in [48] in 1929. The theory has known important developments during the sixties thanks to Zakharov, Filonenco, or Musher see [56, 57, 41], who have described the invariance of particular spectra known as the Kolmogorov-Zakharov (KZ) spectra which represents the average amplitudes to the square of waves, or the number of particles given a wavelength. In more recent works such as [3], the stability of the KZ spectrum has also been studied.

The purpose of these works is to consider each possible wavenumber of the linearised around the zero solution equation and assume what is called the random phase approximation (RPA) which presumes that the phases of the waves corresponding to these wavenumbers are initially independent from each other and taken uniformly distributed over the circle S^1.

A review of general KZ spectra, that is, of the statistics such that the average of the amplitudes to the square are invariant under the flow of some PDEs can be found in [29].

Moreover, different time scales have been observed between the studied PDE and the evolution of average quantities such as the energy, [46, 45].

Another question that arises is how does the law of the statistics itself evolve. This question appeared since the beginning of the theory in [48] and was later developed by Brout and Prigogine in [10].

In more recent papers, not only the phases are supposed independent but also the modulus of the amplitudes (Random phase and amplitude assumption). One can thus wonder whether the waves remain independent as they evolve in time and as they interact due to the non linearity of the equation. A general investigation leads to the preservation of the independence and the distribution of the phases under some conditions up to corrections of order 2 with respect to a small parameter controlling the non linearity, [18].

To be more precise, what is called a statistics is a random variable with value in L^2, or a space linearly spanned by the eigenmodes of the linear equation corresponding to the non linear PDE. This random variable induces a measure on L^2. It can also be seen as two sequences of random variables with values in $\mathbb{R}^+$ and S^1, $A_n \in \mathbb{R}^+$ (for the modulus of the amplitude) and $\varphi_n \in S^1$ for the phase. Then, the random initial data is given by :

$$\sum_n A_n \varphi_n e_n$$

where e_n are the eigenfunctions of the linear operator involved in the studied PDE.

Under the random phase approximation, the φ_n are supposed initially independent from each other and from the A_n, and are supposed uniformly distributed over S^1. Under the random phase and amplitude assumption, the A_n are also supposed independent from each other. The quantity introduced in [18] to study the stability of the statistics is :

$$Z^N\{\lambda, \mu, t\} = \langle \prod_{i=1}^N e^{\lambda_i A_i(t)^2} \varphi_i(t)^{\mu_i} \rangle \tag{5.1}$$

where $\langle\ .\ \rangle$ denotes the mean value with respect to the initial measure induced by the statistics. The integer N corresponds to a certain (large) number of waves and the behaviour of Z^N is studied as N goes to ∞. As time passes by, the values of the random variable A_n and φ_n evolve and interact with each other, and thus, Z^N depends also on t. Notice that this construction highly depends on the choice of the basis $(e_n)_n$. The stability is a control of the difference between $Z^N(t)$ and $Z^N(t = 0)$. However, this generating functional has been chosen mainly for its convenience regarding the problem studied. Here, another one is taken, still for reasons of convenience, but the main purpose remains studying the law of the statistics.

Modulus invariant statistics (KZ spectra) are the ones such that $\langle A_n^2(t)\rangle = \langle A_n^2(0)\rangle$ at least “locally” in n, that is for n of a certain order. The solutions are of the form $\langle A_n^2\rangle = Cn^\beta$.

In this paper, the equation onto which wave statistics are dealt with is the Benjamin - Bona - Mahony equation :

$$\begin{cases} \partial_t \left(1 - \partial_x^2\right) u + \partial_x \left(u + \frac{u^2}{2}\right) = 0 & u \text{ periodic in } x, t \in \mathbb{R} \\ u|_{t=0} = u_0 \in H^s & \text{for some } s \geq 0 \end{cases} . \tag{5.2}$$

This equation is an alternative to KdV in the context of long wavelengths and small amplitudes water waves. The terms of second order in u_x have been replaced by $-u_t$. It has been chosen because it has a so-called linear invariant, the H^1 norm to the square. This invariant permits to construct an initial datum belonging almost surely to L^2, whose law is invariant under the flow of the equation, that is, there is an invariant statistics (measure) μ on L^2 for the BBM equation.

For the measure μ, the questions that generally arise in wave turbulence, are entirely dealt with thanks to its invariance. The squares of the amplitudes are invariant and equal to $\frac{2}{1+n^2} \sim n^{-2}$, the independence remains valid at all time, there is no time scale so to speak for the evolution of the average quantities in general since they are invariant.

This statistics will be slightly perturbed, in a way that shall be defined later, and the investigation is about the evolution of this perturbed statistics μ_V, and what it implies for the evolution of the squares of the amplitudes. This measure depends on a small parameter V, which is a C^2 function representing a potential, whose L^∞ norm and the L^∞ norm of its derivatives are close to 0.

Remark that the unknown is real here and not complex. Thus, the initial statistics considered has been chosen as the real part of a complex statistics satisfying the conditions imposed by wave turbulence, random phase amplitude or random phase approximation.

The generating functional used to study the evolution of the law of the statistics is the characteristic function, which means that the evolution of

$$Z_V(\lambda, t) = E_V(e^{i\langle \lambda, \psi(t)u_0\rangle}) \tag{5.3}$$

is considered, where E_V is the mean value with respect to the perturbed statistics μ_V, or $d\mu_V(u_0)$, $\psi(t)$ is the flow of the BBM equation, $\lambda \in L^2$, and the brackets denote the usual scalar product in L^2.

In fact, there does not seem to exist quantities of type (5.1) in the context of BBM as it is a real valued context. However, (5.3) measures independence of the amplitudes as well as (5.1) and thus seems as natural as (5.1).

It is known from [5, 49] that the BBM equation is globally well posed in H^s for all $s \geq 0$ and there even exist bounds on the L^2 norm of $\psi(t)u_0$. The first thing proved here is the existence of a statistics invariant under the flow of BBM.

Theorem 5. *There exists a measure μ on L^2 invariant under the flow of BBM. The measure μ is a Gaussian vector in infinite dimension. For all $A \subseteq L^2$ measurable (in the sense of the topological σ algebra),*

$$\mu(\psi(t)A) = \mu(A) \ .$$

This statistics μ is taken such that all eigenmodes are independent from each other. The measure μ is of Gibbs type, in the spirit of the works by Lebowitz-Rose-Speer, [39] and Bourgain, [7].

Now a small function V is introduced, and the statistics μ is changed into a statistics μ_V which allows covariance (of order V) between the modes. As it happens, μ_V is built in a way that involves a slightly different linear operator $D_V = (1+V)^{-1/2}(1-\partial_x^2)^{-1}\partial_x(1+V)^{1/2}$ from the operator of BBM ($(1-\partial_x^2)^{-1}\partial_x$) and the perturbed eigenmodes (the projections onto the eigenfunctions of D_V) are independent from each other.

In fact, the change of statistics corresponds to a change of the equation, and the statistics μ_V is invariant under the perturbed flow, the new equation being :

$$\partial_t u_V + D_V(u_V + \frac{(1+V)^{1/2}u_V^2}{2}) = 0$$

as BBM is

$$\partial_t u + (1-\partial_x^2)^{-1}\partial_x \left(u + \frac{u^2}{2} \right) = 0 \ .$$

This equation is globally well-posed and its flow is called ψ_V. The measure μ_V is an infinite dimensional Gaussian vector on L^2 with covariance operator $\sqrt{1+V}(1-\partial_x^2)^{-1}\sqrt{1+V}$.

In the end, there is an estimate regarding the characteristic functions (5.3).

Theorem 6. *Let $\epsilon \in]0, \frac{1}{2}[$, there exist two constants C and c such that for all $\lambda \in L^2$ and all $t \in \mathbb{R}$,*

$$|Z_V(\lambda,t) - Z_V(\lambda,0)| \leq C \left(\|V\|_{L^\infty} + \|\partial_x V\|_{L^\infty} + \|\partial_x^2 V\|_{L^\infty} \right) \|\lambda\|_{L^2} |t|^{5/(2\epsilon)} e^{c|t|^{6/\epsilon - 2}}$$

where $Z_V(\lambda,t)$ is defined by (5.3).

Remark 5.1.1. *This result leads to the stability of the so-called KZ spectrum for this equation, that is, the mean values of the amplitudes to the square differ from their initial values only with order $\left(\|V\|_{L^\infty} + \|\partial_x V\|_{L^\infty} + \|\partial_x^2 V\|_{L^\infty} \right)$ and with the same behaviour in time : for $\epsilon \in]0, 1/2[$, there exist C, c such that for all $n \in \mathbb{N}$ $t \in \mathbb{R}$,*

$$\begin{aligned} &|E_V(|\langle \psi(t)u_0, \cos(nx)\rangle|^2) - E_V(|\langle u_0, \cos(nx)\rangle|^2)| \\ &\leq C \left(\|V\|_{L^\infty} + \|\partial_x V\|_{L^\infty} + \|\partial_x^2 V\|_{L^\infty} \right) |t|^{5/(2\epsilon)} e^{c|t|^{6/\epsilon - 2}} \ . \end{aligned}$$

Remark that $x \mapsto e^{ix}$ can be replaced by any F as long as F is smooth enough, for instance if F is differentiable and its derivative is bounded.

Plan of the paper In Section 2, the existence and invariance under the BBM flow of the measure μ is proved. For that, the techniques used are the same as in [26]. The BBM flow is approached by finite dimensional flows, and the measure by other measures onto finite dimensional spaces, such that the conservation of the approached measures under the approached flows can be actually computed. Then, the limit is taken.

In Section 3, the meaning of the phrase "perturbation of the statistics" is given. The measure μ is a Gaussian vector of diagonal covariance matrix, the perturbed measure μ_V is also a Gaussian vector whose covariance matrix coefficients depend on the Fourier coefficients of the small C^2 parameter V, such that it tends to the covariance of μ when V goes to 0. The measure μ_V is built in a way such that it is a priori invariant under the flow ψ_V of a V perturbed equation. Section 3 also introduces this equation, its invariant and a method to approach it by finite dimensional equations.

Section 4 provides a proof of local well posedness of the perturbed equation, which is mainly a verification that the operator D_V has properties in common with $(1-\partial_x^2)^{-1}\partial_x$. Then, the almost sure global well posedness is exposed along with the invariance of μ_V under the flow $\psi_V(t)$ which leads to the invariance of

$$Z'(\lambda,t) = E_V\left(e^{i\langle \lambda, \psi_V(t)u_0\rangle} \right) \ .$$

In Section 5, the same techniques as in the first sections of [5] are used to get bounds on the L^2 norms of $\psi(t)u_0$ and $\psi_V(t)u_0 - \psi(t)u_0$ in order to finally prove Theorem 6.

5.2 Invariance of the independent Gaussian statistics

Consider a system which can be seen as a statistical repartition of waves. In particular, look at the case when the repartition is a Gaussian onto each mode of the linear equation and those Gaussian variables are independent. It means that two different wavelengths are statistically independent. It evolves through the flow of the BBM equation. As it evolves, the different wavelengths interfere but the statistical repartition remains the same. Namely, the statistics is represented by a measure that is invariant through the flow.

The plan of this section comes as follow : first, the measure is defined, then, its invariance through the linear flow is proved, and then, its invariance through the BBM equation.

5.2.1 Linear invariance

The measure constructed here is a infinite dimensional Gaussian vector whose covariance matrix corresponds to the Laplacian. In finite dimension, it is a Gaussian vector with a covariance matrix representing $(1-\partial_x^2)^{-1}$. This also corresponds to a Brownian motion conditioned by 2π periodicity, $u(2\pi) = u(0)$. Then, the limit is taken.

The equation (5.2) admits

$$\frac{1}{2}\int u\left(1-\partial_x^2\right)udx \tag{5.4}$$

as an invariant.

Then, there exists a measure invariant through the flow of (5.2), as the action of the covariance matrix to the solution is independent from time.

Definition 5.2.1. Let $(c_n)_{n\geq 0}$ and $(s_n)_{n\geq 1}$ be the orthonormal basis of real L^2 with periodic conditions :

$$c_0(x) = \frac{1}{\sqrt{2\pi}}\ ,\ c_n(x) = \frac{1}{\sqrt{\pi}}\cos(nx) \text{ and } s_n(x) = \frac{1}{\sqrt{\pi}}\sin(nx)\ .$$

Let $(g_n)_{n\geq 0}$ and $(h_n)_{n\geq 1}$ be real independent centred normalized Gaussian variables on a probability space $\Omega, \mathcal{A}, P$. For all $M \leq N \in \mathbb{N}$, call

$$\varphi_M^N : \begin{cases} \Omega\times[0,2\pi] \to \mathbb{R} \\ \omega, x \mapsto \sum_{n=M}^{N}\left(\frac{g_n(\omega)}{\sqrt{1+n^2}}c_n(x) + \frac{1}{\sqrt{1+n^2}}s_n(x)\right) \end{cases}$$

with the convention $s_0 = 0$ and $h_0 = 0$.

Define μ_M^N the measure onto E_M^N the Hilbert subspace of L^2 linearly spanned by $\{c_n, s_n \mid n = M, \dots, N\}$ that is the image of φ_M^N.

In [54], some helpful properties for the φ_M^N and μ_M^N are given.

Proposition 5.2.2. *For any $M \geq 0$, the sequence $(\varphi_M^N)_N$ converges in*

$$L^2(\Omega, L^2([0, 2\pi])) \ .$$

Call φ_M it limit, and μ_M the measure on E_M the subset of L^2 linearly spanned by $\{c_n, s_n \mid n \geq M\}$.

As a convention, μ_0 is noted μ.

The following statement holds :

Proposition 5.2.3. *For any open set $U \subseteq E_M$ (for the trace topology of L^2),*

$$\mu_M(U) \leq \liminf_{N\to\infty} \mu_M^N(U \cap E_M^N) \ .$$

What is more, for any $s \in [0, \frac{1}{2}[$, calling B_R^s the closed ball of centre 0 and radius R in H^s, which is a compact set in L^2 when $s > 0$, it comes that :

$$\mu((B_R^s)^c) \leq e^{-a_s R^2}$$

where $a_s = \frac{1}{4}\left(1 + 2\sum_{n\geq 1} \frac{1}{(1+n^2)^{1-s}}\right)$ is a constant with respect to R.

This is enough to show the invariance of the measures μ_M through the linear flow.

Definition 5.2.4. Let $S(t)$, $t \in \mathbb{R}$ be the linear flow of (5.2), that is the flow of :

$$\partial_t \left(1 - \partial_x^2\right) u + \partial_x u = 0 \ . \tag{5.5}$$

This flow is isometric in L^2, but as a matter of fact, it is also isometric in H^s for all s and in particular, for the s that have an interest regarding the measure μ, that is $s \in [0, \frac{1}{2}[$.

What is more, for all M, N, $S(t)E_M^N = E_M^N$ and $S(t)E_M = E_M$, and it is reversible since $S(t_1 + t_2) = S(t_1) \circ S(t_2)$.

Then, thanks to stability, the measure on the finite dimensional subspace E_M^N is invariant under the linear flow. If u is written :

$$u = \sum_{n=M}^{N} (a_n(t)c_n + b_n(t)s_n)$$

then

$$a_n(t) = a_n^0 \cos(\frac{-n}{1+n^2}t) - b_n^0 \sin(\frac{-n}{1+n^2}t) \text{ and } b_n(t) = b_n^0 \cos(\frac{-n}{1+n^2}t) + a_n^0 \sin(\frac{-n}{1+n^2}t) \ .$$

Hence, the measure $e^{-(a^2+b^2)(1+n^2)/2} dadb$ is invariant under the change of variable $a_n^0, b_n^0 \mapsto a_n(t), b_n(t)$.

As

$$d\mu_M^N(u) = d_M^N e^{-\sum_{n=M}^N (a_n^2+b_n^2)(1+n^2)/2} \prod_{k=M}^N da_k db_k \ ,$$

the measure μ_M^N is invariant under the linear flow.

Making a parallel with the proof of the invariance of the measure under the linear flow in [26], one can see that Proposition 5.2.2, the reversibility of $S(t)$ and the fact that it is isometric in L^2 are sufficient to prove the invariance. Hence,

Proposition 5.2.5. *For all M, the measure μ_M on E_M is invariant under the flow of* (5.5).

Remark 5.2.1. *For all M, μ is the measure generated by μ_0^{M-1} and μ_M on the Cartesian product $E_0^{M-1} \times E_M = L^2$.*

5.2.2 Approaching the non linear flow thanks to finite dimension

Using now the approach by Burq-Thomann-Tzvetkov [13] and by Burq and Tzvetkov [15, 16], it is possible to prove the invariance of μ under the flow of (5.2). For that, the non linear flow is approached by flows in finite dimensional spaces instead of L^2. The idea is that it is possible to compute functionals and measurements in finite dimension, not in L^2. Then, to get results on the whole space, compact convergence arguments are used.

Definition 5.2.6. Let Π_N be the orthogonal (on L^2) projector on E_0^N and consider the non linear equation :

$$\begin{cases} \partial_t \left(1-\partial_x^2\right) u + \partial_x \left(u + \Pi_N \frac{(\Pi_N u)^2}{2}\right) = 0 \\ u(0) = u_0 \in L^2 \end{cases} . \tag{5.6}$$

Writing $u_0 = \Pi_N u_0 + (1-\Pi_N)u_0 = v_N^0 + w_N^0$, with $v_N^0 \in E_0^N$ and $w_N^0 \in E_{N+1}$, one sees that the problem (5.6) can be reduced to a linear problem with infinite dimension on w_N^0 and a non linear one with finite dimension on v_N^0, that is $u = v_N + w_N$, $v_N \in E_0^N$, $w_N \in E_{N+1}$ satisfying :

$$\partial_t \left(1-\partial_x^2\right) v_N + \partial_x \left(v_N + \Pi_N \frac{(v_N)^2}{2}\right) = 0$$

and

$$\partial_t \left(1-\partial_x^2\right) w_N + \partial_x w_N = 0 \ .$$

Proposition 5.2.7. *The equation*

$$\partial_t \left(1-\partial_x^2\right) v_N + \partial_x \left(v_N + \Pi_N \frac{(v_N)^2}{2}\right) = 0$$

has a unique global solution on E_0^N and μ_0^N is invariant under its flow, noted ϕ_N.

Proof The local uniqueness and existence of the solution is due to the fact that the non linearity is Lipschitz continuous in finite dimension. The global uniqueness and existence comes from the invariance of the H^1-Sobolev norm (equivalent to the L^2 norm in finite dimension) and then, the invariance of the Lebesgue measure from Liouville's theorem for ODEs, see [51] for the proof and further properties of Hamiltonian flows). Indeed, write $F_N(u) = (1-\partial_x^2)^{-1}\partial_x\left(u + \Pi_N \frac{u^2}{2}\right)$. This function (on E_0^N) derives from a Hamiltonian, see the work of Roumégoux, [49] for the details of the proof, of the form

$$F_N(u) = J\nabla f_N(u)$$

where J is an antisymmetric operator

$$J = (1-\partial_x^2)^{-1}\partial_x$$

and f_N is the function :

$$f_N(u) = \int \frac{u^3}{6} \ .$$

From this Hamiltonian form, it appears that F_N is divergence free. Indeed, indexing some basis of E_0^N by i and writing $F_N = (F_N^i)_i$ in this basis, it comes

$$\begin{aligned} \text{div } F_N &= \sum_i \partial_i F_N^i = \sum_i (J\nabla f_N)^i = \sum_{i,j} \partial_i J_j^i (\nabla f_N)^j \\ &= \sum_{i,j} J_j^i \partial_i \partial_j f_N \end{aligned}$$

and J being antisymmetric, this sum is zero, F_N is divergence free.

Now, since F_N is divergence free, the Jacobian of $\phi_N(t)$ does not depend on t, and so it is 1, the Lebesgue measure is invariant under the flow, which is the Liouville theorem. Indeed, its proof gives

$$D_t\left(\text{jac } \phi_N(t)(u_0)\right) = D_t\left(\det\left(d_{u_0}\phi_N(t)\right)\right) = \left(D_{d_{u_0}\phi_N(t)}\det\right) \circ \left(D_t(d_{u_0}\phi_N(t))\right)$$

and D_t and d_{u_0} commute so

$$D_t(d_{u_0}\phi_N(t)) = d_{u_0}(D_t\phi_N(t)) = d_{u_0}F_N \circ \phi_N(t) = d_{\phi_N(t)u_0}F \circ d_{u_0}\phi_N(t)$$

$$D_t\left(\text{jac } \phi_N(t)(u_0)\right) = \text{ Tr}\left((d_{u_0}\phi_N(t))^{-1} \circ d_{\phi_N(t)u_0 F} \circ d_{u_0}\phi_N(t)\right)$$

$$= \text{ Tr}\left(d_{\phi_N(t)u_0 F}\right) = \text{ div } F(\phi_N(t)u_0) = 0 \ .$$

Then, as the H^1 norm is invariant under the flow,

$$d\mu_0^N(u) = d_0^N e^{-\frac{1}{2}\int u(1-\partial_x^2)u} dL(u)$$

is also invariant under the flow. ◊

Proposition 5.2.8. *The measure $\mu = \mu_0^N \otimes \mu_N$ is invariant under the flow of* (5.6)*, noted ψ_N.*

Proof Let $A \subseteq E_0^N$ and $B \subseteq E_{N+1}$ μ_0^N and μ_N measurable respectively. Then,

$$\mu(\psi_N(t)(A \times B)) = \mu((\phi_N(t)A) \times (S(t)B))$$

$$= \mu_0^N(\phi_N(t)A)\mu_{N+1}(S(t)B) = \mu_0^N(A)\mu_{N+1}(B)$$

$$= \mu(A \times B)$$

thanks to the invariance of μ_0^N under ϕ_N and of μ_{N+1} under $S(t)$.

As the proposition holds for every Cartesian products, it holds on all measurable sets. ◊

5.2.3 Invariance under the non linear flow

Definition 5.2.9. For all $T \in \mathbb{R}_+$, set

$$X_T^s = C([-T, T], H^s)$$

normed by $\| \, . \, \|_{L_t^\infty, H_x^s}$.

The following lemma comes from [49] :

Lemma 5.2.10. *Let $s \geq 0$. There exists a constant C_s depending only on (and increasing with) s such that the flow ψ of the BBM equation* (5.2) *is defined on $[-T, T] \times B_R^s$, as long as $T < \frac{1}{C_s R}$. Moreover, if $(t, u_0) \in [-T, T] \times B_R^s$, then*

$$\|\psi(t)(u_0)\|_{H^s}, \|\psi_N(t)(u_0)\|_{H^s} \leq 2R$$

and, calling, for $\|u\|_{X_T^s} \leq 2R$,

$$A(u)(t) = S(t)u_0 - \frac{1}{2}\int_0^t S(t-s)(1-\partial_x^2)^{-1}\partial_x u^2 ds \; ,$$

for all u, v,

$$\|A(u) - A(v)\|_{X_t^s} \leq 2C_s R t \|u - v\|_{X_T^s} \; .$$

The sequence $\psi_N(t)(u_0)$ converges uniformly in $u_0 \in B_R^s$, $s > 0$ for the topology of X_T^0 with a suitable T.

Lemma 5.2.11. *Let $s \in]0, \frac{1}{2}[$ and $R > 0$. Let $\epsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$, all $u_0 \in B_R^s$ and all $t \in [-\frac{1}{3C_sR}, \frac{1}{3C_sR}]$,*

$$\|\psi(t)u_0 - \psi_N(t)u_0\|_{L^2} \leq \epsilon \ .$$

Proof Let $u_0 \in B_R^s$. Call $u = \psi(t)u_0$ and $u_N = \psi_N(t)u_0$. Then, u is a fixed point for A and u_N for A_N such that :

$$A_N(v)(t) = S(t)u_0 - \frac{1}{2}\int_0^t S(t-s)(1-\partial_x^2)^{-1}\partial_x \Pi_N(\Pi_N v(s))^2 ds$$

that is

$$A_N(v)(t) - S(t)u_0 = \Pi_N\left(A(\Pi_N v)(t) - S(t)u_0\right) \ .$$

Thus,

$$\begin{aligned} u - u_N &= A(u) - A_N(u_N) \\ &= A(u) - S(t)u_0 - (A_N(t)u_N - S(t)u_0) \\ &= A(u) - A(\Pi_N u_N) + (1-\Pi_N)(S(t)u_0) \ . \end{aligned}$$

Hence, with $T = \frac{1}{3C_sR}$,

$$\begin{aligned} \|u - u_N\|_{X_T^0} &\leq \|(1-\Pi_N)S(t)u_0\|_{L^2} + \|A(u) - A(\Pi_N u)\|_{X_T^0} \\ &\leq N^{-s}\|S(t)u_0\|_{H^s} + 2C_0TR\|u - \Pi_N u_N\|_{X_T^0} \ . \end{aligned}$$

Then,

$$\|S(t)u_0\|_{H^s} = \|u_0\|_{H^s} \leq R$$

and

$$\|u - \Pi_N u_N\|_{X_T^0} \leq \|u - u_N\|_{X_T^0} + \|u_N - \Pi_N u_N\|_{X_T^0} \leq \|u - u_N\|_{X_T^0} + N^{-s}\|u_N\|_{X_T^s} \ .$$

Finally,

$$\|u - u_N\|_{X_T^0} \leq \frac{2}{3}\|u - u_N\|_{X_T^0} + N^{-s}\left(\frac{2}{3} + R\right)$$

so there exists N_0 depending only on s and R such that for all $N \geq N_0$:

$$\|u - u_N\|_{X_T^0} \leq \epsilon \ .$$

◊

Lemma 5.2.12. *Let* $s \in]0, \frac{1}{2}[$ *and* $R > 0$*. Let* A *be a measurable set of* L^2 *included in* B_R^s*. Let* $T = \frac{1}{3C_sR}$ *, for all* $t \in [-T, T]$,

$$\mu(\psi(t)A) = \mu(A) \ .$$

Proof Suppose that A is closed. There exists N_0 such that for all $N \geq N_0$ and all $u_0 \in A$,

$$\|\psi(t)u_0 - \psi_N(t)u_0\|_{L^2} \leq \epsilon \ .$$

for all $N \geq N_0$, $\psi(t)A \subseteq \psi_N(t)A + B_\epsilon^0$. Then, ψ_N satisfies, thanks to the continuity and the reversibility of the local flow.

$$\psi_N(t)A + B_\epsilon^0 = \psi_N(t)\psi_N(-t)(\psi_N(t)A + B_\epsilon^0) \subseteq \psi_N(t)(\psi_N(-t)\psi_N(t)A + B_{C\epsilon}^0)$$

$$\psi_N(t)A + B_\epsilon^0 \subseteq \psi_N(t)(A + B_{C\epsilon}^0)$$

with a constant C independent from N. So,

$$\mu(\psi(t)A) \leq \mu\left(\psi_N(t)(A + B_{C\epsilon}^0)\right) = \mu(A + B_{C\epsilon}^0)$$

and, with $\epsilon \to 0$, thanks to the dominated convergence theorem

$$\mu(\psi(t)A) \leq \mu(A) \ .$$

For the reverse inequality, $\psi_N(t)A \subseteq \psi(t)A + B_\epsilon^0$, so

$$\mu(\psi(t)A + B_\epsilon^0) \geq \mu(\psi_N(t)A) = \mu(A) \ .$$

If A is open, then A^c the complementary of A in B_R^s is closed and included in B_R^s. So, the local invariance is true for open sets. Then, the flow being unique and reversible, it is also true for countable disjoint unions, and so for all measurable sets. ◊

Build now a set set onto which μ is invariant under the BBM flow and prove that it is of full measure. Then, as it is of full measure, μ is invariant under the flow of BBM.

Definition 5.2.13. Let $s \in]0, \frac{1}{2}[$ and $R > 0$. Let $R_k = \sqrt{k+1}R$ and $t_k = \frac{1}{3C_s\sqrt{k+1}R}$ and $T_0 = 0$, $T_{n+1} = \sum_{k=0}^{n} t_k$. Call $A_N^n(R) = \psi_N(T_n)^{-1}(B_{R_n}^s) \cup \psi_N(-T_n)^{-1}(B_{R_n}^s)$,

$$A_N(R) = \bigcap_{n\geq 0} A_N^n(R)$$

$$A(R) = \limsup_{N\to\infty} A_N(R) \ .$$

Proposition 5.2.14. *There exists two constants $C > 0$ and $a > 0$ such that for all $R > 2$, $\mu(A(R)^c) \leq Ce^{-aR^2}$.*

Proof Indeed,

$$A(R)^c = \liminf A_N(R)^c$$

$$\mu(A(R)^c) \leq \liminf \mu(A_N(R)^c)$$

then,

$$\mu(A_N(R)^c) \leq \sum_{n\geq 0} \mu(A_N^n(R)^c)$$

and

$$\mu(A_N^n(R)^c) = \mu(\psi_N(T_n)^{-1}(B_{R_n}^s)^c) + \mu(\psi_N(-T_n)^{-1}(B_{R_n}^s)^c) \leq 2\mu((B_{R_n}^s)^c) \leq 2e^{-a(n+1)R^2}$$

thanks to the invariance of μ under ψ_N.

$$\mu(A_N(R)^c) \leq 2\sum_{n\geq 0}(e^{-aR^2})^{(n+1)} \leq Ce^{-aR^2}$$

with C independent from R. Hence,

$$\mu(A(R)^c) \leq Ce^{-aR^2}$$

. ◊

Theorem 5.2.15. *Let C be a μ measurable set of L^2. Then, for all $t \in \mathbb{R}$,*

$$\mu(\psi(t)C) = \mu(C) \ .$$

Proof Let $C_R = C \cap A(R)$. As $A(R)$ is μ measurable, C_R is also measurable and included in $A(R)$.

By induction over n, $\psi(\pm T_n)C_R \subseteq B^s_{R_n}$ and for all $t \in [-T_n, T_n]$,

$$\mu(\psi(t)C_R) = \mu(C_R) .$$

Indeed, for $n = 0$, $T_n = 0$, so $\mu(\psi(0)C_R) = \mu(C_R)$. Then, as $C_R \subseteq A(R)$, for all $u \in C_R$, there exits a sequence $N_k \to \infty$ such that $u \in A_{N_k}(R)$, that is $\psi_{N_k}(T_n)(u) \in B^s_{R_n}$ for all n. In particular, for $n = 0$,

$$u = \psi_{N_k}(0)(u) \in B^s_{R_0}$$

and it will also appear by induction that for all n, $\psi_{N_k}(T_n)u$ converges in L^2 toward $\psi(T_n)(u)$ when k goes to ∞.

For $n \to n+1$, suppose that $\psi(T_n)C_R \subseteq B^s_{R_n}$ and for all $t \in [-T_n, T_n]$, $\mu(\psi(t)C_R) = \mu(C_R)$. As $\psi(T_n)C_R \supseteq B^s_{R_n}$ and $\psi(-T_n)C_R \subseteq B^s_{R_n}$, thanks to Lemma 5.2.12, for all $t \in [0, t_n]$,

$$\mu(\psi(t)\psi(T_n)C_R) = \mu(\psi(T_n)C_R) = \mu(C_R)$$

and

$$\mu(\psi(-t)\psi(-T_n)C_R) = \mu(\psi(-T_n)C_R) = \mu(C_R) ,$$

as $T_{n+1} = T_n + t_n$, the invariance is true for $t \in [-T_{n+1}, T_{n+1}]$. Then for all u in C_R,

$$\|\psi(T_{n+1})(u) - \psi_{N_k}(T_{n+1})(u)\|_{L^2} = \|\psi(t_n)\psi(T_n)(u) - \psi_{N_k}(t_n)\psi(T_n)(u)\|_{L^2} +$$

$$\|\psi_{N_k}(t_n)(\psi(T_n)(u)) - \psi_{N_k}(t_n)(\psi_{N_k}(T_n)(u)\|_{L^2} .$$

Thanks to Lemma 5.2.10, there exists a constant independent from N such that $\|\psi_{N_k}(t_n)(u) - \psi_{N_k}(t_n)(v)\|_{L^2} \leq C\|u - v\|_{L^2}$ as long as $u, v \in B^0_{R_n} \subset B^s_{R_n}$, so

$$\|\psi_{N_k}(t_n)(\psi(T_n)(u)) - \psi_{N_k}(t_n)(\psi_{N_k}(T_n)(u)\|_{L^2} \leq C\|\psi(T_n)(u) - \psi_{N_k}(T_n)(u)\|_{L^2} \to 0$$

by induction hypothesis and

$$\|\psi(t_n)\psi(T_n)(u) - \psi_{N_k}(t_n)\psi(T_n)(u)\|_{L^2} \to 0$$

when $k, N_k \to \infty$ thanks to Lemma 5.2.11. So, $\psi_{N_k}(T_{n+1})(u) \in B^s_{R_{n+1}}$ converges toward $\psi(T_{n+1})u$ in L^2 and as $B^s_{R_{n+1}}$ is compact in L^2, $\psi(T_{n+1})u \in B_{R_{n+1}}$.

The induction is proved.

Then, for all $t \in \mathbb{R}$, as $T_n = \sum_{k=1}^n \frac{1}{3C_s R\sqrt{k}} \to \infty$, there exists n such that $t \in [-T_n, T_n]$, so

$$\mu(\psi(t)C_R) = \mu(C_R) .$$

Finally,

$$\mu(\psi(t)C) \geq \mu(\psi(t)C_R) = \mu(C_R)$$

and

$$\mu(C) \leq \mu(C_R) + \mu(A(R)^c) \leq \mu(C_R) + Ce^{-aR^2} \leq \mu(\psi(t)C) + Ce^{-cR^2}$$

it comes

$$\mu(\psi(t)C) \geq \mu(C)$$

and

$$\mu(C) = \mu(\psi(-t)\psi(t)C) \geq \mu(\psi(t)C) \ ;$$

hence

$$\mu(\psi(t)C) = \mu(C) \ .$$

◊

5.3 A new measure and new equations

Now that the statistics μ has been proved to remain invariant through the BBM flow, perturb it a little bit and see if μ is stable. That is, build a statistics μ_V depending on a small parameter V close to 0 and analyse the evolution in time of μ_V, see that it remains close to its initial statistics, and so close to μ.

The new statistics μ_V will admit two different interpretations, depending on the point of view. First, regarding the measure itself, it adds some correlations between the wavelengths. With the statistics μ, the different wavelengths were all independent from each other (the Gaussian variables had been taken independent), with μ_V two different wavelengths will be all the more depending on each other that their wavelengths are close.

The statistics μ_V are also the invariant statistics for the BBM equation onto which the unknown u has been replaced by $\sqrt{1+Vu}$. Developing this expression to the first order in V, one gets a new equation corresponding to adding some external potential or a dispersive term, like frictional or shearing resistance.

5.3.1 Perturbation of the measure

The measure μ shall now be perturbed.

Definition 5.3.1. Let V be a C^2 periodic function. Set $\| \,.\, \|_\infty = \| \,.\, \|_{L^\infty} + \|\partial_x \,.\, \|_{L^\infty} + \|\partial_x^2 \,.\, \|_{L^\infty}$ and suppose that $\|V\|_\infty \leq 1/2$. The operator multiplication by V, also noted V, is defined from L^2 to L^2 and its norm satisfies :

$$\|V\|_0 = \|V\|_{\mathcal{L}(L^2,L^2)} \leq \|V\|_{L^\infty} \ .$$

Proposition 5.3.2. *As $\|V\|_\infty$ is strictly less than 1 and the operator V (multiplication by V) is self-adjoint, the operator on L^2*

$$(1+V)^{-1/2}$$

is well defined and its norm is less than

$$\|(1+V)^{-1/2}\|_0 \leq \frac{1}{\sqrt{1-\|V\|_{L^\infty}}} \leq \sqrt{2}\ .$$

Remark 5.3.1. *The function V is the small parameter by definition, but if one looks at $g = \sqrt{1+V}-1$, it is also a small parameter in the same norm.*

Definition 5.3.3. Let B_N be the matrix of $\Pi_N(1+V)^{-1/2}H^{-1}\Pi_N$ in the orthonormal basis

$$(c_0, c_n, s_n)_{1\leq n\leq N}$$

where $H = (1-\partial_x^2)^{1/2}$, that is the coefficients of B_N are given by

$$(B_N)_{n,m} = \begin{cases} \langle c_n, (1+V)^{-1/2}H^{-1}c_m\rangle & \text{if } n,m\leq N \\ \langle c_n, (1+V)^{-1/2}H^{-1}s_{m-N}\rangle & \text{if } n\leq N\ ,\ m\geq N+1 \\ \langle s_{n-N}, (1+V)^{-1/2}H^{-1}c_m\rangle & \text{if } m\leq N\ ,\ n\geq N \\ \langle s_{n-N}, (1+V)^{-1/2}H^{-1}s_{m-N}\rangle & \text{otherwise.} \end{cases}$$

where $\langle\ ,\ \rangle$ is the scalar product in E_N.

Definition 5.3.4. Call $g^N = (g_0, \ldots, g_N, h_1, \ldots, h_N)$ and let

$$\alpha^N = B_N g^N = (\alpha_0^N, \ldots, \alpha_N^N, \beta_0^N, \ldots, \beta_N^N)\ .$$

The vector α^N is a real centred Gaussian vector of covariance matrix :

$$B_N B_N^*\ .$$

Definition 5.3.5. Set

$$\varphi_V^N = \sum_{n=0}^N \alpha_n^N c_n + \sum_{n=1}^N \beta_n^N s_n\ .$$

The map φ_V^N defines a measure μ_V^N on E_0^N.

Proposition 5.3.6. *Let $0\leq s<\frac{1}{2}$. The sequence φ_V^N converges in $L^2(\Omega, H^s)$. Its limit is called φ_V and induces a measure on L^2 called μ_V. Besides, calling E_V the mean value with respect to μ_V, $E_V(\|u\|_{L^2}^2)$ is uniformly (in V) bounded.*

Proof Let $N \geq M > 0$. The norm of $\varphi_V^N - \varphi_V^M$ is such that :

$$\|\varphi_V^N - \varphi_V^M\|^2_{L^2_\omega, H^s_x} = E(\|\varphi_V^N - \varphi_V^M\|^2_{H^s})$$

$$= E\left(\sum_{n=M+1}^{N} (1+n^2)^s(\alpha_n^N)^2 + (1+n^2)^s(\beta_n^N)^2 + \sum_{n=0}^{M}(1+n^2)^s(\alpha_n^N - \alpha_n^M)^2 + (1+n^2)^s(\beta_n^N - \beta_n^M)^2\right) .$$

The first sum corresponds to the trace :

$$Tr((1-\Pi_M)H^s B_N^* B_N H^s(1-\Pi_M)) = Tr\{(1-\Pi_M)H^{s-1}\Pi_N(1+V)^{-1/2}\Pi_N(1+V)^{-1/2}H^{s-1}(1-\Pi_M)\}$$

$$= Tr\left((1-\Pi_M)H^{2(s-1)}\Pi_N(1+V)^{-1/2}\Pi_N(1+V)^{-1/2}\right)$$

$$\leq Tr\left((1-\Pi_M)H^{2(s-1)}\right)\|\Pi_N(1+V)^{-1/2}\Pi_N(1+V)^{-1/2}\|$$

$$\leq \sum_{n=M+1}^{N} \frac{1}{(1+n^2)^{1-s}}\frac{2}{1-\|V\|_{L^\infty}}$$

$$\leq 4\sum_{n\geq M+1}\frac{1}{(1+n^2)^{1-s}} ,$$

which goes to 0 when $M \to \infty$.

The second is :

$$Tr(\Pi_M H^s(B_N - B_M)^*(B_N - B_M)H^s\Pi_M) =$$

$$Tr(\Pi_M H^{s-1}((1+V)^{-1/2}(\Pi_N - \Pi_M)(1+V)^{-1/2}\Pi_M)H^{s-1}\Pi_M)$$

since H^{-1}, Π_N and Π_M commute.

Then, use the fact that $\Pi_M = (\Pi_M - \Pi_{M/2}) + \Pi_{M/2}$ to get that the trace is less than :

$$\|(1+V)^{-1/2}\|^2_{L^2} Tr((\Pi_N - \Pi_{M/2})H^{2(s-1)}) + Tr(\Pi_{M/2}((1+V)^{-1/2}(\Pi_N - \Pi_M)(1+V)^{-1/2}\Pi_{M/2})) .$$

The first trace is less than

$$C\sum_{m>M/2}\frac{1}{(1+m^2)^{1-s}}$$

which goes to 0 when $M \to \infty$, the second is less than the sum to the square of the Fourier coefficients of $(1+V)^{-1/2}$ of wavelengths bigger than $M/2$. Indeed, if $g \in E_0^{M/2}$ and $h \in E_M$ then $hg \in E_{M/2+1}$. As $(1+V)^{-1/2}$ is C^1, the series of its Fourier coefficients is absolutely convergent and thus the sum of its Fourier coefficients of wavelengths bigger than $M/2$ goes to 0 when M goes to ∞.

So, the sequence φ_V^N is a Cauchy sequence in $L^2(\Omega, H^s)$, hence it converges toward a certain φ_V in H^s.

What is more,

$$E_V(\|u\|_{L^2}^2) = E(\|\varphi_V\|_{L^2}^2) \leq 4 \sum \frac{1}{1+n^2} \ .$$

◇

Example 5.3.7. The covariance between two waves is given by $E(\alpha_n^N \alpha_m^N)$ or the mean value of any combination of α_k^N and β_l^N with $k, l = n$ or m. In particular,

$$E(\alpha_n^N \alpha_m^N) = (B_N B_N^*)_{n,m} = \sum_{k=0}^{2N+1} (B_N)_{n,k} (B_N)_{m,k}$$

$$= \sum_{k=0}^{N} \frac{1}{1+k^2} \langle c_n c_k, (1+V)^{-1/2} \rangle \langle c_m c_k, (1+V)^{-1/2} \rangle + \sum_{k=1}^{N} \frac{1}{1+k^2} \langle c_n s_k, (1+V)^{-1/2} \rangle \langle s_m c_k, (1+V)^{-1/2} \rangle$$

which involves k bigger than $|n-m|/2$ or Fourier coefficients of $(1+V)^{-1/2}$ of wavelengths bigger than $|n-m|/2$. So, the dependence between two waves decreases quite quickly when the difference between the wavelengths increases.

Remark that as $(1+V)^{-1} = \sum (-V)^k$ since V is small, then, for $n \neq m$, $E(\alpha_n^N \alpha_m^N)$ has no zero order in V, it is at least as small as V itself.

5.3.2 Perturbation of the flow

Now, there is an equation whose flow is invariant under the perturbed measure. In finite dimensional approximation, the linear operator $(1 - \partial_x^2)$ is replaced by $(B_N^{-1})^* B_N^{-1}$ (the matrix that appears in the law of α^N as $\Pi_N (1-\partial_x^2) \Pi_N$ was the matrix that appeared in the law of g^N) on E_0^N and $(1-\Pi_N)(1-\partial_x^2)(1-\Pi_N)$ on its orthogonal.

Definition 5.3.8. Let W_N be the operator on E_0^N whose matrix is $(B_N^{-1})^* B_N^{-1}$ in the basis

$$\{c_0, \dots, c_N, s_1, \dots, s_N\}$$

and V_N the operator $\Pi_N \sqrt{1+V} \Pi_N$ on E_0^N such that $W_N = V_N (1-\partial_x^2) V_N$.

On E_0^N, the law of μ_V^N is given by :

$$d\mu_V^N \left(u := \sum a_n c_n + b_n s_n \right) = d_N^V e^{-\frac{1}{2} \int u W_N u} \prod da_n db_n$$

where

$$d_N^V = \sqrt{\det W_N}(2\pi)^{-(2N+1)/2}$$

is a normalization factor.

Remark 5.3.2. *The operator V_N has an inverse on E_0^N and satisfies for all $s, N, V, u \in E_0^N$:*

$$\|V_N u\|_{H^s} \leq 2\|u\|_{H^s} \text{ and } \|V_N^{-1} u\|_{H^s} \leq 4\|u\|_{H^s} .$$

Proof Call I_N the identity of E_0^N and remark that

$$\|(V_N - I_N)u\|_{H^s} \leq \|\Pi_N \sqrt{1+V} - 1\|_{L^\infty}\|u\|_{H^s}$$

and that

$$\|\Pi_N \sqrt{1+V} - 1\|_{L^\infty} \leq \frac{\|V\|_\infty}{\sqrt{1-\|V\|_\infty}} \leq \sqrt{2}\|V\|_\infty$$

so

$$\|V_N u\|_{H^s} \leq \left(1 + \frac{\sqrt{2}}{2}\right)\|u\|_{H^s} \leq 2\|u\|_{H^s}$$

and

$$\|V_N^{-1} u\|_{H^s} \leq \frac{1}{1 - \sqrt{2}/2}\|u\|_{H^s} \leq 4\|u\|_{H^s} .$$

◊

Proposition 5.3.9. *The equation*

$$\begin{cases} \partial_t W_N u + V_N \partial_x (V_N u + \Pi_N \frac{(V_N u)^2}{2}) = 0 \\ u|_{t=0} = u_0 \in E_0^N \end{cases} \tag{5.7}$$

admits a unique global solution $u(t) = \phi_V^N(t)(u_0)$ and

$$E_V(t) = \frac{1}{2}\int u(t) W_N u(t)$$

is invariant under this flow, it does not depend on time t.

Proof The existence and uniqueness of the local flow is given by Cauchy-Lipschitz theorem, as

$$W_N^{-1} = V_N^{-1} H^{-2} V_N^{-1}$$

and the derivation of E_V gives

$$\dot{E}_V = \int u \partial_t (W_N u(t)) = -\int u V_N \partial_x (V_N u + \Pi_N \frac{(V_N u)^2}{2})$$

$$\int \partial_x (V_N u) \left(V_N u + \frac{(V_N u)^2}{2} \right) = \int \partial_x \left(\frac{(V_N u)^2}{2} + \frac{(V_N u)^3}{6} \right) = 0$$

since W_N and V_N are self-adjoint and ∂_t and W_N commute.

What is more, as $\|V\|_\infty \leq 1/2$, W_N is strictly positive, then $\sqrt{E_V}$ is a norm on E_0^N equivalent to all other norms on E_0^N, so the flow is global. ◊

Proposition 5.3.10. *The measure μ_V^N is invariant under ϕ_V^N.*

To prove this proposition, Liouville's theorem is used and so it is required to give the equation (5.7) its Hamiltonian form.

Lemma 5.3.11. *The equation* (5.7) *admits a Hamiltonian formulation.*

Proof Call J_N the operator on E_0^N :

$$J_N = W_N^{-1} V_N \partial_x V_N^{-1} = V_N^{-1} H^{-2} \partial_x V_N^{-1} \ .$$

This operator is antisymmetric, since V_N and H are self-adjoint, ∂_x is antisymmetric and H and ∂_x commute. The equation (5.7) can be written :

$$\partial_t u + J_N \left(V_N^2 u + V_N \Pi_N \frac{(V_N u)^2}{2} \right) = 0 \ .$$

Writing $u = \frac{1}{\sqrt{2\pi}} \sum_{n=-N}^{N} u_n e^{inx}$ and $(V_N)_j^k = \frac{1}{2\pi} \int e^{-ikx} V_N(e^{ijx}) = \frac{1}{2\pi} \int e^{-i(k-j)x} \sqrt{1+V}$, call

$$H(u_{-N}, \ldots, u_N) = H_1 + H_2$$

with

$$H_1 = \frac{1}{2} \int (V_N u)^2 \text{ and } H_2 = \int \frac{1}{6} \int (V_N u)^3 \ ,$$

and

$$F^n(u_{-N}, \ldots, u_N) = F_1^n + F_2^n$$

with

$$F_1^n = \frac{1}{\sqrt{2\pi}} \int e^{-inx} V_N^2 u \text{ and } F_2^n = \frac{1}{\sqrt{2\pi}} \int e^{-inx} V_N \Pi_N \frac{(V_N u)^2}{2} .$$

The function H_1 can be rewritten :

$$H_1 = \frac{1}{2} \sum_{k+l=0} \sum_{n,m=-N}^{N} 1_{|k|\leq N} 1_{|l|\leq N} (V_N)_n^k (V_N)_m^l u_n u_m$$

so its complex derivative with respect to u_n is :

$$\frac{dH_1}{du_n} = \sum_{k+l=0} \sum_{m=-N}^{N} 1_{|k|\leq N} 1_{|l|\leq N} (V_N)_n^k (V_N)_m^l u_m .$$

As V_N is self-adjoint : $(V_N)_n^k = \overline{(V_N)_k^n}$. As V_N and u are real, $(V_N)_m^l = \overline{(V_N)_{-m}^{-l}}$ and $u_m = \overline{u_{-m}}$ so

$$\frac{dH_1}{du_n} = \overline{\sum_{k,m=-N}^{N} (V_N)_k^n (V_N)_m^k u_m} = \overline{F_1^n}$$

or

$$\frac{dH_1}{d\overline{u_n}} = F_n^1$$

therefore $F_1 = V_N^2 u = \nabla H_1$. The function H_2 can be rewritten

$$H_2 = \frac{1}{6} \sum_{k+l+j=0} \sum_{n,m,y=-N}^{N} (V_N)_n^k (V_N)_m^l (V_N)_y^j u_n u_m u_y$$

so its complex derivative with respect to u_n is :

$$\begin{aligned} \frac{dH_2}{du_n} &= \frac{1}{2} \sum_{k+l+j=0} \sum_{m,y=-N}^{N} (V_N)_n^k (V_N)_m^l (V_N)_y^j u_m u_y \\ &= \overline{\frac{1}{2} \sum_{k+l+j=0} \sum_{m,y=-N}^{N} (V_N)_k^n (V_N)_m^{-l} (V_N)_y^{-j} u_m u_y} \\ &= \overline{\frac{1}{2} \sum_{k+l+j=0} (V_N)_k^n (V_N u)_{-l} (V_N u)_{-j}} = \overline{\frac{1}{2} \sum_{k+l+j=0} (V_N)_k^n (V_N u)_{-l-j}^2} = \overline{F_2^n} . \end{aligned}$$

Therefore, the equation (5.7) is written :

$$\partial_t u = -J_N \nabla_u H$$

it has a Hamiltonian form. ◊

Proof of Proposition 5.3.10. The equation being Hamiltonian, the Lebesgue measure on E_0^N, that is, $\prod da_n db_n$ when $u \in E_0^N$ is written $a_0 c_0 + \sum_{n=1}^N a_n c_n + b_n s_n$ is invariant through its flow ϕ_V^N, see the proof of Proposition 5.2.7.

Then, the measure μ_V^N is given by

$$d\mu_V^N(u) = d_V^N e^{-\int u W_N u} da_0 \prod_{n=1}^N da_n db_n$$

with d_V^N a normalisation factor that depends only on the eigenvalues of W_N and thus does not depend on time. What is more

$$E_V = \int \phi_V^N(t) u_0 W_N \phi_V^N(t) u_0$$

does not either depend on time. So the measure μ_V^N is invariant through the flow of the equation (5.7). ◊

Definition 5.3.12. Let ν_V^N the measure on L^2 defined as $\mu_V^N \otimes \mu_{M+1}$.

Proposition 5.3.13. *The equation*

$$\begin{cases} \partial_t((1-\Pi_N)H^2(1-\Pi_N) + W_N)u + \partial_x(1-\Pi_N)u + V_n\partial_x\left(V_N u + \frac{(V_N u)^2}{2}\right) = 0 \\ u|_{t=0} = u_0 \in L^2 \end{cases} \tag{5.8}$$

admits a unique global solution $\psi_V^N(t)(u_0)$.

What is more, ν_V^N is invariant under ψ_V^N.

Proof If u is decomposed as $u = v + w = \Pi_N u + (1-\Pi_N)u$ then the equation (5.8) is equivalent to

$$\partial_t H^2 w + \partial_x w = 0 \text{ with } w_{t=0} = (1-\Pi_N)u_0$$

and

$$\partial_t W_N v + V_N \partial_x \left(V_N v + \Pi_N \frac{(V_N v)^2}{2} \right) \text{ with } v_{t=0} = \Pi_N u_0 \ .$$

Hence the flow $\psi_V^N(t) = \phi_V^N(t) + S(t)$ globally exists and is unique and the measures of the Cartesian products are invariant under the flow, so the measure ν_V^N is invariant for all measurable sets. ◊

5.3.3 Properties of the new measure

In order to prove the invariance of the perturbed measure under the infinite dimensional perturbed flow, first estimate the measure of some compact sets in L^2 (the same as for μ). The proof of the following proposition is greatly resembling what has been done by Tzvetkov in [54].

Proposition 5.3.14. *Let U be an open set of L^2. The measures of U satisfy :*

$$\mu_V(U) \leq \liminf_{N\to\infty} \nu_V^N(U) \ .$$

Proof Let

$$\widetilde{\varphi}_V^N = \varphi_V^N + \varphi - \varphi_0^N \ .$$

This sequence converges in L^2 and thus almost surely in ω toward φ_V. Indeed,

$$E(\|\varphi_V - \widetilde{\varphi}_V^N\|_{L^2}^2) \leq C\left(E(\|\varphi_V - \varphi_V^N\|_{L^2}^2) + E(\|\varphi - \varphi_0^N\|_{L^2}^2)\right) \to_{N\to\infty} 0 \ .$$

Let Z be the set included in Ω such that $\widetilde{\varphi}_V^N$ converges.
Let

$$A = \{\omega \in Z \mid \varphi_V(\omega) \in U\} = (\varphi_V)^{-1}(U) \cap Z$$

$$A^N = \{\omega \in Z \mid \widetilde{\varphi}_V^N(\omega) \in U\} = (\widetilde{\varphi}_V^N)^{-1}(U) \cap Z \ .$$

As $P(Z) = 1$,

$$\mu_V(U) = P(A) \text{ and } \nu_V^N(U) = P(A^N) \ .$$

If $\omega \in A$, as U is open, there exists $\epsilon > 0$ such that $\varphi_V(\omega) + B_\epsilon \subseteq U$. Then, there exists N_0, such that for all $N \geq N_0$,

$$\widetilde{\varphi}_V^N(\omega) \in \varphi_V(\omega) + B_\epsilon \subseteq U$$

as the sequence $\widetilde{\varphi}_V^N$ converges toward φ_V in L_x^2. So, for all $N \geq N_0$,

$$\omega \in A^N$$

$$\omega \in \liminf A^N$$

$$A \subseteq \liminf A^N \ .$$

Hence, thanks to Fatou lemma,

$$\mu_V(U) = P(A) \leq P(\liminf A^N) \leq \liminf P(A^N) = \liminf \nu_V^N(U) .$$

◊

The same property also holds for the μ_V^N :

Proposition 5.3.15. *Let U be an open set of L^2. The measures of U satisfy :*

$$\mu_V(U) \leq \liminf_{N\to\infty} \mu_V^N(U \cap E_0^N) .$$

Proof The proof is very similar to the one above. It uses the convergence of the φ_V^N toward φ_V. ◊

Proposition 5.3.16. *Let $s \in [0, \frac{1}{2}[$. There exists C, c two constants such that for all $R \geq 0$, the measure of the complementary of the closed ball in H^s of radius R satisfies :*

$$\mu_V((B_R^s)^c) \leq Ce^{-cR^2} .$$

Proposition 5.3.17. *There exist C, c two constants such that for all $N_0 \in \mathbb{N}$ and $R \geq 0$:*

$$\mu_V(\{u_0 \mid \|(1 - \Pi_{N_0})u_0\|_{L^2} > R\}) \leq Ce^{-cN_0R^2} .$$

The proofs of the previous propositions are very similar, hence they shall be proved in parallel. For this, the following lemma should prove itself useful.

Lemma 5.3.18. *Let $(a_n)_{n\in\mathbb{Z}}$ such that $\sum_n \frac{a_n^2}{1+n^2} < \infty$. Then*

$$P(|\alpha_0^N a_0 + \sum_{n=1}^N \alpha_n^N a_n + \beta_n^N a_{-n}| > \lambda) \leq 2e^{-c_1\lambda^2/(\sum_n \frac{a_n^2}{1+n^2})} .$$

Proof If $\sum_n \frac{a_n^2}{1+n^2} = 0$ then the inequality is satisfied. Else, for all $t > 0$,

$$P(\alpha_0^N c_0 + \sum_{n=1}^N \alpha_n^N a_n + \beta_n^N a_{-n} > \lambda) = P(e^{t\alpha_0^N a_0 + \sum_{n=1}^N \alpha_n^N a_n + \beta_n^N a_{-n}} > e^{t\lambda})$$

$$P(\alpha_0^N c_0 + \sum_{n=1}^N \alpha_n^N a_n + \beta_n^N a_{-n} > \lambda) \leq e^{-t\lambda} E(e^{t\alpha_0^N a_0 + \sum_{n=1}^N \alpha_n^N a_n + \beta_n^N a_{-n}}) \tag{5.9}$$

Compute this average.

$$E(e^{t\alpha_0^N a_0+\sum_{n=1}^N \alpha_n^N a_n+\beta_n^N a_{-n}}) = \int d_N dx^N e^{t\langle a^N, B_N x^N\rangle} e^{-\langle x^N, x^N\rangle/2}$$

where x^N is of size $2N+1$. So,

$$E(e^{t\alpha_0^N a_0+\sum_{n=1}^N \alpha_n^N a_n+\beta_n^N a_{-n}}) = e^{t^2\langle B_N^* a^N, B_N^* a^N\rangle/2} \ . \tag{5.10}$$

Then, B_N corresponds to the operator $\Pi_N(1+V)^{-1/2}H^{-1}\Pi_N$ so, writing $u_0 = a_0$ and for all $n>0$,

$$u_n = \frac{a_n - ia_{-n}}{2} \ , \ u_{-n} = \frac{a_n + ia_{-n}}{2}$$

such that :

$$a_0 + \sum_{n=1}^N a_n c_n + a_{-n} s_n = \sum_n u_n e^{inx}$$

and

$$(1+V)^{-1/2} = \sum_n V_n e^{inx} \ ,$$

$B_N^* a^N$ corresponds to

$$\sum_n \frac{1_{|n|\le N}}{\sqrt{1+n^2}} \left(\sum_k V_k u_{n-k} 1_{|n-k|\le N} \right)$$

$$\langle B_N^* a^N, B_N^* a^N\rangle = \sum_{n,k_1,k_2} \frac{1_{|n|\le N}}{1+n^2} \overline{V}_{k_1} V_{k_2} \overline{u}_{n-k_1} u_{n-k_2} 1_{|n-k_1|\le N} 1_{|n-k_2|\le N}$$

$$\le \sum_{k_1,k_2,n} |V_{k_1}|\,|V_{k_2}| \frac{|u_{n-k_1}|}{\sqrt{1+n^2}} \frac{|u_{n-k_1}|}{\sqrt{1+n^2}} \frac{|u_{n-k_2}|}{\sqrt{1+n^2}}$$

Then, use that for all n and all k

$$\frac{1}{\sqrt{1+n^2}} \le \frac{\sqrt{2(1+k^2)}}{\sqrt{1+(n-k)^2}}$$

$$\langle B_N^* a^N, B_N^* a^N\rangle \le \sum_{n,k_1,k_2} \sqrt{2(1+k_1^2)}|V_{k_1}| \sqrt{2(1+k_2^2)}|V_{k_2}| \frac{|u_{n-k_1}|}{\sqrt{1+(n-k_1)^2}} \frac{|u_{n-k_2}|}{\sqrt{1+(n-k_2)^2}}$$

$$\sum_n \frac{|u_{n-k_1}|}{\sqrt{1+(n-k_1)^2}} \frac{|u_{n-k_2}|}{\sqrt{1+(n-k_2)^2}} \le \sum_n \frac{|u_n|^2}{1+n^2} = \sum_n \frac{a_n^2}{1+n^2}$$

$$\langle B_N^* a^N, B_N^* a^N \rangle \leq \sum_n \frac{a_n^2}{1+n^2} \left(\sum_k |V_k| \sqrt{2(1+k^2)} \right)^2$$

Now, as V is C^2 and its norm $\|V\|_\infty \leq 1/2$, $(1+V)^{-1/2}$ is also C^2 so the sum :

$$\sum_k |V_k| \sqrt{2(1+k^2)} \leq \sqrt{2} \left(V_k^2 (1+k^2)^2 \right)^{1/2} \left(\sum_k \frac{1}{1+k^2} \right)^{1/2} \leq C \|V\|_\infty$$

converges and is bounded uniformly in V.

Finally, we have that

$$\langle B_N^* a^N, B_N^* a^N \rangle \leq C \sum_n \frac{a_n^2}{1+n^2} \ .$$

Thus, by combining (5.9) and (5.10), we get

$$P(\alpha_0^N c_0 + \sum_{n=1}^N \alpha_n^N a_n + \beta_n^N a_{-n} > \lambda) \leq e^{-t\lambda} e^{t^2 \langle B_N^* a^N, B_N^* a^N \rangle / 2} \ ,$$

Hence, we get

$$P(\alpha_0^N a_0 + \sum_{n=1}^N \alpha_n^N a_n + \beta_n^N a_{-n} > \lambda) \leq e^{-t\lambda} e^{Ct^2 \sum \frac{a_n^2}{1+n^2}/2}$$

With $t = \frac{\lambda}{C\sum(a_n^2/(1+n^2))}$,

$$P(\alpha_0^N a_0 + \sum_{n=1}^N \alpha_n^N a_n + \beta_n^N a_{-n} > \lambda) \leq e^{-\frac{\lambda^2}{2C\sum(a_n^2/(1+n^2))}} \ ,$$

and with the same kind of arguments,

$$P(\alpha_0^N a_0 + \sum_{n=1}^N \alpha_n^N a_n + \beta_n^N a_{-n} < -\lambda) \leq e^{-\frac{\lambda^2}{2C\sum \frac{a_n^2}{1+n^2}}} \ ,$$

so

$$P(|\alpha_0^N a_0 + \sum_{n=1}^N \alpha_n^N a_n + \beta_n^N a_{-n}| > \lambda) \leq 2e^{-\frac{\lambda^2}{2C\sum \frac{a_n^2}{1+n^2}}} \ ,$$

◇

Lemma 5.3.19. *There exists C_1 such that for all $q \geq 1$, and all sequences a_n and all N,*

$$\|a_0\alpha_0^N + \sum_{n=1}^{N} a_n\alpha_n^N + a_{-n}\beta_n^N\|_{L^q_\omega} \leq \sqrt{C_1 q \sum_n \frac{a_n^2}{1+n^2}}\ .$$

Proof

$$\|a_0\alpha_0^N + \sum_{n=1}^{N} a_n\alpha_n^N + a_{-n}\beta_n^N\|^q_{L^q_\omega} = \int q\lambda^{q-1} P(|\alpha_0^N c_0 + \sum_{n=1}^{N} \alpha_n^N a_n + \beta_n^N a_{-n}| > \lambda) d\lambda$$

$$\leq \int q\lambda^{q-1} 2e^{-c_1\lambda^2/2\sum_n \frac{a_n^2}{1+n^2}}$$

By a change a variable $y = \frac{\sqrt{c_1}\lambda}{\sqrt{\sum_n a_n^2/(1+n^2)}}$,

$$\|a_0\alpha_0^N + \sum_{n=1}^{N} a_n\alpha_n^N + a_{-n}\beta_n^N\|^q_{L^q_\omega} \leq 2\left(\sum_n \frac{(a_n)^2}{c_1(1+n^2)}\right)^{q/2} \int qy^{q-1}e^{-y^2/2}dy$$

$$\|a_0\alpha_0^N + \sum_{n=1}^{N} a_n\alpha_n^N + a_{-n}\beta_n^N\|_{L^q_\omega} \leq \left(4\sum_n \frac{(a_n)^2}{c_1(1+n^2)}\right)^{1/2} \left(\int qy^{q-1}e^{-y^2/2}dy\right)^{1/q}$$

The integral which depends on q ,

$$\left(\int qy^{q-1}e^{-y^2/2}dy\right)^{1/q}$$

can be bounded uniformly in q as long as q is taken in a compact. For instance, for all $q \in [1,3]$, by bounding y^{q-1} by y^2 if $y > 1$ and 1 if $y \leq 1$, we get

$$\left(\int qy^{q-1}e^{-y^2/2}dy\right)^{1/q} \leq 2\sqrt{2\pi} = C_1'$$

If $q \geq 3$, by integration by parts, we get the induction formula

$$\int qy^{q-1}e^{-y^2/2}dy = q(q-2)\int y^{q-3}e^{-y^2/2}$$

With $K = \lceil \frac{q-3}{2} \rceil$, and by induction, we get that for all $k \leq K$:

$$\int qy^{q-1}e^{-y^2/2}dy = \prod_{j=0}^{k-1}(q-2j)(q-2k)\int y^{q-2k-1}e^{-y^2/2}dy$$

thus for K,

$$\int qy^{q-1}e^{-y^2/2}dy = \prod_{k=0}^{K-1}(q-2k)(q-2K)\int y^{q-2K-1}e^{-y^2/2}dy$$

and as K has been chosen such that $q - 2k \in [1, 3]$,

$$\int q y^{q-1} e^{-y^2/2} dy \le q^K C_1 \le q^{q/2} C_1' \ .$$

With this bound on the integral depending on q, we can conclude :

$$\|a_0 \alpha_0^N + \sum_{n=1}^{N} a_n \alpha_n^N + a_{-n} \beta_n^N \|_{L^q_\omega} \le \left(C_1 q \sum_n \frac{(a_n)^2}{(1+n^2)} \right)^{1/2}$$

◇

Proof of Propositions 5.3.16,5.3.17. By definition of μ_V^N as the image measure of P by φ_V^N,

$$\mu_V^N((B_R^s)^c \cap E_0^N) = P(\|\varphi_V^N(\omega)\|_{H^s} > R) = P(\|H^s \varphi_V^N(\omega)\|_{L^2} > R) \ .$$

Hence, for all $q \ge 2$,

$$\mu_V^N((B_R^s)^c \cap E_0^N) = P(\|H^s \varphi_V^N(\omega)\|_{L^2}^q > R^q)$$
$$\le R^{-q} E(\|H^s \varphi_V^N(\omega)\|_{L^2}^q) = R^{-q} \|H^s \varphi_V^N(\omega)\|^q_{L^q_\omega, L^2_x}$$

Thanks to Minkowski inequality,

$$\mu_V^N((B_R^s)^c \cap E_0^N) \le R^{-q} \|H^s \varphi_V^N\|^q_{L^2_x, L^q_\omega} \ .$$

Similarly,

$$\mu_V^N(\{u_0 \mid \|(1-\Pi_{N_0})u_0\|_{L^2} > R\}) \le R^{-q} \|(1-\Pi_{N_0})\varphi_V^N\|^q_{L^2_x, L^q_\omega} \ .$$

Then, as

$$H^s \varphi_V^N = \alpha_0^N c_0(x) + \sum_n \left((1+n^2)^{s/2} c_n(x) \alpha_n^N + (1+n^2)^{s/2} s_n(x) \beta_n^N \right)$$

we get

$$\|H^s \varphi_V^N(x)\|_{L^q_\omega} \le \sqrt{C_1 q \left(c_0(x)^2 + \sum_n \frac{c_n(x)^2 + s_n(x)^2}{(1+n^2)^{1-s}} \right)}$$

by taking the L^2_x norm of this expression

$$\|H^s \varphi_V^N(x)\|_{L^2_x, L^q_\omega} \le \sqrt{C_1 q \left(1 + \sum_n \frac{2}{(1+n^2)^{1-s}} \right)}$$

and for the same reasons

$$\|(1-\Pi_{N_0})\varphi_V^N\|_{L^2_x,L^q_\omega} \leq \sqrt{C_1 q \sum_{n>N_0} \frac{2}{1+n^2}} \leq \sqrt{\frac{2C_1}{N_0}} \ .$$

As $s < \frac{1}{2}$, $1-s > \frac{1}{2}$, the series converges. We get

$$\|H^s \varphi_V^N(x)\|_{L^2_x,L^q_\omega} \leq \sqrt{C_2 q}$$

where C_2 depends on s but not on N.

Finally,

$$\mu_V^N((B_R^s)^c \cap E_0^N) \leq \left(\frac{C_2 q}{R^2}\right)^{q/2} \ .$$

Also,

$$\mu_V^N(\{u_0 \mid \|(1-\Pi_{N_0})u_0\|_{L^2} > R\}) \leq \left(\frac{2C_1 q}{N_0 R^2}\right)^{q/2} \ .$$

For $R \leq \sqrt{2eC_2}$, see that with $c = \frac{1}{2eC_2}$,

$$\mu_V^N((B_R^s)^c \cap E_0^N) \leq 1 \leq e^{1/2} e^{-cR^2} = Ce^{-cR^2}$$

and if $R \geq \sqrt{2eC_2}$, by replacing q with $\frac{R^2}{eC_2} \geq 2$,

$$\mu_V^N((B_R^s)^c \cap E_0^N) \leq e^{-q/2} = e^{-R^2/(2eC_2)} = e^{-cR^2} \leq Ce^{-cR^2}$$

hence, for all R and all N

$$\mu_V^N((B_R^s)^c \cap E_0^N) \leq Ce^{-cR^2} \ .$$

With the same kind of arguments,

$$\mu_V^N(\{u_0 \mid \|(1-\Pi_{N_0})u_0\|_{L^2} > R\}) \leq C^{e^{-cN_0R^2}} \ .$$

Now see that B_R^s is closed in L^2 so $(B_R^s)^c$ and $\{u_0 \mid \|(1-\Pi_{N_0})u_0\|_{L^2} > R\}$ are open in L^2,

$$\mu_V((B_R^s)^c)) \leq \liminf \mu_V^N((B_R^s)^c \cap E_0^N) \leq Ce^{-cR^2}$$

and

$$\mu_V(\{u_0 \mid \|(1-\Pi_{N_0})u_0\|_{L^2} > R\}) \leq Ce^{-cN_0R^2} \ .$$

◊

5.4 Convergence of the flows

To study the invariance of the perturbed measure, the property of local uniform convergence of the "finite dimensional" flows on compact sets is needed.

5.4.1 Properties of the finite dimensional operators

First, investigate on the different operators involved.

Lemma 5.4.1. *Let $s < \frac{1}{2}$ and $s_1 > 0$. Let K be the operator defined as :*

$$K = (1 - \partial_x^2)^{-1}\partial_x \ .$$

There exists C such that for all u, v in L^2 and g a linear operator defined on L^1 such that

$$\|g\| := \sup_{n\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} |g_m^n| = \sup_{n\in\mathbb{Z}} \sum_m |\frac{1}{2\pi} \int e^{-inx} g(e^{imx})|$$

is finite, and for all $N \geq 1$,

1. $\|K(g(uv)\|_{H^s} \leq C\|g\| \, \|u\|_{L^2}\|v\|_{L^2}$,
2. *if u, v are in H^{s_1},* $\|K(g(1-\Pi_N)uv) \leq CN^{-s_1}\|g\| \, \|u\|_{H^{s_1}}\|v\|_{H^{s_1}}$.

Proof Write

$$u(x) = \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} u_k e^{ikx}$$

and

$$v(x) = \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} v_k e^{ikx}$$

their Fourier series. As uv belongs to L^1,

$$uv(x) = \frac{1}{2\pi} \sum_{k\in\mathbb{Z}} \left(\sum_l u_{k-l} v_l \right) e^{ikx}$$

$$g(uv)(x) = \frac{1}{\sqrt{2\pi}} \sum_{m\in\mathbb{Z}} e^{imx} \sum_{k\in\mathbb{Z}} g_k^m \sum_{l\in\mathbb{Z}} u_{k-l} v_l \ .$$

Now, give an upper bound of the n-th Fourier coefficient c_n of $g(uv)$,
Reversing the sums over k and l gives :

$$|c_n| \leq \sum_l |v_l| \sum_k |g_k^n|\ |u_{k-l}|$$

and by a Cauchy-Schwartz inequality

$$|c_n| \leq \|v\|_{L^2} \left(\sum_l \left(\sum_k |g_k^n|\ |u_{k-l}| \right)^2 \right)^{1/2} .$$

Now

$$\sum_l \left(\sum_k |g_k^n|\ |u_{k-l}| \right)^2 = \sum_{l,k_1,k_2} |g_{k_1}^n|\ |g_{k_2}^n|\ |u_{k_1-l}||u_{k_2-l}|$$

$$= \sum_{k_1,k_2} |g_{k_1}^n|\ |g_{k_2}^n| \sum_l |u_{k_1-l}||u_{k_2-l}| .$$

Use Cauchy-Schwartz inequality a second time to get :

$$\sum_l |u_{k_1-l}||u_{k_2-l}| \leq \|u\|_{L^2}^2$$

so

$$|c_n| \leq \|u\|_{L^2} \|v\|_{L^2} \sqrt{\sum_{k_1,k_2} |g_{k_1}^n||g_{k_2}^n|}$$

but

$$\sum_{k_1,k_2} |g_{k_1}^n||g_{k_2}^n| = \left(\sum_k |g_k^n| \right)^2 \leq \|g\|^2$$

hence the result. Indeed,

$$\|K(g(uv))\|_{H^s} = \left(\sum_n \frac{n^2}{(1+n^2)^{2-s}} |c_n|^2 \right)^{1/2} \leq \|u\|_{L^2} \|v\|_{L^2} \|g\| \sqrt{\sum_k \frac{k^2}{(1+k^2)^{2-s}}}$$

and the series converges.

For the third one, c_k the k-th Fourier coefficient of $g((1-\Pi_N)(uv))$ is given by

$$c_k = \sum_{l,m} g_m^k 1_{|m|>N} u_{m-l} v_l .$$

See that if $|m| > N$ then or $|m-l| > N/2$, or $|l| > N/2$, so $1_{|m|>N} \leq 1_{|m-l|>N/2} + 1_{|l|>N/2}$ and

$$|c_k| \le \sum_{l,m} |g^k_m| 1_{|m-l|>N/2} |u_{m-l}||v_l| + \sum_{l,m} |g^k_m||u_{m-l}| 1_{|l|>N/2} |v_l| \ .$$

Estimate the second sum as the two of them are symmetrical.

$$\sum_{l,m} |g^k_m||u_{m-l}| 1_{|l|>N/2}|v_l| \le \|(1-\Pi_{N/2})v\|_{L^2} \left(\sum_l \left(\sum_m |g^k_m||u_{m-l}| \right)^2 \right)^{1/2}$$

$$\le (N/2)^{-s_1} \|v\|_{H^{s_1}} \|u\|_{L^2} \|g\| \le (N/2)^{-s_1} \|v\|_{H^{s_1}} \|u\|_{H^{s_1}} \|g\| \ .$$

Hence the result.

◇

Definition 5.4.2. Let $W = \sqrt{1+V} H^2 \sqrt{1+V}$ and $D = W^{-1}(1+V)^{1/2}\partial_x(1+V)^{1/2} = (1+V)^{-1/2}K(1+V)^{1/2}$. Call also $D_N = V_N^{-1} K V_N$ and $K_N = \Pi_N K \Pi_N$.

Lemma 5.4.3. *Let $s < 1/2$. The operators D and D_N are defined and continuous from L^2 to L^2 and there exists C such that for all V, N :*

1. *for all $u, v \in L^2$, $\|K_N(uv)\|_{H^s} \le C\|u\|_{L^2}\|v\|_{L^2}$,*
2. *for all $u, v \in L^2$, $\|D_N V_N^{-1}(\Pi_N(uv))\|_{H^s} \le C\|u\|_{L^2} \ \|v\|_{L^2}$,*
3. *for all $u \in H^s$, $t \in \mathbb{R}$, $\|e^{-tD_N}u\|_{L^2} \le e^{c|t|}\|u\|_{H^s}$,*
4. *for all $u, v \in L^2$, $\|D((1+V)^{-1/2}uv)\|_{H^s} \le C\|u\|_{L^2} \ \|v\|_{L^2}$,*
5. *for all $u \in H^s$, $t \in \mathbb{R}$, $\|e^{-tD}u\|_{H^s} \le e^{c|t|}\|u\|_{H^s}$.*

Proof The first, second and fourth inequalities are consequences of the previous lemma with $g = \Pi_N$ or $g = Id_{L^2}$, using the fact that the norms of the operators V_N, V_N^{-1}, $(1+V)^\alpha$ are uniformly bounded in V and N.

To obtain the third or the fifth one, observe that

$$D_N = V_N^{-1} K V_N \text{ and } D = (1+V)^{-1/2}K(1+V)^{1/2}$$

so the norm of D_N as an operator is uniformly bounded in V and N.

The main problem is that D_N or D are not antisymmetric, so e^{-tD_N} or e^{-tD} can not be an isometry. Nevertheless setting $f(t) = \|e^{-tD_N}u\|_{H^s}$ and using Gronwall lemma, as

$$f(t) \le \int_{t'=0}^t \|D_N e^{-t'D_N}u\|_{H^s} dt' \le c\int_0^t f(t')dt'$$

$$f(t) \le f(0)e^{c|t|} = \|u\|_{H^s} e^{c|t|} \ .$$

◇

5.4.2 Local existence and convergence of the finite dimensional perturbed flows

Show now the local well posedness of the perturbed equations and the uniform convergence of the $2N+1$ dimensional solutions toward the infinite dimensional one on compact sets.

Definition 5.4.4. Let $u_0 \in L^2$ and A_V^N and A_V be defined on X_T^0 as

$$A_V^N(u) = e^{-t(1-\Pi_N)K}(1-\Pi_N)u_0 + e^{-tD_N}\Pi_N u_0 + \int_0^t e^{-(t-s)D_N} D_N V_N^{-1}\Pi_N \frac{(V_N\Pi_N u(s))^2}{2} ds$$

and

$$A_V(u) = e^{-tD}u_0 + \int_0^t e^{-(t-s)D} D(1+V)^{-1/2}\frac{((1+V)^{1/2}u(s))^2}{2} ds \ .$$

Proposition 5.4.5. *There exists C independent from $u_0 \in H^s$ and N such that for all $T \leq 1$, $u, v \in X_T^s$, $s < 1/2$,*

1. $\|A_V^N(u)\|_{X_T^s} \leq C\left(\|u_0\|_{H^s} + T\|u\|_{X_T^s}^2\right)$,
2. $\|A_V(u)\|_{X_T^s} \leq C\left(\|u_0\|_{H^s} + T\|u\|_{X_T^s}^2\right)$,
3. $\|A_V^N(u) - A_V^N(v)\|_{X_T^s} \leq CT\left(\|u\|_{X_T^s} + \|v\|_{X_T^s}\right)\|u-v\|_{X_T^s}$,
4. $\|A_V(u) - A_V(v)\|_{X_T^s} \leq CT\left(\|u\|_{X_T^s} + \|v\|_{X_T^s}\right)\|u-v\|_{X_T^s}$.

Proof Write $A_V^N(u) = I + II + III$ with

$$I = e^{-t(1-\Pi_N)K}(1-\Pi_N)u_0 \ , \ II = e^{-tD_N}\Pi_N u_0$$

and

$$III = \int_0^t e^{-(t-s)D_N} D_N V_N^{-1}\Pi_N \frac{(V_N\Pi_N u(s))^2}{2} ds \ .$$

Using that Π_N and K commute, we get :

$$\|I\|_{X_T^s} \leq \|(1-\Pi_N)u_0\|_{H^s} \leq \|u_0\|_{H^s} \ .$$

Using Lemma 5.4.3, 3., II, we get that :

$$\|II\|_{X_T^s} \leq e^{cT}\|u_0\|_{H^s} \leq C\|u_0\|_{H^s} \ .$$

For III, we start with :

$$\|III\|_{H^s} \leq \int_0^t \|e^{-(t-s)D_N} D_N V_N^{-1}\Pi_N \frac{(V_N\Pi_N u(s))^2}{2}\|_{ds} \ .$$

But, as $\|e^{-(t-s)D_N} f\|_{H^s} \le C\|f\|_{H^s}$ as long as $|t-s| \le 1$, we get

$$\|e^{-(t-s)D_N} D_N V_N^{-1} \Pi_N \frac{(V_N \Pi_N u(s))^2}{2}\|_{H^s} \le C\|D_N V_N^{-1} \Pi_N \frac{(V_N \Pi_N u(s))^2}{2}\|_{H^s}$$

Then, we use Lemma 5.4.3,2. to get the bound :

$$\|e^{-(t-s)D_N} D_N V_N^{-1} \Pi_N \frac{(V_N \Pi_N u(s))^2}{2}\|_{H^s} \le C\|V_N \Pi_N u(s)\|_{L^2}^2$$

and the uniform bound in N and V of V_N to get

$$\|e^{-(t-s)D_N} D_N V_N^{-1} \Pi_N \frac{(V_N \Pi_N u(s))^2}{2}\|_{H^s} \le C\|u\|_{X_T^s}^2 .$$

Hence, taking the integral over time :

$$\|III\|_{H^s} \le CT\|u\|_{X_T^s}^2$$

so

$$\|A_V^N(u)\|_{X_T^s} \le C(\|u_0\|_{H^s} + T\|u\|_{X_T^s}^2) .$$

The same proof holds for 2.

For 3. and 4. , compute $A_V^N(u) - A_V^N(v)$:

$$\begin{aligned} A_V^N(u) - A_V^N(v) &= \int_0^t e^{-(t-s)D_N} D_N V_N^{-1} \Pi_N \frac{(V_N \Pi_N u(s))^2 - (V_N \Pi_N v(s))^2}{2} ds \\ &= \int_0^t e^{-(t-s)D_N} D_N V_N^{-1} \Pi_N \frac{(V_N \Pi_N (u+v))(V_N \Pi_N (u-v))}{2} ds \end{aligned}$$

Hence, with the same computation as before for III :

$$\|A_V^N(u) - A_V^N(v)\|_{X_T^s} \le CT\|u+v\|_{X_T^s}\|u-v\|_{X_T^s} \le CT\left(\|u\|_{X_T^s} + \|v\|_{X_T^s}\right)\|u-v\|_{X_T^s} .$$

◊

Proposition 5.4.6. *Let $R > 0$, and $0 \le s < 1/2$, there exists C_s such that for all $u_0 \in B_R^s$, the flows ψ_V^N of* (5.8) *and ψ_V of :*

$$\begin{cases} \partial_t\left((1+V)^{1/2}(1-\partial_x^2)(1+V)^{1/2}\right)u + (1+V)^{1/2}\partial_x\left((1+V)^{1/2}u + \frac{((1+V)^{1/2}u)^2}{2}\right) = 0 \\ u|_{t=0} = u_0 \end{cases} \tag{5.11}$$

are defined for $t \in [-T, T]$ with $T = \frac{1}{8C_s^2 R}$ and satisfy

$$\|\psi_V(u_0)\|_{X_T^s} , \ \|\psi_V^N(u_0)\|_{X_T^s} \le 2C_s R .$$

Proof For all $u_0 \in B_R^s$ and $u, v \in X_T^s$ with $\|u\|_{X_T^s}, \|v\|_{X_T^s} \leq 2C_s R$,

$$\|A_V(u)\|_{X_T^s}, \|A_V^N(u)\|_{X_T^s} \leq C_s(R + T(2C_s R)^2) \leq 2C_s R$$

and

$$\|A_V(u) - A_V(v)\|_{X_T^s}, \|A_V^N(u) - A_V^N(v)\|_{X_T^s} \leq C_s(4C_s R)T\|u - v\|_{X_T^s} = \frac{1}{2}\|u - v\|_{X_T^s}$$

so both A_V^N and A_V have a unique fixed point in the ball of radius $2C_s R$ in X_T^s. ◊

As for BBM, there is a property of uniform convergence on compact sets :

Lemma 5.4.7. *Let $\epsilon > 0$ and $R > 0$. There exists $N_0 \in \mathbb{N}$ such that for all $u_0 \in B_R^s$, for all $N \geq N_0$, and $T \leq \frac{1}{C_2 R}$,*

$$\|\psi_V(u_0) - \psi_V^N(u_0)\|_{X_T^0} \leq \epsilon \ .$$

Proof Let $u = \psi_V(u_0)$ and $u^N = \psi_V^N(u_0)$,

$$\|u - u^N\|_{X_T^s} = \|A_V(u) - A_V^N(u^N)\|_{X_T^0} = \|I + II + III + IV + V\|_{X_T^0}$$

with

$$I = (1 - \Pi_N)S(t)u_0$$

$$II = (e^{-tD} - e^{-tD_N})u_0$$

$$III = \int_0^t \left(e^{-(t-s)D}D(1+V)^{-1/2} - e^{-(t-s)D_N}D_N V_N^{-1}\Pi_N\right)\frac{((1+V)^{1/2}u)^2(s)}{2}ds \ ,$$

$$IV = \int_0^t e^{-(t-s)D_N}D_N V_N \Pi_N\left(\frac{((1+V)^{1/2}u)^2(s)}{2} - \frac{(V_N\Pi_N u)^2(s)}{2}\right)ds$$

and

$$V = \int_0^t e^{-(t-s)D_N}D_N V_N \Pi_N\left(\frac{(V_N u)^2(s)}{2} - \frac{(V_N u^N)^2(s)}{2}\right)ds \ .$$

First, remark that

$$\|(D - D_N)(uv)\|_{L^2} \leq C_s N^{-s}\|u\|_{H^s}\|v\|_{H^s} \ .$$

Indeed, $D - D_N$ can be written,

$$D - D_N = \left((1+V)^{-1/2} - V_N^{-1}\right) K (1+V)^{1/2} + V_N^{-1} K \left((1-\Pi_N)(1+V)^{1/2}\right)$$

$$+ V_N^{-1} K (1-\Pi_N)(1+V)^{1/2}(1-\Pi_N)$$

As operators, the multiplication by $(1+V)^{1/2}$ and V_N are quite close. Computing the difference between their inverse gives

$$\|(1+V)^{-1/2} - V_N^{-1}\|_{H^s\to L^2} \leq \|1-\Pi_N\|_{H^s\to L^2}\left(1 + \sum_k k\|(1+V)^{-1/2} - 1\|^k_{H^s\to H^s}\right) \leq CN^{-s}$$

as $\|(1+V)^{-1/2} - 1\|_{H^s\to H^s} \leq \|(1+V)^{-1/2} - 1\|_{L^\infty} \leq \frac{1}{\sqrt{2}}$.

Therefore, it appears that :

$$\|\left((1+V)^{-1/2} - V_N^{-1}\right) K (1+V)^{1/2}(uv)\| \leq CN^{-s}\|K(1+V)^{1/2}(uv)\|_{H^s}$$

and as $\sum_m |(1+V)^{1/2}|_m^n = \sum_m |(1+V)^{1/2}|_{n-m}$ is the sum of the Fourier coefficients of $\sqrt{1+V}$ which is C^2 with a uniform bound on its second derivative, it comes that

$$\|K(1+V)^{1/2}(uv)\|_{H^s} \leq C_s\|u\|_{L^2}\|v\|_{L^2} \leq C_s\|u\|_{H^s}\|v\|_{H^s}$$

with C_s a constant that neither depends on N nor on V :

$$\|\left((1+V)^{-1/2} - V_N^{-1}\right) K (1+V)^{1/2}(uv)\| \leq C_s N^{-s}\|u\|_{H^s}\|v\|_{H^s} .$$

As the norm of V_N as an operator from H^s to H^s is bounded uniformly in V and K and Π_N commute,

$$\|V_N^{-1} K \left((1-\Pi_N)(1+V)^{1/2}\right)\|_{H^s} \leq C\|K\left((1-\Pi_N)(1+V)^{1/2}\right)(uv)\|_{H^s}$$

$$\leq C_s N^{-s}\|(1+V)^{1/2}u\|_{L^2}\|v\|_{L^2} \leq C_s N^{-s}\|u\|_{H^s}\|v\|_{H^s} .$$

The same goes for $V_N^{-1}K(1-\Pi_N)(1+V)^{1/2}(1-\Pi_N)$ as the sum of the Fourier coefficients of $(1+V)^{1/2}$ are uniformly bounded in V so

$$\|(D - D_N)(uv)\|_{L^2} \leq C_s N^{-s}\|u\|_{H^s}\|v\|_{H^s} .$$

After what the I and II are less than $C_s N^{-s} R$, III and IV are less than $C_s N^{-s} R^2$ and V is less than :

$$V \leq C_s T R \|u - u^N\|_{X_T^0} .$$

Indeed,

$$\|II\|_{X_T^0} \leq T\|(D - D_N)e^{-tD}u_0\|_{L^2} + T\|D_N II\|_{X_T^0}$$

$$\leq CTN^{-s}\|u_0\|_{H^s} + CT\|II\|_{X_T^0}$$

so for T small enough, $T \leq \frac{1}{2C_s R}$, the uniform convergence is satisfied. ◊

5.4.3 Invariance of the perturbed measure under the perturbed flow

Show now that the perturbed measure is invariant trough the perturbed flow. For that, the techniques used are basically the same as in the first section, in particular regarding the local invariance.

Lemma 5.4.8. *Let $s \in]0, \frac{1}{2}[$ and $R > 0$. Let A be a measurable set of L^2 included in B_R^s. Let $T = \frac{1}{CR}$ with C depending on s big enough , for all $t \in [-T, T]$,*

$$\mu(\psi_V(t)A) = \mu(A) \ .$$

Proof Use the invariance of ν_V^N through ψ_V^N, the uniform convergence of ψ_V^N toward ψ_V, the uniform continuity of the flows ψ_V^N and ψ_V and the fact that for all open set U :

$$\mu_V(U) \leq \liminf \nu_V^N(U) \ .$$

Indeed, if A is closed, as $A + B'_\epsilon$ is open (the B' denotes the open ball in L^2) :

$$\mu_V(\psi_V(t)(A + B'_\epsilon)) \leq \mu_V(\psi_V(t)(A) + B'_{C\epsilon}) \leq \liminf \nu_V^N(\psi_V(t)A + B'_{C\epsilon})$$

Then, use that $\psi_V(t)A \subseteq \psi_V^N(t)A + B'_\epsilon$ above a certain N.

$$\mu_V(\psi_V(t)(A + B'_\epsilon)) \leq \liminf \nu_V^N(\psi_V^N(t)AB'_{C\epsilon}) \leq \limsup \nu_V^N(\psi_V^N(t)A + B_{C\epsilon})$$

and as the flow is locally continuous in L^2,

$$\mu_V(\psi_V(t)(A + B'_\epsilon)) \leq \limsup \nu_V^N(\psi_V^N(t)(A + B_{C'\epsilon}))$$

and as ν_V^N invariant through ψ_V^N,

$$\mu_V(\psi_V(t)(A + B'_\epsilon)) \leq \limsup \nu_V^N(A + B_{C'\epsilon}) \leq \mu_V(A + B_{C\epsilon})$$

and by the dominated convergence theorem when ϵ goes to 0,

$$\mu_V(\psi_V(t)(A)) \leq \mu_V(A) \ .$$

For the reverse inequality, consider that above a certain N, $\psi_V^N(t)A \subseteq \psi_V(t)A + B'_\epsilon$, so

$$\mu_V(A + B'_\epsilon) \leq \liminf \nu_V^N(A + B'_\epsilon) \leq \liminf \nu_V^N(\psi_V^N(t)A + B'_{C\epsilon})$$

$$\mu_V(A+B'_\epsilon) \leq \liminf \nu_V^N(\psi_V(t)A + B'_{C'\epsilon}) \leq \limsup \nu_V^N(\psi_V(t)A + B_{C'\epsilon}) \leq \mu_V(\psi_V(t)A + B_{C'\epsilon})$$

and by DCT when ϵ goes to 0, as A is closed and thus $\psi(t)A = (\psi(-t))^{-1}A$ is closed too :

$$\mu_V(A) \leq \mu_V(\psi(t)A) \ .$$

Hence the lemma is true for closed sets. For all the measurable sets, see that B_R^s is closed in L^2 so the property passes to the complementary and the countable unions thanks to the uniqueness of the local flow. $\diamond$

Then, define the sets where the solution exists globally in time in quite the same way as in the first section.

Definition 5.4.9. Let $R > 1$ and $R_n = \sqrt{n+1}R$ for $n \geq 0$ and $t_n = \frac{1}{3C_s\sqrt{n}R}$ for $n \geq 1$, $T_n = \sum_{k=1}^n t_n$. Call

$$A_{V,n}^N(R) = \{\varphi_V(\omega) | \varphi_V^N(\omega) \in \phi_V^N(T_n)^{-1}(B_{R_n}^s)\}$$
$$A_{V,-n}^N(R) = \{\varphi_V(\omega) | \varphi_V^N(\omega) \in \phi_V^N(-T_n)^{-1}(B_{R_n}^s)\}$$

and then

$$A_V^N(R) = \bigcap_{n\in\mathbb{Z}} A_{V,n}^N(R)$$

$$A_V(R) = \limsup_{N\to\infty} A_V^N(R)$$

and even

$$A_V = \bigcup_{M\geq 2} A_V(M) \ .$$

Proposition 5.4.10. *The set $A_V(R)$ is such that its complementary satisfies :*

$$\mu_V(A_V(R)^c) \leq Ce^{-cR^2}$$

and thus

$$\mu_V(A_V^c) = 0 \ .$$

Proof Consider the sets restricted to the ω such that φ_V^N converges in H^s, given that the sequence converges in $L^2(\Omega, H^s)$ and thus almost surely.

It appears that

$$\mu_V(A_{V,n}^N(R)^c) = P(\varphi_V^N(\omega) \notin \phi_V^N(T_n)^{-1}(B_{R_n}^s))$$

$$\mu_V(A^N_{V,n}(R)^c) = \mu^N_V(\left(\phi^N_V(T_n)^{-1}(B^s_{R_n})\right)^c)$$

$$\mu_V(A^N_{V,n}(R)^c) \leq \mu^N_V(\phi^N_V(T_n)^{-1}(B^s_{R_n})^c)$$

and μ^N_V is invariant through the flow ϕ^N_V so

$$\mu_V(A^N_{V,n}(R)^c) \leq 2\mu^N_V(B^s_{R_n}) \leq Ce^{-cR_n^2} \leq Ce^{-c(n+1)R^2} \ .$$

Then,

$$\mu_V(A^c_{V,n}) \leq \liminf \mu_V(A^N_{V,n}(R)^c) \leq Ce^{-c(n+1)R^2}$$

$$\mu_V(A_V(R)^c) \leq C \sum_{n\geq 1} 2e^{-cnR^2} \leq C' e^{-cR^2}$$

and

$$\mu_V(A_V(R)^c) \leq \liminf \mu_V(A^N_V(R)) \leq Ce^{-cR^2}$$

$$\mu_V(A^c_V) = 0 \ .$$

◊

Proposition 5.4.11. *The flow ψ_V is unique and globally defined on A_V and the measure μ_V is invariant through this flow.*

Proof As it happens, the proof is roughly the same as in Proposition 5.2.15. It uses however the fact that $\varphi^N_V + \varphi_{N+1}$ converges almost surely toward φ_V. Indeed, to study the convergence in H^s at the times $T_n \to \infty$, see that

$$\psi_V(T_n)(\varphi_V(\omega))$$

is the H^s limit when $N \to \infty$ of

$$\phi^N_V(T_n)(\varphi^N_V(\omega))$$

as

$$\begin{aligned}\psi_V(T_n)(\varphi_V(\omega)) - \phi^N_V(T_n)(\varphi^N_V(\omega)) &= \psi_V(T_n)(\varphi_V(\omega)) - \psi^N_V(\varphi_V(\omega)) + \psi^N_V(\varphi_V(\omega)) - \phi^N_V(T_n)(\varphi^N_V(\omega)) \\ &= \psi_V(T_n)(\varphi_V(\omega)) - \psi^N_V(\varphi_V(\omega)) + \psi^N_V(\varphi_V(\omega)) - \psi^N_V(\varphi^N_V(\omega) + \varphi_{N+1}(\omega)) + S(t)\varphi_{N+1}(\omega)\end{aligned}$$

and as $\psi^N_V(T_n)$ is continuous the sequence converges in H^s. ◊

5.5 Evolution of characteristic functionals

The statistics μ_V are not too much changed by the flow of the original BBM equation. Though, to investigate about those changes, build the characteristic functional of $\psi(t)(\mu_V)$. Estimations on the characteristic functionals seem relevant, in the sense that they contain all the information about the image measure, and they give precise estimates regarding the small parameter V.

5.5.1 Definition of the generating functionals

Introduce now the definition of the generating functionals.

Definition 5.5.1. Let $\lambda \in L^2$ and let $\langle \lambda, u\rangle$ the scalar product in L^2 of u and λ. Call then $Z_V(\lambda)$ the quantity :

$$Z_V(\lambda) = E_V\left(e^{i\langle \lambda, u\rangle}\right)$$

where E_V denotes the average over μ_V.

Remark 5.5.1. *This functional is the characteristic function of μ_V.*

When V is equal to 0, this functional is equal to

$$Z_{V=0}(\lambda) = e^{-\|\lambda\|^2_{H^{-1}}/2}$$

adopting the convention

$$\|\lambda\|_{H^{-1}} = \left(\frac{1}{2\pi}\langle \lambda, c_0\rangle^2 + \frac{1}{2\pi}\sum_{n\geq 1}\frac{1}{1+n^2}(\langle \lambda, c_n\rangle^2 + \langle \lambda, s_n\rangle^2)\right)^{1/2} .$$

When V is different from 0,

$$Z_V(\lambda) = e^{-\|(1+V)^{-1/2}\lambda\|_{H^{-1}}} .$$

Introduce now the generating functional monitoring the behaviour of the BBM flow.

Definition 5.5.2. Let $Z_V(t,\lambda)$ be the quantity :

$$Z_V(t,\lambda) = E_V(e^{i\langle \lambda, \psi(t)u\rangle}) .$$

Remark 5.5.2. *First, see that if ψ is replaced by ψ_V, this quantity remains the same in time, as μ_V is invariant through ψ_V and $\psi_V(t)$ is almost surely defined.*

Then, it is sufficient to study the interaction between the different waves since the covariance between two modes is given by the behaviour of Z as :

$$E_V(\langle \psi_V(t)u_0, c_n\rangle\langle \psi_V(t)u_0, c_n\rangle) = -D^2Z|_{\lambda=0}(c_n)(c_m) .$$

where the right hand term is the second order differential of Z at the point $\lambda = 0$ under the directions c_n and c_m. And those quantities are well defined.

5.5.2 Closeness of the flows

First, prove the global existence of the BBM and the perturbed flow, as in [5], along with some useful estimates.

Definition 5.5.3. For all $u_0 \in L^2$ and $T \in \mathbb{R}$ call

$$N(u_0, T) = \min\{N \in \mathbb{N} \mid \|(1-\Pi_N)u_0\|_{L^2} \leq \frac{1}{C(1+|T|)}\}$$

where C is the constant involved in the L^2 local well posedness of the BBM and the perturbed flow.

Proposition 5.5.4. *Let $s \in [0, \frac{1}{2}[$ and $\sigma \in]\frac{1}{2}, 1]$. There exists C such that for all $u_0 \in H^s$, the flows ψ and ψ_V are globally defined in L^2, and for all $T \in \mathbb{R}$,*

$$\|\psi(t)u_0\|_{L^2}\ ,\ \|\psi_V(t)u_0\|_{L^2} \leq C + CN(u_0,T)^{(1+\sigma-2s)/2}\|u_0\|_{H^s}\ .$$

Proof Fix $T \in \mathbb{R}$ and let $v_0 = (1-\Pi_{N_0})u_0$ and $w_0 = \Pi_{N_0}u_0$. Thanks to the local well-posedness, $\psi(t)v_0$ and $\psi_V(t)v_0$ are defined in $[-T,T]$ in L^2 and they satisfy

$$\|\psi(t)v_0\|_{X_T^0}\ ,\ \|\psi_V(t)v_0\|_{X_T^0} \leq C\|v_0\|_{L^2} \leq \frac{C}{1+|T|}\ .$$

Call $v = \psi(t)v_0$ and $v_V = \psi_V(t)v_0$. Consider now the equations

$$\partial_t(1-\partial_x^2)w = -\partial_x\left(w + vw + \frac{w^2}{2}\right) \tag{5.12}$$

and

$$\partial_t W w_V = -(1+V)^{1/2}\partial_x\left((1+V)^{1/2}w_V + (1+V)w_V v_V + (1+V)\frac{w_V^2}{2}\right)\ . \tag{5.13}$$

Those equations are well posed in H^1 as long as v and v_V exist and have a priori bounds. Indeed, calling

$$f(t) = \|w(t)\|_{H^\sigma}\|w(t)\|_{H^1} \text{ and } f_V(t) = \|w_v(t)\|_{H^\sigma}\|w_v(t)\|_{H^1}$$

it comes that for $t \in [-T,T]$,

$$f(t) \leq \|w(t)\|_{H^1}^2 \leq |\int_0^t \int_x v(\partial_x w)w| \leq \|v\|_{X_T^0}\int_0^t \|w\|_{H^1}\|w\|_{L^\infty}$$

and thanks to Sobolev embedding theorem ($\sigma > 1/2$)

$$f(t) \leq \|v\|_{X_T^0}\int_0^t f(t)$$

also, with C a constant independent from V,

$$f_V(t) \le 2\int_0^t \int_x w_V W w_V \le C\|v_V\|_{X_T^0} \int_0^t f_V(t) \ .$$

Since w_0 is in H^1, the equations (5.12) and (5.13) are well posed on $[-T, T]$ with initial datum w_0 and

$$f(t), f_V(t) \le e^{C\frac{|T|}{1+|T|}} \|w_0\|_{H^1} \|w_0\|_{H^\sigma} \ .$$

Now, it appears that

$$\|w_0\|_{H^1} \le N(u_0, T)^{1-s} \|u_0\|_{H^s} \text{ and } \|w_0\|_{H^\sigma} \le N(u_0, T)^{\sigma - s} \|u_0\|_{H^s}^2$$

so

$$f(t), f_V(t) \le CN(u_0, T)^{1+\sigma-2s} \|u_0\|_{H^s} \ .$$

The functions $u = v + w$ and $u_V = v_V + w_V$ are solution respectively of the BBM and the perturbed flow with initial datum u_0 and

$$\|u(T)\|_{L^2} \le \|v\|_{X_T^0} + \|w(T)\|_{L^2} \le \frac{C}{1+|T|} + f(t)^{1/2} \le C + CN^{(1+\sigma-2s)/2} \|u_0\|_{H^s}$$

so

$$\|\psi(t)u_0\|_{L^2},\ \|\psi_V(t)u_0\|_{L^2} \le C + CN^{(1+\sigma-2s)/2} \|u_0\|_{H^s} \ .$$

$\diamond$

Proposition 5.5.5. *Let $s \in [0, \frac{1}{2}[$ and $\sigma \in]\frac{1}{2}, 1]$. There exist C such that for all $u_0 \in H^s$ and $T \in \mathbb{R}$,*

$$\|\psi_V(T)u_0 - \psi(T)u_0\|_{L^2} \le C\|V\|_\infty (1 + N^{1+\sigma-2s}\|u_0\|_{H^s}^2 + \|u_0\|_{L^2}) e^{c(1+N^{(1+\sigma-2s)/2}\|u_0\|_{H^s})|T|}$$

where $N = N(u_0, T)$ has been defined at the beginning of the subsection, by Definition 5.5.3.

Proof First, compute $\|(K-D)(uv)\|_{L^2}$ for all $u, v \in L^2$.

$$K - D = (1 - (1+V)^{-1/2})K + (1+V)^{1/2}K(1 - (1+V)^{1/2}) \ .$$

As $\|(1+V)^{-1/2} - 1\|_{L^\infty} \le C\|V\|_\infty$ and $\|(1+V)^{1/2} - 1\|_{L^\infty} \le C\|V\|_\infty$, the following result is ensured : for all $u, v \in L^2$,

$$\|(K-D)(uv)\|_{L^2} \le C\|V\|_\infty \|u\|_{L^2}\ \|v\|_{L^2} \ .$$

Then, let $v \in L^2$ and $g(t) = \|e^{-tD}v - e^{-tK}v\|_{L^2}$, then

$$g'(t) = \|(K-D)e^{-tK}v + D(e^{-tK}v - e^{-tD}v)\|_{L^2} \leq \|K-D\|_0\|v\|_{L^2} + \|D\|_0 g(t)$$

$$g(t) \leq g(0) + \|K-D\|_0\|v\|_{L^2}|t|e^{|t|\,\|D\|_0}$$

and $g(0) = 0$. So, the following inequality applies : there exist C, c two constants such for all $v \in L^2$, all $V \in C^1$, and all $t \in \mathbb{R}$,

$$\|e^{-tD}v - e^{-tK}v\|_{L^2} \leq C\|V\|_\infty\|v\|_{L^2}e^{c|t|}\ .$$

Write

$$u_V = \psi_V(t)u_0 = e^{-tD}u_0 + \int_O^t e^{(s-t)D}D\frac{(1+V)^{1/2}u_V^2(s)}{2}ds$$

and

$$u = \psi(t)u_0 = e^{-tK}u_0 + \int_0^t e^{(s-t)K}K\frac{(\psi(s)u_0)^2(s)}{2}ds\ .$$

Let now

$$f(t) = \|u_V - u\|_{L^2}$$

$$f(t) \leq \|e^{-tD}u_0 - e^{-tK}u_0\|_{L^2} + \|\int_0^t \left(e^{(s-t)D}D\frac{(1+V)^{1/2}u_V^2(s)}{2} - e^{(s-t)K}K\frac{u^2(s)}{2}d\right)\|_{L^2}$$

The integral term is less than :

$$\|\int_0^t \left(e^{(s-t)D}D\frac{(1+V)^{1/2}u_V^2(s)}{2} - e^{(s-t)K}K\frac{u^2(s)}{2}d\right)\|_{L^2} \leq I + II + III + IV$$

with

$$\begin{aligned}
I &= \|\int_0^t \left(e^{(s-t)D}D\frac{(1+V)^{1/2}u_V^2(s)}{2} - e^{(s-t)K}D\frac{(1+V)^{1/2}u_V^2(s)}{2}\right)\|_{L^2} \\
II &= \|\int_0^t \left(e^{(s-t)K}D\frac{(1+V)^{1/2}u_V^2(s)}{2} - e^{(s-t)K}K\frac{(1+V)^{1/2}u_V^2(s)}{2}\right)\|_{L^2} \\
III &= \|\int_0^t \left(e^{(s-t)K}K\frac{(1+V)^{1/2}u_V^2(s)}{2} - e^{(s-t)K}K\frac{u_V^2(s)}{2}\right)\|_{L^2} \\
IV &= \|\int_0^t \left(e^{(s-t)K}K\frac{u_V^2(s)}{2} - e^{(s-t)K}K\frac{u^2(s)}{2}\right)\|_{L^2}
\end{aligned}$$

Estimate now the different terms. For all $t \in [-T, T]$,

$$I \leq \int_0^t C\|V\|_\infty \|D(\frac{(1+V)^{1/2}(u_V)^2(s)}{2} e^{c|t-s|}\|_{L^2} ds \leq C\|V\|_\infty (1 + N^{1+\sigma-2s}\|u_0\|_{H^s}^2) e^{c|t|}$$

As K is antisymmetric, e^{tK} is isometric in L^2 so :

$$II \leq \int_0^t \|(D-K)\left(\frac{\sqrt{1+V}u_V^2(s)}{2}\right)\|_{L^2} ds \leq C\|V\|_\infty |t| (1 + N^{1+\sigma-2s}\|u_0\|_{H^s}^2)$$

$$II \leq C\|V\|_\infty (1 + N^{1+\sigma-2s}\|u_0\|_{H^s}^2) e^{c|t|}$$

$$III \leq \int_0^t \left(\|K \frac{((1+V)^{1/2}-1)u_V^2(s)}{2}\|_{L^2}\right) \leq C\|V\|_\infty |t| (1 + N^{1+\sigma-2s}\|u_0\|_{H^s}^2)$$

$$III \leq C\|V\|_\infty (1 + N^{1+\sigma-2s}\|u_0\|_{H^s}^2) e^{c|t|}$$

$$IV \leq \int_0^t C\|u(s) + u_V(s)\|_{L^2} \|u(s) - u_V(s)\|_{L^2} \leq C(1 + N^{(1+\sigma-2s)/2}\|u_0\|_{H^s}) \int_0^t f(s) ds \ .$$

To sum up, for all $t \in [-T, T]$, $f(t)$ is less than :

$$f(t) \leq C\|V\|_\infty \left(\|u_0\|_{L^2} + 1 + N^{1+\sigma-2s}\|u_0\|_{H^s}^2\right) e^{c|t|} + C(1 + N^{(1+\sigma-2s)/2}\|u_0\|_{H^s}) \int_0^t f(s) ds$$

so

$$f(t) \leq C\|V\|_\infty \left(\|u_0\|_{L^2} + 1 + N^{1+\sigma-2s}\|u_0\|_{H^s}^2\right) e^{c|t|} e^{c(1+N^{(1+\sigma-2s)/2}\|u_0\|_{H^s})|t|}$$

$$f(t) \leq C\|V\|_\infty \left(\|u_0\|_{L^2} + 1 + N^{1+\sigma-2s}\|u_0\|_{H^s}^2\right) e^{c(1+N^{(1+\sigma-2s)/2}\|u_0\|_{H^s})|t|} \ .$$

◇

5.5.3 Evolution of the perturbed statistics

Now see that the law of $\psi(t)u_0$ is not too different from the law of u_0. For this use the generating functional

$$Z_V(\lambda, t) = E_V(e^{i\langle \lambda, \psi(t)u_0 \rangle})$$

and prove that it is quite close to its initial data.

Theorem 5.5.6. *Let $\epsilon \in]0, \frac{1}{2}[$. There exists C, c such that for all $V \in C^2$ with $\|V\|_\infty < \frac{1}{2}$, all $\lambda \in L^2$, and all $t \in \mathbb{R}$,*

$$|Z_V(\lambda, t) - Z_V(\lambda, 0)| \leq C\|V\|_\infty \|\lambda\|_{L^2} e^{c|t|^{6/\epsilon - 3}} \ .$$

Remark 5.5.3. *The function $x \mapsto e^{ix}$ used in the generating functional could have been replaced by any C^1 function f whose derivative is bounded.*

Proof First using the invariance of the measure μ_V through the flow ψ_V, it comes that for all $t \in \mathbb{R}$,

$$Z_V(\lambda, 0) = E_V(e^{i\langle \lambda, u_0\rangle}) = E_V(e^{i\langle \lambda, \psi_V(t) u_0\rangle})$$

so

$$|Z_V(\lambda, t) - Z_V(\lambda, 0)| = |E_V\left(e^{i\langle \lambda, \psi(t) u_0\rangle} - e^{i\langle \lambda, \psi(t) u_0\rangle}\right)| \ .$$

It appears then that

$$|Z_V(\lambda, t) - Z_V(\lambda, 0)| \leq E_V\left(|\langle \lambda, \psi(t)u_0 - \psi_V(t) u_0\rangle|\right) \leq \|\lambda\|_{L^2} E_V(\|\psi(t)u_0 - \psi_V(t)u_0\|_{L^2}) \ .$$

As $\epsilon < \frac{1}{2}$, one can take s in $]\frac{1+2\epsilon}{4}, \frac{1}{2}[$ and $\sigma = 2s - \epsilon > \frac{1}{2}$ such that $1 + \sigma - 2s = 1 - \epsilon$. Apply then the Proposition 5.5.5 with such σ and s :

$$|Z_V(\lambda, t) - Z_V(\lambda, 0)| \leq C\|\lambda\|_{L^2}\|V\|_\infty E_V\left((1 + N^{1-\epsilon}\|u_0\|_{H^s}^2 + \|u_0\|_{L^2}) e^{c(1 + N^{(1-\epsilon)/2}\|u_0\|_{H^s})|t|}\right)$$

$$\leq C\|\lambda\|_{L^2}\|V\|_\infty e^{c|t|} E_V\left((1 + N^{1-\epsilon}\|u_0\|_{H^s}^2 + \|u_0\|_{L^2}) e^{cN^{(1-\epsilon)/2}\|u_0\|_{H^s}|t|}\right) \ .$$

Remember that $N = N(u_0, |t|)$ is the smallest integer such that the solutions of BBM and of the perturbed BBM are given by local well posedness on $[-|t|, |t|]$ with initial datum $(1 - \Pi_N)u_0$.

Using that $N^{(1-\epsilon)/2}\|u_0\|_{H^s} \leq N^{1-\epsilon/2} + \|u_0\|_{H^s}^{2-\epsilon}$ the mean value :

$$E_V\left((1 + N^{1+\sigma-2s}\|u_0\|_{H^s}^2 + \|u_0\|_{L^2}) e^{cN^{(1-\epsilon)/2}\|u_0\|_{H^s}|t|}\right) \leq \sqrt{I.1\ I.2}\sqrt{II.1\ II.2}\sqrt{III.1\ III.2}$$

with

$$I.1 = E_V(e^{2c\|u_0\|_{H^s}^{2-\epsilon}|t|}), II.1 = E_V(\|u_0\|_{H^s}^4 e^{2c\|u_0\|_{H^s}^{2-\epsilon}|t|}), III.1 = E_V(\|u_0\|_{L^2}^2 e^{2c\|u_0\|_{H^s}^{2-\epsilon}|t|})$$

and

$$I.2 = III.2 = E_V(e^{2cN^{1-\epsilon/2}|t|}), II.2 = E_V(N^{2-2\epsilon}e^{2cN^{1-\epsilon/2}|t|}) \ .$$

First, remember that there exists $c' > 0$ and C' such that

$$\mu_V\left(\{u_0 \mid \|u_0\|_{H^s} > R\}\right) \leq C' e^{-cR^2}$$

so

$$I.1, II.1, III.1 \leq C_\epsilon e^{c_\epsilon |t|^{2/\epsilon}} \ .$$

Then,

$$P(N > N_0) = \mu_V \left(\{u_0 \mid \|(1 - \Pi_{N_0}) u_0\|_{L^2} \geq \frac{1}{C(1 + |t|)}\} \right) \leq C' e^{-c' N_0/(1+|t|)^2}$$

so

$$I.2, II.2, III.2 \leq C_\epsilon e^{c_\epsilon |t|^{6/\epsilon - 3}} \ .$$

Since $\epsilon < \frac{1}{2}$, it appears that $\frac{6}{\epsilon} - 3 > \frac{2}{\epsilon}$, so in the end , there exist two constants C_ϵ, c_ϵ such that

$$|Z_V(\lambda, t) - Z_V(\lambda, 0)| \leq C_\epsilon \|\lambda\|_{L^2} \|V\|_\infty e^{c_\epsilon |t|^{\frac{6}{\epsilon} - 3}} \ .$$

◇

Remark 5.5.4. *The averages of the products of the amplitudes admit the same kind of estimates. For instance, calling*

$$(\alpha_V^2)_{n,m}(t) = E_V(\langle c_n, \psi(t) u_0 \rangle \, \langle c_m, \psi(t) u_0 \rangle) \ ,$$

it appears that

$$|(\alpha_V^2)_{n,m}(t) - (\alpha_V^2)_{n,m}(0)| \leq C'_\epsilon \|V\|_\infty e^{c'_\epsilon |t|^{\frac{6}{\epsilon} - 3}}$$

only with different constants.

Chapitre 6

On the propagation of weakly nonlinear random dispersive wave

Ce chapitre est issu d'un article en collaboration avec Nikolay Tzvetkov, [28].

6.1 Introduction

In this paper we study several basic dispersive models with random periodic initial data such that the different Fourier modes are independent random variables. Motivated by the vast Physics literature on related topics (see e.g. [56]), we then study how much the Fourier modes of the solution at later times remain decorrelated, and how much the mean values of the amplitudes to the square of the Fourier modes vary with time. Our results are sensitive to the resonances associated with the dispersive relation and to the particular choice of the initial data.

All the models we will be interested in can be injected in the following general framework. Consider the equation

$$(\partial_t + L)u + \varepsilon J(u^2) = 0, \tag{6.1}$$

posed on the torus $\mathbb{T}^d$ of dimension d with an initial datum being a random variable that shall be described later. In (6.1), $\varepsilon \ll 1$ since we want to investigate about the effect of a weak non linearity over the behaviour of the statistics related to the random initial datum. We suppose that u is real valued and L and J are linear maps which are defined as Fourier multipliers by

$$\widehat{Lu}(n) = -i\omega(n)\,\hat{u}(n), \quad \widehat{Ju}(n) = i\varphi(n)\,\hat{u}(n), \quad \forall\, n \in \mathbb{Z}^d,$$

where $\widehat{\cdot}$ denotes the Fourier transform on $\mathbb{T}^d$ and $\omega, \varphi : \mathbb{Z}^d \mapsto \mathbb{R}$ are supposed to be such that

$$\omega(0, n') = \varphi(0, n') = 0, \quad \forall\, n' \in \mathbb{Z}^{d-1}, \tag{6.2}$$

with the natural convention in the case $d = 1$. We suppose that the variable on $\mathbb{T}^d$ is given by $x = (x_1, \dots, x_d)$. Then, thanks to the assumption (6.2), we obtain that we can consider solutions of (6.1) such that $\int_{\mathbb{T}} u(t, x) dx_1 = 0$. We also suppose that ω, φ are odd functions. Observe that under the last assumption L and J send real valued functions to real valued functions. Set

$$\mathbb{D}^d = \{n = (n_1, \dots, n_d) \in \mathbb{Z}^d \mid n_1 \neq 0\}.$$

For $s \in \mathbb{R}$, we introduce the Sobolev spaces H^s of real functions having zero x_1 mean value :

$$H^s = \{u(x) = \sum_{n \in \mathbb{Z}^d} e^{in \cdot x} u_n \mid u_n = \overline{u_{-n}}, \quad \int_{\mathbb{T}} u(x) dx_1 = 0, \quad \sum_{n \in \mathbb{D}^d} |n|^{2s} |u_n|^2 < \infty\}$$

where $|n| = \sum_j |n_j|$. In this work we shall always make use of these Sobolev spaces H^s since they are the ones adapted to our models. In all our examples the equation (6.1) is globally well-posed in some H^s and thus there will be no difficulty caused by the problem of the existence of the dynamics.

Let us describe the dispersive models which can be written under the form (6.1) we will be interested in. They all appear in the modelling of long, small amplitude dispersive waves with a possible weak transverse perturbation. The first example is the KdV equation

$$\partial_t u + \partial_x^3 u + \partial_x (u^2) = 0$$

which corresponds to (6.1) in the case $d = 1$ with $\omega(n) = n^3$ and $\varphi(n) = n$ (with the convention $x = x_1$ and $n = n_1$ is the case $d = 1$). The KdV is globally well posed in H^s, $s \geq -1$ (see [34], for earlier results we refer to [7, 21, 35]).

A second example again in the case $d = 1$ is an alternative of the KdV model, derived by Benjamin-Bona-Mahony (BBM equation) which can be written as

$$\partial_t u + \partial_x u - \partial_t \partial_x^2 u + \partial_x (u^2) = 0.$$

The BBM equation corresponds to (6.1) with $-\omega(n) = \varphi(n) = n/(1 + n^2)$. The BBM equation is globally well-posed in H^s, $s \geq 0$ (see [5, 49]).

Our two dimensional models will be the famous Kadomtsev-Petviashvili (KP) equations. In fact there are two models according to the impact of the surface tension. The first one is the KP-II equation which corresponds to a weak surface tension and can be written as

$$\partial_t u + \partial_{x_1}^3 u + \partial_{x_1}^{-1} \partial_{x_2}^2 u + \partial_{x_1} (u^2) = 0.$$

The KP-II equation corresponds to (6.1) in the case $d = 2$ with $\omega(n_1, n_2) = n_1^3 - n_2^2 / n_1$ if $n_1 \neq 0$, $\omega(0, n_2) = 0$ and $\varphi(n_1, n_2) = n_1$. The KP-II equation is globally well-posed in H^s, $s \geq 0$ (see [6]).

Finally, the KP-I equation

$$\partial_t u + \partial_{x_1}^3 u - \partial_{x_1}^{-1} \partial_{x_2}^2 u + \partial_{x_1} (u^2) = 0.$$

corresponds to (6.1) with $\omega(n_1, n_2) = n_1^3 + n_2^2/n_1$ if $n_1 \neq 0$, $\omega(0, n_2) = 0$ and $\varphi(n_1, n_2) = n_1$. The KP-I equation is globally well-posed if the data is in H^s, $s \geq 2$ (see [31] and also [32]).

Next, we describe the random initial data we shall deal with. With $\mathbb{D}_+^d = \{n \in \mathbb{D}^d \mid n_1 > 0\}$, let $(g_n)_{n \in \mathbb{D}_+^d}$ be a sequence of independent identically distributed complex random variables such that

$$E(g_n) = 0, \quad E(|g_n|^2) = 1$$

and such that there exist to positive constants c and C such that for all $\gamma \in \mathbb{R}$,

$$E(e^{\gamma \mathrm{Re}(g_n)}) \leq C\, e^{c\gamma^2}, \quad E(e^{\gamma \mathrm{Im}(g_n)}) \leq C\, e^{c\gamma^2}, \tag{6.3}$$

where E is the expectation. We also suppose that the distribution of g_n is invariant under the multiplication by $e^{i\theta}$ with $12\theta \neq 0[2\pi]$. Note that under these assumptions, $E(g_n^2)$ is equal to 0. Further consequencies of this property will be used in the sequel.

Remark 6.1.1. A typical example of random variables satisfying our assumptions are the (complex) Gaussian random variables, i.e. $g_n = \frac{1}{\sqrt{2}}(h_n + i l_n)$, with $h_n, l_n \in \mathcal{N}(0, 1)$. Another example coming from the Physics literature is what is known as random phase approximation, that is, g_n is written $g_n = \chi_n A_n$, where χ_n is uniformly distributed on S^1 and A_n is a non-negative random variable independent from χ_n, and $E(A_n^2) = 1$. In all these examples the symmetry assumption on g_n holds with any angle $\theta \neq 0$. In order to ensure (6.3), we can suppose that the distribution μ of A_n satisfies

$$\int_0^\infty e^{\gamma r} d\mu(r) \leq C e^{c\gamma^2} .$$

For instance, the last property holds true if μ is compactly supported.

Next, for $n \in \mathbb{D}_+^d$ set $g_{-n} = \overline{g_n}$. Let $\lambda = (\lambda_n)_{n \in \mathbb{D}_+^d}$ be a sequence of complex numbers such that

$$\sum_{n \in \mathbb{D}_+^d} |n|^{2s} |\lambda_n|^2 < \infty \tag{6.4}$$

for some s depending on L and J such that the equation (6.1) is globally well-posed in H^s. Set for all $n \in \mathbb{D}_+^d$, $\lambda_{-n} = \overline{\lambda_n}$. Set

$$u_0(x) = \sum_{n \in \mathbb{D}^d} g_n \lambda_n e^{in \cdot x} \tag{6.5}$$

Thanks to our assumption on $(\lambda_n)_{n \in \mathbb{D}_+^d}$, we have that $u_0 \in H^s$ almost surely. Moreover, it is real valued. Let $u(\varepsilon; t, x)$ be the solution of

$$\begin{cases} (\partial_t + L)\, u + \varepsilon J(u^2) = 0 \\ u_{|t=0} = u_0. \end{cases}$$

Consider the expansion of $u(\varepsilon; t, x)$ as a Fourier series,

$$u(\varepsilon; t, x) = \sum_{n \in \mathbb{D}^d} u_n(\varepsilon; t) e^{in \cdot x} .$$

Set $S(t) = e^{-tL}$. Then clearly $u(0;t,x) = S(t)u_0$ and

$$u_n(0;t) = e^{i\omega(n)t} g_n \lambda_n .$$

In particular, thanks to our assumption on the random variables g_n,

$$E(\overline{u_m(0;t)}u_n(0;t)) = \delta_n^m |\lambda_n|^2 , \quad \forall\, t \in \mathbb{R}. \tag{6.6}$$

Our aim is to expand the quantity $E(\overline{u_m(\varepsilon;t)}u_n(\varepsilon;t))$ in ε and see how much (6.6) survives in the nonlinear setting.

In order to state our result, we introduce the following notations. We set

$$\Delta_n^{k,l} = \omega(k) + \omega(l) - \omega(n)$$

which corresponds to the pulsation associated to the three waves interaction $k + l \to n$ when $k + l = n$. Next we set :

$$F_n^{k,l}(t) = \int_0^t e^{i\Delta_n^{k,l}\tau} d\tau .$$

Here is our main result.

Theorem 7. *Consider* (6.1)*, in the cases of the KP-II, BBM and the KP-I equations, with initial data given by* (6.5) *of typical Sobolev regularity H^s, $s > 3/8$ for BBM, $s > 2$ for $KP-II$ and $s > 3$ for $KP-I$. Then*

$$E(\overline{u_m(\varepsilon;t)}u_n(\varepsilon;t)) = \delta_n^m |\lambda_n|^2 + \delta_n^m \varepsilon^2 G_n(\lambda,t) + \varepsilon^3 R(\varepsilon;t,m,n)$$

where $G_n(\lambda,t)$ is given by $G_n(\lambda,0) = 0$ and

$$\begin{aligned}
\partial_t G_n(\lambda,t) \quad &= \quad 4\varphi(n) \sum_{k+l=n} Re(-F_n^{k,l}(-t))\Big(\varphi(n)|\lambda_k|^2|\lambda_l|^2 - \varphi(k)|\lambda_n|^2|\lambda_l|^2 - \varphi(l)|\lambda_n|^2|\lambda_k|^2\Big) \\
&\quad + (E(|g_n|^4) - 2)\Big(2\delta_n^{2q} Re(-F_n^{q,q}(-t))\varphi^2(n)|\lambda_q|^4 - 4Re(-F_n^{2n,-n}(-t))\varphi(2n)\varphi(n)|\lambda_n|^4\Big)
\end{aligned}$$

and besides $G_n(\lambda,t)$ and $R(\varepsilon;t,m,n)$ satisfy the following estimates. There exists $C > 0$ such that for every $\varepsilon \in (0,1]$, every $|t| \le \frac{1}{C\varepsilon}$, every m,n,

$$|G_n(\lambda,t)| \le C|t|\,|n|^{-\beta(s)}, \qquad |R(\varepsilon;t,m,n)| \le C \min(|n|,|m|)^{-1}\,|t|(1+|t|)$$

in the case of the BBM equation, with $\beta(s) = 2 + 2s$ if $s \ge 1/2$ and $\beta(s) = 1 + 4s$ otherwise,

$$|G_n(\lambda,t)| \le C|t|\,|n|^{-2s}, \qquad |R(\varepsilon;t,m,n)| \le C \max(|n|,|m|)|t|(1+|t|)$$

in the case of the KP-II equation, and

$$|G_n(\lambda,t)| \le Ct^2|n|^{2-2s}, \qquad |R(\varepsilon;t,m,n)| \le C \max(|n|,|m|)|t|^3$$

in the case of the KP-I equation.

Remark 6.1.2. *The formula for G_n does not depend on the analytic structure of L and J. However, the existence and the bounds of the remainder R and the bounds on G_n depends on their analytic properties, along with the choice for s. In general, the computations of the different orders in this papers does not involve the regularity of the initial data, but the computations of the bounds does.*

It is remarkable that in the case of the BBM equation, if $\lambda_k = \frac{1}{\sqrt{1+k^2}}$ and $E(|g_n|^4) = 2$ then $G_n(\lambda, t) = 0$. This goes with the fact, due to the first author [25], that the measure (on $H^{1/2-}$) induced by

$$\sum_n \frac{g_n}{\sqrt{1+n^2}} e^{inx},$$

g_n Gaussians, is invariant by the flow of BBM. Indeed, this measure is a renormalization of the formal measure

$$e^{-\|u\|^2_{H^1}} du$$

and the H^1 norm of the solution of BBM is conserved by the evolution. In this particular case for λ_n and g_n the terms of higher order should also vanish as shows the next proposition.

Proposition 6.1.1. *Consider the BBM equation. Let $\lambda_k = \frac{1}{\sqrt{1+k^2}}$ and let $g_n = \frac{1}{\sqrt{2}}(h_n + il_n)$, with $h_n, l_n \in \mathcal{N}(0,1)$. Then with the notations of Theorem 7,*

$$G_n(\lambda, t) = R(\varepsilon; t, m, n) = 0, \qquad \forall\,(m,n),\ \forall\, t,\ \forall\, \varepsilon.$$

However, in the proof of Theorem 7, the computation of G_n depends only on $E(|g_n|^2)$ and $E(|g_n|^4)$, which gives a larger framework for almost remaining decorelated initial data. The assumptions on the random variables that they have large Gaussian deviation estimates is imposed in order get the analytic bounds on $|R(\varepsilon; t, m, n)|$.

Remark 6.1.3. The idea underlying the computation of G_n comes from the theory of wave turbulence and the notion of statistical equilibrium. Indeed, as stochastic laws invariant through the flow of one conservative Hamiltonian PDE tend to be quite rare, and to broaden our views on the topic, statistical equilibrium is defined as the next best thing, that is, a law whose moments of order 2, i.e. the $E(|u_n|^2)$ are unchanged by the evolution in time. With $F_n((y_m)_{m\in\mathbb{D}^d}, t) = \partial_t G_n((\sqrt{y_m})_m, t)$, if we replace $|\lambda_m|^2$ by $E(|u_m|^2)+O(\varepsilon^2)$ in the following expression of $\partial_t E(|u_n|^2)$, we formally get that :

$$\partial_t E(|u_n|^2) = \varepsilon^2 \partial_t G_n(\lambda, t) + O(\epsilon^3) = \varepsilon^2 F_n((E(|u_m|^2))_m, t) + O(\varepsilon^3)\ .$$

Hence, by neglecting the remainder because of its order in ε, we have a closed equation on the $E(|u_m|^2)$ detecting statistical equilibrium :

$$\forall t, \forall n,\ \partial_t E(|u_n|^2) = F_n((E(|u_m|^2))_m, t) = 0\ .$$

When one takes the weak limit of F_n when t goes to ∞, only the resonance terms remain. In this sense,

$$\forall n\ ,\ \lim_{t\to\infty} F_n((E(|u_m|^2))_m, t) = 0$$

is the kinetic equation corresponding to statistical equilibrium or KZ spectra in the wave turbulence theory. Namely, if for instance $E(|g_n|^4) = 2$ then the equation can be written as $\forall n$,

$$\sum_{k+l=n,\Delta_n^{k,l}=0} \Big(\varphi(n)(E(|u_k|^2))(E(|u_l|^2)) - \varphi(k)(E(|u_n|^2))(E(|u_l|^2)) - \varphi(l)(E(|u_k|^2))(E(|u_n|^2))\Big) = 0\ .$$

One can see that it depends only on the $E(|u_n|^2)$, meaning that the solution is invariant through dephasing on each wave length. We did not make a serious effort in either finding solutions of this equation or of the more general

$$\forall n\ \forall t\ ,\ G_n(\lambda, t) = 0 \tag{6.7}$$

but we believe that the solutions of the first one, would they exist, would be consistent with the KZ spectra and the actual status of the wave turbulence theory. These solutions would act as a statistics into which the difference between $E(|u_n(\varepsilon;t)|^2)$ and its initial value would be negligible.

Remark 6.1.4. As a matter of a simple observation, inspired by the discussion on the BBM equation, we have that if $E(|g_n|^4) = 2$ then $|\lambda_k|^2 = \varphi(k)/k_1$, is a solution of (6.7). In such a situation the quantity $E(|u_n(\varepsilon;t)|^2)$ is the same as its initial value at $t = 0$ up to a correction of order ε^3, at least for times of order 1.

Remark 6.1.5. It also seems that one should be able to get similar results for

$$E(\overline{u_{m_1}...u_{m_k}}u_{n_1}...u_{n_k}))\ ,$$

with $k > 1$, thus approaching the more general law of the solution instead of only the covariances between the amplitudes of the Fourier modes.

Let us observe that the results of Theorem 7 for the KP equation equations also apply for the KdV equation, by considering data independent of the transverse variable x_2. The result for the BBM and KP-II equations is stronger compared to the result for the KP-I equation thanks to the absence of resonance interactions.

The regularity assumptions of Theorem 7 are more restrictive the ones required by the well-posedness results quoted above. It is a natural open question whether in Theorem 7 one can cover the weaker regularity assumptions of the well-posedness results. We reckon that some new phenomenons may occur at low regularities.

Let us now explain the main ingredients of the proof of Theorem 7. The first step is to get deterministic bounds on the first two Picard iterations. Similarly to the Cauchy problem analysis, the presence of resonances plays an important role in the control of the second iteration. We use some algebraic cancellations of the average between the first and the second iterations. Similar computations appear in the Physics literature. The main novelty in our work is the control on the remainder (once one singles out the first two iterations). Here we use an energy method based on a conservation law together with the exponential integrability of the first two iterations for times of order $\lesssim \varepsilon^{-1}$.

The remaining part of this paper is organized as follows. In the next section we prove Proposition 6.1.1. In the subsequent sections, we prove Theorem 7.

6.2 Proof of Proposition 6.1.1

Denote by μ the measure on $H^{1/2-}$ induced by the map

$$\omega \longmapsto \sum_n \frac{g_n(\omega)}{\sqrt{1+n^2}} e^{inx} \equiv u^\omega .$$

Denote by $\Phi(t)$ the global flow of the BBM on L^2 defined in [49]. Thanks to [25],

$$\int_{H^{1/2-}} F(u)d\mu(u) = \int_{H^{1/2-}} F(\Phi(t)(u))d\mu(u), \quad \forall\, t \in \mathbb{R},\ \forall\, F \in L^1(d\mu). \tag{6.8}$$

Denote by Π_n the projection to the n'th Fourier mode. Then

$$\begin{aligned} E(\overline{u_m(\varepsilon;t)}u_n(\varepsilon;t)) &= \int_\Omega \overline{\Pi_m\Phi(t)(u^\omega)}\Pi_n\Phi(t)(u^\omega)dp(\omega) \\ &= \int_{H^{1/2-}} \overline{\Pi_m\Phi(t)(u)}\Pi_n\Phi(t)(u)d\mu(u) . \end{aligned}$$

Using (6.8) with $F(u) = \overline{\Pi_m(u)}\Pi_n(u)$, we get

$$\begin{aligned} E(\overline{u_m(\varepsilon;t)}u_n(\varepsilon;t)) &= \int_{H^{1/2-}} \overline{\Pi_m(u)}\Pi_n(u)d\mu(u) \\ &= \int_\Omega \overline{\Pi_m(u^\omega)}\Pi_n(u^\omega)dp(\omega) \\ &= \delta_n^m|\lambda_n|^2 . \end{aligned}$$

This completes the proof of Proposition 6.1.1.

6.3 Deterministic estimates for the expansion at order 2 of the solutions

In this section u_0 is a deterministic H^s function. Consider (6.1) with data u_0. We suppose that

$$u_0(x) = \sum_{n\in\mathbb{D}^d} a_n e^{in\cdot x}, \quad a_{-n} = \overline{a_n} .$$

Let us expand the solution of (6.1) with data u_0 at order 2 in ε. For simplification in the computations, let

$$v(\varepsilon;t,x) \equiv S(-t)u(\varepsilon,t,x) = \sum_{n\in\mathbb{D}^d} v_n(\varepsilon,t)e^{in\cdot x}$$

such that v satisfies :

$$\partial_t v = -\varepsilon S(-t)J\big((S(t)v)^2\big)$$

with initial datum u_0. Write then

$$v(\varepsilon;t,x) = u_0(x) + \varepsilon b(t,x) + \varepsilon^2 c(\varepsilon;t,x)$$

and

$$v_n(\varepsilon;t) = a_n + \varepsilon b_n(t) + \varepsilon^2 c_n(\varepsilon;t)$$

with

$$b(t,x) = -\int_0^t S(-\tau)J((S(\tau)u_0)^2)d\tau\ .$$

Then

$$b_n(t) = -i\varphi(n)\sum_{k+l=n} a_k a_l F_n^{k,l}(t)\,.$$

The next no resonance lemma plays a key role in out analysis.

Lemma 6.3.1. *Let $k+l=n$. Then KP-II and BBM present no resonances, that is for KP-II*

$$|\Delta_n^{k,l}| \geq 3|n_1 k_1 l_1| \neq 0$$

and for BBM

$$\Delta_n^{k,l} = \frac{nkl(k^2+l^2+kl+3)}{(1+n^2)(1+k^2)(1+l^2)} \neq 0\ .$$

The proof is a straightforward computation. The consequence of this lemma is that the norm of b can be bounded independently from t for KP-II and BBM.

Lemma 6.3.2. *Suppose that for KP-II, the initial datum u_0 belongs to H^s with $s > \frac{1}{2}$, then there exists C independent of t and u_0 such that*

$$||b(t)||_{H^s} \leq C||u_0||_{H^s}^2\ .$$

For BBM, suppose that the initial datum is in H^s, with $s > \frac{1}{4}$ and let σ such that $0 \leq \sigma < 2s - \frac{1}{2}$, then if $s \leq 1$,

$$||b(t)||_{H^\sigma} \leq C||u_0||_{H^s}^2\ ,$$

and if $s > \frac{1}{2}$,

$$||b(t)||_{H^s} \leq C||u_0||_{H^s}^2\ .$$

Proof We use the form of b_n to give the bound

$$|n|^{2s}|b_n|^2 \leq |n|^{2s}|\varphi(n)|^2\sum_{k,l}|a_k a_{n-k}\overline{a_l a_{n-l}}||F_n^{k,n-k}(t)|\ |F_n^{l,n-l}(t)|\ .$$

Then, as $s \geq 0$, $|n|^s \leq C_s(|k|^s + |n-k|^s)$, and using the facts that the sum is symmetric in k, $n-k$ and l, $n-l$ and that there is no resonances then $|F_n^{k,n-k}(t)| \leq |\frac{2}{\Delta_n^{k,n-k}}|$,

$$|n|^{2s}|b_n|^2 \leq C_s|\varphi(n)|\sum_{k,l}|a_k|\ |n-k|^s|a_{n-k}||a_l|\ |n-l|^s|a_{n-l}|\frac{1}{|\Delta_n^{k,n-k}\Delta_n^{l,n-l}|}\ .$$

For KP-II, $\frac{|\varphi(n)|}{|\Delta_n^{k,n-k}|} \leq \frac{1}{3|k_1|}$, thus by summing over n and using a Cauchy-Schwartz inequality :

$$||b||_{H^s}^2 \leq \sum_{k,l} \frac{|a_k|}{|k_1|} \frac{|a_l|}{|l_1|} \sum_n |n-k|^s |a_{n-k}| \, |n-l|^s |a_{n-l}|$$

$$||b||_{H^s} \leq C_s ||u_0||_{H^s} \Big| \sum_k \frac{|a_k|}{|k_1|} \Big| \leq C_s ||u_0||_{H^s}^2 \sqrt{\sum_{k \in \mathbb{D}^d} \frac{1}{|k_1|^2 \, |k|^{2s}}}$$

and as $d = 2$, the series converges as long as $s > \frac{1}{2}$.

In the case of BBM, we have :

$$\Delta_n^{k,l} = \frac{nkl(k^2 + l^2 + kl + 3)}{(1+n^2)(1+k^2)(1+l^2)} \ .$$

As for $s \in [-1, 1]$, we have :

$$|k|^{s+1} |l|^{1-s} \leq k^2 + l^2 \leq 2(k^2 + l^2 + kl) \ ,$$

we conclude that

$$|\Delta_n^{k,l}| \geq C \frac{|n|}{1+n^2} \frac{|k|^{s+2}}{1+k^2} \frac{|l|^{2-s}}{1+l^2} \geq C |\varphi(n)| \frac{|k|^s}{|l|^s} \ .$$

Hence,

$$\frac{|\varphi(n)|}{|\Delta_n^{k,l}|} \leq C |l|^s |k|^{-s} \ .$$

Let us now bound the H^σ norm of b in terms of the H^s norm of u_0. Since

$$b_n = -i\varphi(n) \sum_{k+l=n} a_k a_l F_n^{k,l}(t)$$

we have that

$$|b_n| \leq |\varphi(n)| \sum_{k+l=n} |a_k| \, |a_l| \frac{2}{|\Delta_n^{k,l}|}$$

Using that for $\sigma \geq 0$, $|n|^\sigma \leq C_\sigma (|k|^\sigma + |l|^\sigma)$ and using the symmetry of the sum over k and l :

$$|n|^\sigma |b_n| \leq C \sum_{k+l=n} |k|^\sigma |a_k| \, |a_l| \, \frac{|\varphi(n)|}{|\Delta_n^{k,l}|} \ .$$

We then use the bound on $\frac{|\varphi(n)|}{|\Delta_n^{k,l}|}$ to write

$$|n|^\sigma |b_n| \leq C \sum_{k+l=n} |k|^{\sigma - s} |a_k| \, |l|^s |a_l| \ .$$

and therefore

$$||b||_{H^\sigma}^2 \leq C \sum_n \sum_{k,j} |k|^{\sigma-s} |a_k| \, |n-k|^s |a_{n-k}| |j|^{\sigma-s} |a_j| \, |n-j|^s |a_{n-j}| \ .$$

By reversing the order of the sums and using a Cauchy-Schwartz inequality on the sum over n :

$$\|b\|_{H^\sigma} \leq C\|u_0\|_{H^s}\Big(\sum_k |k|^{\sigma-s}|a_k|\Big)$$

and since

$$\sum_k |k|^{\sigma-s}|a_k| \leq \left(\sum_k |k|^{2\sigma-4s}\right)^{1/2} \|u_0\|_{H^s}$$

and the series converges if $s > \frac{1}{4} + \frac{\sigma}{2}$, we get :

$$\|b\|_{H^\sigma} \leq C\|u_0\|^2_{H^s} \ .$$

For $s \geq 1$ we simply use $|\Delta_n^{k,l}| \geq |\varphi(n)|$ and an argument similar to the one for KP-II yields the claimed bound. This completes the proof of Lemma 6.3.2. ◊

Remark 6.3.1. *The arguments we presented here for the KP-II equation relax the assumption* $s > 1$ *to* $s > 1/2$ *in [53].*

For the KP-I equation, there are resonances and hence a much weaker statement holds.

Lemma 6.3.3. *For KP-I, it appears that for* $s > 1$

$$\|b\|_{H^{s-1}} \leq C|t| \ \|u_0\|^2_{H^s} \ .$$

Proof Use the expression of b to get the bound

$$\|b(t)\|_{H^{s-1}} \leq \int_0^t \|J\left((S(\tau)u_0)^2\right)d\tau\|_{H^{s-1}} \leq C_s \int_0^t \|S(\tau)u_0\|_{L^\infty}\|S(\tau)u_0\|_{H^s} \leq C_s|t| \ \|u_0\|^2_{H^s}$$

since as $d = 2$ and $s > 1$, the Sobolev embedding $H^s \subset L^\infty$ holds. ◊

Lemma 6.3.4. *The map c satisfies, in every case :*

$$\partial_t c = -S(-t)J\left(2S(t)u_0S(t)b + \varepsilon((S(t)b)^2 + 2S(t)u_0S(t)c)) + \varepsilon^2 2S(t)bS(t)c + \varepsilon^3(S(t)c)^2\right).$$

Proof It comes from a combination of the equations satisfied by v and b. ◊

We now would like to prove that c is of order 0 in ε but that its order in time depends on the cases, whether the equation displays resonances or not.

Lemma 6.3.5. *For KP equations, one can bound the L^2 norm of c. In the case of KP-I (with resonances), it comes if $s > 3$*

$$\|c(t)\|_{L^2} \le C\left(t^2\|u_0\|_{H^s}^3 + \varepsilon|t|^3\|u_0\|_{H^s}^4\right)e^{c\varepsilon|t|\,\|u_0\|_{H^s}(1+\varepsilon|t|\,\|u_0\|_{H^s})}\ .$$

And for KP-II, it comes if $s > 2$

$$\|c(t)\|_{L^2} \le C|t|\left(\|u_0\|_{H^s}^3 + \varepsilon\|u_0\|_{H^s}^4\right)e^{c\varepsilon|t|\,\|u_0\|_{H^s}(1+\varepsilon\|u_0\|_{H^s})}\ .$$

For BBM, the relevant quantity is the H^1 norm of c, it comes if $s > 3/8$:

$$\|c(t)\|_{H^1} \le C|t|(\|u_0\|_{H^s}^3 + \varepsilon\|u_0\|_{H^s}^4)e^{c\varepsilon|t|\,\|u_0\|_{H^s}(1+\varepsilon\|u_0\|_{H^s})}\ .$$

Proof Calling $E(t) = \frac{1}{2}\|c(t)\|_{L^2}$ for KP and $E(t) = \frac{1}{2}\|c(t)\|_{H^1}$ for BBM,

$$E(t)\partial_t E(t) = I + II + III$$

with

$$I = -\int cS(-t)\partial_{x_1}\left(2S(t)u_0S(t)b + \varepsilon(S(t)b)^2\right)\ ,$$

$$II = -\int cS(-t)\partial_{x_1}\left(\varepsilon 2S(t)u_0S(t)c + \varepsilon^2 2S(t)bS(t)c\right)$$

and

$$III = -\varepsilon^3\int cS(-t)\partial_{x_1}(S(t)c)^2 = -\frac{\varepsilon^3}{3}\int\partial_{x_1}(S(t)c)^3 = 0\ .$$

For KP equations, it appears that

$$I(t) \le C\|S(t)c\|_{L^2}\|\partial_{x_1}\left(2S(t)u_0S(t)b + \varepsilon(S(t)b)^2\right)\|_{L^2}$$

and therefore

$$I(t) \le CE(t)\left(\|\partial_{x_1}S(t)u_0\|_{L^2}\|S(t)b\|_{L^\infty} + \|\partial_{x_1}S(t)b\|_{L^2}\|S(t)u_0\|_{L^\infty} + \|\partial_{x_1}S(t)b\|_{L^2}\|S(t)b\|_{L^\infty}\right)$$

Using that the H^s norms are invariant through the flow $S(t)$, as $s \ge 1$ in both cases,

$$\|\partial_{x_1}S(t)u_0\|_{L^2} \le \|S(t)u_0\|_{H^1} = \|u_0\|_{H^1} \le \|u_0\|_{H^s}$$

$$I(t) \le CE(t)\left(\|u_0\|_{H^s}\|S(t)b\|_{L^\infty} + \|b\|_{H^1}\|S(t)u_0\|_{L^\infty} + \varepsilon\|b\|_{H^1}\|S(t)b\|_{L^\infty}\right)$$

and using the fact that $\int f\partial_{x_1}(fg) = \frac{1}{2}\int f^2\partial_{x_1}g$, for KP we have

$$II(t) = -\int(S(t)c)^2\partial_{x_1}\varepsilon S(t)u_0 - \int(S(t)c)^2\varepsilon^2\partial_{x_1}S(t)b$$

$$II(t) \le C\|S(t)c\|_{L^2}^2\left(\varepsilon\|\partial_{x_1}S(t)u_0\|_{L^\infty} + \varepsilon^2\|\partial_{x_1}S(t)b\|_{L^\infty}\right)$$

and thus

$$II(t) \le CE(t)^2 \left(\varepsilon \|u_0\|_{H^s} + \varepsilon^2 \|\partial_{x_1} S(t)b\|_{L^\infty}\right) .$$

Then, for KP-I, use the fact that for $s > 3$ ($s - 2 > 1$), H^{s-2} injects itself in L^∞

$$\|\partial_{x_1} S(t)b\|_{L^\infty} \le C\|\partial_{x_1} S(t)b\|_{H^{s-2}} \le C\|S(t)b\|_{H^{s-1}} \le C|t|\|u_0\|^2_{H^s}$$

and thus

$$\partial_t E(t) \le C\left(|t|\ \|u_0\|^3_{H^s} + \varepsilon t^2 \|u_0\|^4_{H^s}\right) + CE(t)\left(\varepsilon\|u_0\|_{H^s} + \varepsilon^2 |t|\ \|u_0\|^2_{H^s}\right) .$$

With a Gronwall lemma,

$$\|c(t)\|_{L^2} \le C\left(t^2 \|u_0\|^3_{H^s} + \varepsilon |t|^3 \|u_0\|^4_{H^s}\right) e^{\varepsilon c|t|\ \|u_0\|_{H^s} + \varepsilon^2 t^2 \|u_0\|^2_{H^s}} .$$

For KP-II, use the fact that for $s > 2$, $H^{s-1} \subset L^\infty$,

$$\|\partial_{x_1} S(t)b\|_{L^\infty} \le C\|\partial_{x_1} S(t)b\|_{H^{s-1}} \le C\|S(t)b\|_{H^s} \le C\|u_0\|^2_{H^s}$$

and thus

$$\partial_t E(t) \le C\left(\|u_0\|^3_{H^s} + \varepsilon\|u_0\|^4_{H^s}\right) + CE(t)\left(\varepsilon\|u_0\|_{H^s} + \varepsilon^2 \|u_0\|^2_{H^s}\right)$$

and therefore

$$\|c(t)\|_{L^2} \le C\left(|t|\|u_0\|^3_{H^s} + \varepsilon|t|\ \|u_0\|^4_{H^s}\right) e^{\varepsilon c|t|\ \|u_0\|_{H^s} + \varepsilon^2 |t|\ \|u_0\|^2_{H^s}} .$$

For BBM, c satisfies :

$$2(1 - \partial_x^2)\partial_t c = -S(-t)\partial_x \left(2S(t)u_0 S(t)b + \varepsilon(S(t)b)^2 + \varepsilon 2S(t)u_0 c + \varepsilon^2 2S(t)bS(t)c + \varepsilon^3 (S(t)c)^2\right) .$$

Since $s > \frac{3}{8}$, we can choose σ in $]\frac{1}{4}, 2s - \frac{1}{2}[$ if $s \le 1$ and $\sigma = s$ otherwise. We have then that :

$$\partial_t \|c(t)\|_{H^1} \le C\left(\|S(t)u_0 S(t)b\|_{L^2} + \varepsilon\|S(t)b\|^2_{L^4} + \|c(t)\|_{H^1}(\varepsilon\|S(t)u_0\|_{L^2} + \varepsilon^2\|S(t)b\|_{L^2})\right)$$

and as $s, \sigma > 1/4$, the Sobolev embeddings $H^s \subset L^4$ and $H^\sigma \subset L^4$ hold,

$$\|S(t)u_0 S(t)b\|_{L^2} \le \|S(t)u_0\|_{L^4}\|S(t)b\|_{L^4} \le C\|S(t)u_0\|_{H^s}\|S(t)b\|_{H^\sigma} \le C\|u_0\|^3_{H^s}$$

Finally,

$$\|c(t)\|_{H^1} \le C|t|(\|u_0\|^3_{H^s} + \varepsilon\|u_0\|^4_{H^s}) e^{|t|(\varepsilon\|u_0\|_{H^s} + \varepsilon^2\|u_0\|^2_{H^s})} .$$

This completes the proof of Lemma 6.3.5. ◊

Remark 6.3.2. One may also establish estimates for higher order derivatives of c by the classical energy method. This method does not give the cancellation of the term III above and thus the restriction of the time for which the estimate holds depends on u_0 and thus on the probability event of which u_0 is a representation. In particular, it is not clear to us how to exploit in general such an estimate in the context of the study of the decorrelation of the Fourier modes of the solution. Nevertheless, by using random variables g_n with values in a compact set, we should be able to use the energy method with a time of validity that does not depend on the probability event. For instance, one can use $g_n = \chi_n A_n$ where χ_n is uniformly distributed on S^1 and is independent from A_n, where A_n is non-negative, compactly supported and $E(A_n^2) = 1$.

6.4 Probabilistic properties

In this section u_0 is given by (6.5). We now want to prove that until time of order ε^{-1}, the maps a, b and c are of order 0 in ε. For that, we use the following proposition :

Proposition 6.4.1. *There exist C, c two positive constants such that for all $R > 0$, the probability for the initial datum to have a H^s norm bigger than R satisfies :*

$$P(\|u_0\|_{H^s} \geq R) \leq Ce^{-cR^2} .$$

Proof We first observe that (6.3) together with the zero mean value assumption imply that

$$E(e^{\gamma \mathrm{Re}(g_n)}) \leq e^{c\gamma^2}, \quad E(e^{\gamma \mathrm{Im}(g_n)}) \leq e^{c\gamma^2} . \tag{6.9}$$

First, we notice that thanks to (6.3), we only need to get (6.9) for small value of $|\gamma|$, say $|\gamma| \leq 1$ Next, we apply (6.3) with $\gamma = \pm\alpha$ to get

$$E(e^{\alpha|\mathrm{Re}(g_n)|}) + E(e^{\alpha|\mathrm{Im}(g_n)|}) < \infty.$$

Now, we use that there exist two positive constants C_1 and C_2 such that for every $|\gamma| \leq 1$ and every $x \in \mathbb{R}$,

$$|e^{\gamma x} - 1 - \gamma x| \leq C_1\gamma^2 e^{C_2|x|} .$$

Thanks to the zero mean value assumption on g_n, the above analysis implies that there exists a constant A such that

$$E(e^{\gamma \mathrm{Re}(g_n)}) \leq 1 + A\gamma^2 \leq e^{c\gamma^2},$$

provided $c \geq A$. A similar argument applies for the imaginary part of g_n. Thus, we indeed have (6.9) and we are in a position to apply [15, Lemma 3.1].

By separating the real and the imaginary parts, using [15, Lemma 3.1], we obtain that there exist two positive constants C and c such that for every $y \geq 0$ and every sequence (a_n),

$$P(|\sum_n a_n g_n| \geq y) \leq Ce^{-cy^2/(\sum_n |a_n|^2)}$$

Indeed, if $a_n = \alpha_n + i\beta_n$ and $g_n = h_n + il_n$, then,

$$|\sum_n a_n g_n| \leq |\sum_n \alpha_n h_n| + |\sum_n \alpha_n l_n| + |\sum_n \beta_n h_n| + |\sum_n \beta_n l_n|$$

and therefore,

$$\begin{aligned} P(|\sum_n a_n g_n| \geq y) \quad &\leq \quad P(|\sum_n \alpha_n h_n| \geq y/4) + P(|\sum_n \alpha_n l_n| \geq y/4) \\ &\quad + P(|\sum_n \beta_n h_n| \geq y/4) + P(|\sum_n \beta_n l_n| \geq y/4) \end{aligned}$$

and then we can apply the [15, Lemma 3.1] on each term of the right handside. Remark that since the g_n are independent from each other, so are the h_n and the l_n, even though h_n is not necessarly independent from l_n.

We deduce from that that the L^q norm (in the probability space) of $\sum a_n g_n$ satisfies :

$$\|\sum_n a_n g_n\|_{L^q}^q \leq \Big(Cq\sum_n |a_n|^2\Big)^{q/2}$$

with C independent from a_n and q. Indeed, this property is due to a change of variable and an induction on q. First, we have that :

$$\|\sum_n a_n g_n\|_{L^q}^q = \int qy^{q-1}P(|\sum a_n g_n| \geq y)dy \leq \int Cqy^{q-1}e^{-cy^2/\sum |a_n|^2}dy$$

With $z = \frac{y}{\sqrt{\sum |a_n|^2}}$,

$$\|\sum_n a_n g_n\|_{L^q}^q \leq \Big(\sum |a_n|^2\Big)^{q/2} C(q)$$

with

$$C(q) = C\int qz^{q-1}e^{-cz^2}dz\ .$$

By integration by parts, we get :

$$C(q+2) = \frac{q+2}{2c}C(q)$$

and then using that $C(q)$ is bounded uniformly in q for $q \in [1,3]$, we get

$$C(q) \leq C\left(\frac{q}{2c}\right)^{q/2} \leq (Cq)^{q/2}$$

and consequently

$$\|\sum_n a_n g_n\|_{L^q}^q \leq \Big(Cq\sum |a_n|^2\Big)^{q/2} \tag{6.10}$$

Then, we use that :

$$P(\|u_0\|_{H^s} \geq R) = P(\|u_0\|_{H^s}^q \geq R^q) \leq R^{-q}E(\|u_0\|_{H^s}^q) = R^{-q}\|u_0\|_{L_P^q,H_x^s}^q$$

where L_P^q denotes the L^q norm in the probability space and H_x^s the H^s norm in the physical space.

For $q \geq 2$ and thanks to Minkowski inequality,

$$\|u_0\|_{L_P^q,H_x^s} = \|\sum |n|^s\lambda_n g_n e^{inx}\|_{L_P^q,L_x^2} \leq \|\sum |n|^s\lambda_n g_n e^{inx}\|_{L_x^2,L_P^q}$$

Hence, using (6.10) and (6.4), we get

$$\|u_0\|_{L_P^q,H_x^s} \leq \|\Big(Cq\sum_n |\lambda_n|^2\ |n|^{2s}\Big)^{1/2}\|_{L_x^2} = C\sqrt{q}\Big(\sum_n |\lambda_n|^2\ |n|^{2s}\Big)^{1/2} \leq C\sqrt{q}$$

This in turn implies the bound

$$P(\|u_0\|_{H^s} \geq R) \leq \Big(\frac{Cq}{R^2}\Big)^{q/2} .$$

Set $q(R) = e^{-1}\frac{R^2}{C}$ such that what inside the parenthesis in the above expression is equal to e^{-1} in the particular case $q = q(R)$. If R is such that $q(R) \geq 2$ then we have that :

$$P(\|u_0\|_{H^s} \geq R) \leq e^{-q(R)/2} = e^{-cR^2} .$$

Let R_0 be defined by $2 = e^{-1}\frac{R_0^2}{C}$, i.e. $R_0 = \sqrt{2eC}$. For $R \in [0, R_0]$, we can simply write

$$P(\|u_0\|_{H^s} \geq R)e^{cR^2} \leq e^{cR_0^2}$$

Therefore

$$P(\|u_0\|_{H^s} \geq R) \leq e^{cR_0^2}\, e^{-cR^2} \ , \quad \forall\, R \geq 0.$$

This completes the proof of Proposition 6.4.1. ◊

As a consequence of Proposition 6.4.1, we get the following exponential integrability statement.

Lemma 6.4.2. *There exists $\delta_0 > 0$ such that*

$$E(e^{\delta_0\|u_0\|^2_{H^s}}) < \infty .$$

We deduce from Lemma 6.3.5 and Lemma 6.4.2 , that as long as $|t|$ is bounded by ε^{-1} the norm of c can be bounded in probability. Indeed,

Proposition 6.4.3. *For all $p \geq 1$ there exists C_p such that for all $|t| \leq \frac{1}{C_p\varepsilon}$, we have the bounds :*

$$E(\|c(t)\|^p_{L^2})^{1/p} \leq C_p t^2$$

for KP-I,

$$E(\|c(t)\|^p_{L^2})^{1/p} \leq C_p |t|$$

for KP-II,

$$E(\|c(t)\|^p_{H^1})^{1/p} \leq C_p |t|$$

for BBM.

Proof This comes from the fact that

$$E(\|u_0\|^p_{H^s})$$

is bounded for all p and that

$$E(e^{c\varepsilon|t|\|u_0\|_{H^s}})\ ,\ E(e^{c\varepsilon^2|t|\|u_0\|^2_{H^s}})\ ,\ E(e^{c\varepsilon^2t^2\|u_0\|^2_{H^s}})$$

are bounded uniformly in t as long as $t^2c\varepsilon^2$ is less than the δ_0 defined in Lemma 6.4.2. ◊

We next collect some properties of the random variables (g_n).

Lemma 6.4.4. *Under our assumption on (g_n), with the n_j belonging to $\mathbb{D}^d$,*

$$E(g_{n_1}g_{n_2}) = \delta_{n_1}^{-n_2}$$

and

$$E(g_{n_1}g_{n_2}g_{n_3}) = 0.$$

Moreover

$$E(g_{n_1}g_{n_2}g_{n_3}g_{n_4}) = 0,$$

unless $n_1 = -n_j$ for some $j \in \{2,3,4\}$ and $n_k = -n_l$ for the two indexes k,l in the set $\{1,2,3,4\}/\{1,j\}$. Moreover $E(g_{n_1}g_{n_2}g_{n_3}g_{n_4}) = 1$ if $n_1 \neq n_k$ and $n_1 \neq -n_k$. Finally

$$E(g_{n_1}g_{n_2}g_{n_3}g_{n_4}) = E(|g_{n_1}|^4)$$

if $n_1 = n_k$ or $n_1 = -n_k$.

The proof of this lemma follows by using the independence assumption via a careful case by case study. In particular, we use that under our assumption of symmetry of the distribution we have that

$$E(g_n^3) = E(|g_n|^2 g_n) = 0\ ,\ E(g_n^4) = E(|g_n|^2 g_n^2) = E(|g_n|^2 \overline{g_n}^2) = E(\overline{g_n}^4) = 0\ .$$

For instance, for a product of three gs, we use that if $|n_1|$ is different from $|n_2$ and $|n_3|$ then g_{n_1} is independent from $g_{n_2}g_{n_3}$, thus

$$E(g_{n_1}g_{n_2}g_{n_3}) = E(g_{n_1})E(g_{n_2}g_{n_3}) = 0$$

as the mean value of g_{n_1} is null. Otherwise, by symmetry between the indexes, $|n_1| = |n_2| = |n_3|$, hence

$$E(g_{n_1}g_{n_2}g_{n_3})$$

is equal to either

$$E(g_{n_1}\overline{g_{n_1}^2}) = 0$$

or

$$E(g_{n_1}^2\overline{g_{n_1}}) = 0$$

or

$$E(g_{n_1}^3) = 0\ .$$

6.5 Expansion of the covariances

In this section, we complete the proof of Theorem 7. Using the fact that a, b, and c are of order 0 in ε as long as $t \lesssim \varepsilon^{-1}$ we would like to develop the covariances of the amplitudes of the different wavelengths. Let $d_n^m(t)$ be defined as

$$d_n^m(t) = E(\overline{v_m}(t)v_n(t)) \ ,$$

with $v_n(t) = u_n(t)e^{-i\omega(n)t}$. Then we have the following statement.

Proposition 6.5.1. *We have that*

$$\partial_t d_n^m(t) = \delta_n^m \varepsilon^2 \partial_t G_n(\lambda, t) + \varepsilon^3 r(\varepsilon; t, m, n),$$

where $t\, r(\varepsilon; t, m, n)$ satisfies the bounds for $R(\varepsilon; t, m, n)$ announced in the statement of Theorem 7.

Proof Let us compute the time derivative of $d_n^m(t)$. Since v_n satisfies

$$\partial_t v_n(t) = -i\varepsilon\varphi(n) \sum_{k+l=n} v_k v_l e^{i\Delta_n^{k,l} t} \ ,$$

it comes

$$\partial_t d_n^m(t) = i\varepsilon\varphi(m)E\Big(\sum_{k+l=m} \bar{v}_k \bar{v}_l v_n e^{-i\Delta_m^{k,l} t}\Big) - i\varepsilon\varphi(n)E\Big(\sum_{k+l=n} v_k v_l \bar{v}_m e^{i\Delta_n^{k,l} t}\Big) \ . \tag{6.11}$$

In the cases of the KP equations, the term of last order, that is the term of order 7 in ε will involve three occurrences of c, and since we only have a bound for c in L^2, we will not be able to bound

$$\varphi(n)E\Big(\sum_{k+l=n} c_k c_l \bar{c}_m e^{i\Delta_n^{k,l} t}\Big)$$

by some function depending on the time and not on n, m (see Remark 6.3.2)

By inserting $v_n = v_n(\varepsilon) = a_n + \varepsilon b_n + \varepsilon^2 c_n(\varepsilon)$ in (6.11) we distinguish different cases according to the power of ε.

First, it is clear that the term of order 0 in the expression of $\partial_t d_n^m$ is 0.

Then the term of order 1 is also 0 since it involves three occurrences of a : $a_k a_l a_n$, and $a_k = \lambda_k g_k$. Thus, we can apply Lemma 6.4.4. This cancellation is frequently used in the Physics literature on the subject.

We will describe the term of order 2 later.

The term of order 3 involves combinations of 1 c and 2 a or 2 b and 1 a. Hence, in the KP-I case it is less than $C\max(|n|, |m|)t^2$. In the case of KP-II, because of the different estimate on b, it is less than $C\max(|n|, |m|)|t|$. A similar analysis applies in the BBM case to get the bound $C(\min(|m|, |n|)^{-1}|t|$.

Let us describe this bound in the particular case of combinations between 1 c and 2 a, the other ones resulting from similar computations. If we replace one occurence of v by c and two occurences of v by a in the expression

$$i\varphi(m)E\Big(\sum_{k+l=m} \bar{v}_k \bar{v}_l v_n e^{-i\Delta_m^{k,l} t}\Big) - i\varphi(n)E\Big(\sum_{k+l=n} v_k v_l \bar{v}_m e^{i\Delta_n^{k,l} t}\Big)$$

we get

$$i\varphi(m)E\Big(\sum_{k+l=m}(\overline{a}_k\overline{a}_l c_n+\overline{a}_k\overline{c}_l a_n+\overline{c}_k\overline{a}_l a_n)\,e^{-i\Delta_m^{k,l}t}\Big)-$$
$$i\varphi(n)E\Big(\sum_{k+l=n}(a_k a_l\overline{c}_m+a_k c_l\overline{a}_m+c_k a_l\overline{a}_m)\,e^{i\Delta_n^{k,l}t}\Big)\,.$$

For KP, we can bound the L^2 norm of c, hence, since by a Cauchy-Schwartz inequality on the a :

$$\Big|\sum_{k+l=m}\overline{a}_k\overline{a}_l c_n e^{-i\Delta_m^{k,l}t}\Big|\le|c_n|\,\|u_0\|_{L^2}^2\le\|c\|_{L^2}\|u_0\|_{L^2}^2$$

and by taking its expectation and circular arguments for the other terms, we get the bound on this term :

$$3\,(|\varphi(n)|+|\varphi(m)|)\,E(\|c\|_{L^2}^3)^{1/3}E(\|u_0\|_{H^1}^3)^{2/3}\ .$$

For KP-I, $E(\|c\|_{L^2}^3)^{1/3}$ is bounded by Ct^2 and for KP-II it is bounded by $C|t|$. Hence, as $\varphi(n)=n_1$, this term is bounded by $C\max(|n|,|m|)t^2$ for KP-I and $C\max(|n|,|m|)|t|$ for KP-II. For BBM, $E(\|c\|_{L^2}^3)^{1/3}\le E(\|c\|_{H^1}^3)^{1/3}$ is bounded by $C(1+|t|)$. Hence, as $|\varphi(n)|\le|n|^{-1}$, this term is bounded by $C(\min(|m|,|n|)^{-1}|t|$.

The third order in ε also involves combinations of 2 b and 1 a. In this case, the order in time for KP-II and BBM is 0. Hence, the term of third order is bounded by $C\max(|n|,|m|)t^2$ for KP-I, $C\max(|n|,|m|)(1+|t|)$ for KP-II and $C\min(|n|,|m|)^{-1}(1+|t|)$ for BBM.

The term of order 4 involves combinations of 1 c, 1 b and 1 a or 3 b. Hence, in the KP-I case it is less than $C\max(|n|,|m|)|t|^3$. A similar analysis applies in the KP-II and BBM cases.

The term of order 5 involves combinations of 1 a and 2 c or 2 b and 1 c. Hence, in the KP-I case it is less than $C\max(|n|,|m|)|t|^4$. Again a similar analysis applies in the KP-II and BBM cases.

The term of order 6 involves combinations of 1 b and 2 c. Hence, in the KP-I case it is less than $C\max(|n|,|m|)|t|^5$ and a similar analysis applies in the KP-II and BBM cases.

Finally the term of order 7 involves combinations of 3 c. Hence, it is less than $C\max(|n|,|m|)|t|^6$ in the KP-I case and $C\max(|n|,|m|)|t|^3$ in the KP-II case.

Since t is less than ε^{-1}, we have that all estimates in the KP-I case are $O(\max(|n|,|m|)\varepsilon^3t^2)$. In the KP-II case they are $O(\max(|n|,|m|)\varepsilon^3(1+|t|))$ and in the BBM case $O((\min(|m|,|n|)^{-1}\varepsilon^3(1+|t|))$.

Let us compute the term of order 2. As it involves 2 a and 1 b, two sums of different nature (and their complex conjugate when inverting n and m) appear in it :

$$V_n^m(t)=i\varphi(m)E(\Big(\sum_{k+l=m}\overline{a}_k\overline{a}_l b_n e^{-i\Delta_m^{k,l}t}\Big)$$

and

$$W_n^m(t)=i\varphi(m)E(\Big(\sum_{k+l=m}\overline{b}_k\overline{a}_l a_n e^{-i\Delta_m^{k,l}t}\Big)$$

which appears twice because of the symmetry between k and l. The term of order 2 is therefore equal to

$$V_n^m(t) + \overline{V}_m^n(t) + 2(W_n^m(t) + \overline{W}_m^n(t)) \; .$$

By replacing $b_n(t)$ by its value

$$b_n(t) = -i\varphi(n) \sum_{j+q=n} a_j a_q F_n^{j,q}(t)$$

we get

$$V_n^m(t) = \varphi(n)\varphi(m) \sum_{k+l=m} \sum_{j+q=n} E(\overline{a}_k \overline{a}_l a_j a_q) e^{-i\Delta_m^{k,l} t} F_n^{j,q}(t) \; .$$

We now recall that thanks to Lemma 6.4.4, $E(\overline{a}_k \overline{a}_l a_j a_q)$ is equal to zero unless we can pair the indexes. We can not pair k with l or we will have $k = l$, $m = 0$, and $m \neq 0$ since $m_1 \neq 0$ by hypothesis, but we can pair k with j and l with q or k with q and l with j. In both case we have $n = m$. As long as $k \neq l$, we have :

$$E(\overline{a}_k \overline{a}_l a_j a_q) = |\lambda_k|^2 |\lambda_l|^2$$

otherwise $2k = n$ and

$$E(\overline{a}_k \overline{a}_l a_j a_q) = \delta_n^{2k} |\lambda_k|^4 E(|g_k|^4) \; .$$

Using our assumptions on the random variables and that $e^{-i\Delta_n^{k,l} t} F_n^{k,l}(t) = -F_n^{k,l}(-t)$ we get

$$V_n^m(t) = 2\delta_n^m \varphi^2(n) \sum_{k+l=n} |\lambda_k|^2 |\lambda_l|^2 (-F_n^{k,l}(-t)) + (E(|g_n|^4) - 2)\delta_n^m \varphi^2(n) \delta_n^{2q} (-F_n^{q,q}(-t)) |\lambda_q|^4 \; .$$

Next,

$$W_n^m(t) = -\varphi(m) \sum_{k+l=m} \sum_{j+q=k} \varphi(k) E(\overline{a}_j \overline{a}_q \overline{a}_l a_n) e^{-i\Delta_m^{k,l} t} \overline{F_k^{j,q}(t)} \; .$$

Here, we can pair j with l and q with n or j with n and q with l but not j with q. In both case, we can do the computation (with changing the indexes) :

$$n = q = k - j = k + l = m$$

As long as $l \neq n$, we get :

$$E(\overline{a}_k \overline{a}_l a_j a_q) = |\lambda_n|^2 |\lambda_l|^2$$

but otherwise

$$E(\overline{a}_k \overline{a}_l a_j a_q) = |\lambda_n|^4 E(|g_n|^4) \; .$$

Again, using our assumptions on the random variables, we get

$$\begin{aligned} W_n^m(t) \quad = \quad & -2\delta_n^m \varphi(n) \sum_{k+l=n} \varphi(k) |\lambda_n|^2 |\lambda_l|^2 e^{-i\Delta_n^{k,l} t} \overline{F_k^{n,-l}(t)} \\ & + (E(|g_n|^4) - 2)\delta_n^m \varphi(2n)\varphi(n) e^{-i\Delta_n^{2n,-n} t} \overline{F_{2n}^{n,n}(t)} |\lambda_n|^4 \end{aligned}$$

and since

$$F_k^{n,-l}(t) = \overline{F_n^{k,l}(t)},$$

we arrive at

$$\begin{aligned} W_n^m(t) &= -2\delta_n^m \varphi(n) \sum_{k+l=n} \varphi(k)|\lambda_n|^2|\lambda_l|^2 e^{-i\Delta_n^{k,l}t}(-F_n^{k,l}(-t)) \\ &\quad +(E(|g_n|^4)-2)\delta_n^m \varphi(2n)\varphi(n)(-F_n^{2n,-n}(-t))|\lambda_n|^4 \ . \end{aligned}$$

Combining the previous formulae implies the claimed expression for the second order. This completes the proof of Proposition 6.5.1. ◊

Observe that

$$d_n^m(t) = e^{-it(\omega(n)-\omega(m))}\, E(\overline{u_m}(t)u_n(t))$$

and therefore

$$E(\overline{u_m}(t)u_n(t)) - E(\overline{u_m}(0)u_n(0)) = e^{it(\omega(n)-\omega(m))}d_n^m(t) - d_n^m(0).$$

If $m = n$, it suffices to employ the fundamental theorem of calculus to the function d_n^m and to apply Proposition 6.5.1. If $m \neq n$ then one has that $d_n^m(0) = 0$ and hence one may writre

$$e^{it(\omega(n)-\omega(m))}d_n^m(t) - d_n^m(0) = e^{it(\omega(n)-\omega(m))}(d_n^m(t) - d_n^m(0))$$

and apply again the fundamental theorem of calculus in combination with Proposition 6.5.1.

Let us now bound $G_n(\lambda, t)$. For KP-I, we use the fact that $\varphi(n) = n_1$ and $|F_n^{k,l}(t)| \leq |t|$. Then, as the term involving $E(|g_n|^4)$ is included in the sum,

$$|\partial_t G_n(t)| \leq C|t|\Big(|n|^2 \sum_{k+l=n} |\lambda_k|^2|\lambda_l|^2 + |n||\lambda_n|^2 \sum_{k+l=n} |k|\ |\lambda_l|^2\Big),$$

$$\sum_{k+l=n} |k|\ |\lambda_l|^2 \leq (|n| \sum_l |\lambda_l|^2 + \sum_l |l||\lambda_l|^2) \leq C|n|\ \|u_0\|_{H^s}^2 \leq C|n|,$$

$$|n|^{2s} \sum_{k+l=n} |\lambda_k|^2|\lambda_l|^2 \leq 2 \sum_{k+l=n} |k|^{2s}|\lambda_k|^2|\lambda_l|^2 \leq 2\max(|\lambda_l|^2)\|u_0\|_{H^s}^2 \leq C$$

and since $|\lambda_n|^2|n|^{2s}$ is bounded,

$$|\partial_t G_n(t)| \leq C|t|\ |n|^{2-2s}, \quad |G_n(t)| \leq C|t|^2|n|^{2-2s} \ .$$

For KP-II, use that $\varphi(n) = n_1$ and $|F_n^{k,l}(t)| \leq \frac{C}{|n_1k_1l_1|}$. Hence

$$|\partial_t G_n(t)| \leq C\Big(|n_1| \sum_{k+l=n} \frac{|\lambda_k\lambda_l|^2}{|k_1l_1|} + |\lambda_n|^2 \sum_l \frac{|\lambda_l|^2}{|l_1|}\Big),$$

$$|n|^{2s}|n_1|\sum_{k+l=n}\frac{|\lambda_k\lambda_l|^2}{|k_1l_1|}\leq 2\sum_{k+l=n}\frac{|\lambda_l|^2}{|l_1|}|k|^{2s}|\lambda_k|^2<\infty,$$

$$\sum_l\frac{|\lambda_l|^2}{|l_1|}<\infty$$

and since $|\lambda_n|^2|n|^{2s}$ is bounded :

$$|\partial_t G_n(t)|\leq C|n|^{-2s},\quad |G_n(t)|\leq C|t|\ |n|^{-2s}\ .$$

For BBM, we have that $\varphi(n)=\frac{n}{1+n^2}$, so $|\varphi(n)|\leq |n|^{-1}$ and

$$|F_n^{k,l}(t)|\leq\frac{2}{|\Delta_n^{n,k}|}\leq 2|n|\frac{|\varphi(n)|}{|\Delta_n^{k,l}|}\leq C|n|\frac{|k|^\alpha}{|l|^\alpha}$$

for all α between -1 and 1.

Let us now compute the bound for $G_n(t)$. Set $\beta(s)=1+4s$ if $s\in]\frac{3}{8},\frac{1}{2}]$ and $\beta(s)=2+2s$ otherwise. We have

$$\left|\partial_t G_n(t)\right|\leq C|\varphi(n)|\sum_{k+l=n}|\mathrm{Re}(-F_n^{k,l}(t))|\left(|\varphi(n)||\lambda_k|^2|\lambda_l|^2+|\varphi(k)||\lambda_n|^2|\lambda_l|^2+|\varphi(l)||\lambda_k|^2|\lambda_n|^2\right)$$

$$|\partial_t G_n(t)|\leq C(I+II)$$

with

$$I=|\varphi(n)|\sum_{k+l=n}|\mathrm{Re}(-F_n^{k,l}(t))\varphi(n)|\ |\lambda_k|^2|\lambda_l|^2$$

and

$$II=|\varphi(n)|\sum_{k+l=n}|\mathrm{Re}(-F_n^{k,l}(t))\varphi(k)|\ |\lambda_n|^2|\lambda_l|^2\ .$$

By symmetry over k and l, we get that :

$$|n|^{\beta(s)}|I|\leq C\sum_{k+l=n}|\varphi(n)||k|^{\beta(s)-1}|\mathrm{Re}(-F_n^{k,l}(t))||\lambda_k|^2|\lambda_l|^2$$

Hence, by using the bound on $F_n^{k,l}$

$$|n|^{\beta(s)}|I|\leq C\sum_{k+l=n}|k|^{\beta(s)-1}\frac{|k|^\alpha}{|l|^\alpha}|\lambda_k|^2|\lambda_l|^2$$

For $s\leq\frac{1}{2}$, choose $\alpha=-2s$, ie $\beta(s)-1+\alpha=2s$,

$$|n|^{\beta(s)}|I|\leq C\sum_{k+l=n}|k|^{2s}|l|^{2s}|\lambda_k|^2|\lambda_l|^2$$

which is bounded by the sum of the $|k|^{2s}|\lambda_k|^2$ to the square. For $s \geq 1/2$, choose $\alpha = -1$,

$$|n|^{\beta(s)}|I| \leq C \sum_{k+l=n} |k|^{2s}|l|^2|\lambda_k|^2|\lambda_l|^2 .$$

For the second term, we have :

$$|n|^{\beta(s)}|II| \leq II.i + II.ii$$

with, as $|n|^{2s}|\lambda_n|^2$ is bounded

$$II.i = C|\varphi(n)| \sum_{k+l=n} |k|^{\beta(s)-2s}|\mathrm{Re}(-F_n^{k,l}(t))||\varphi(k)||\lambda_l|^2$$

and

$$II.ii = C|\varphi(n)| \sum_{k+l=n} |l|^{\beta(s)-2s}|\mathrm{Re}(-F_n^{k,l}(t))||\varphi(k)||\lambda_l|^2$$

We get

$$II.i \leq C \sum_{k+l=n} |k|^{\beta(s)-\alpha-1-2s}|l|^{\alpha}|\lambda_l|^2$$

With $\alpha = 2s$ for $s \leq 1/2$,

$$II.i \leq C \sum_{k+l=n} |l|^{2s}|\lambda_l|^2 < \infty$$

and with $\alpha = 1$ otherwise,

$$II.i \leq C \sum_{k+l=n} |l|^{2}|\lambda_l|^2 < \infty$$

We also have

$$II.ii \leq C \sum_{k+l=n} |l|^{\beta(s)-2s-\alpha}|k|^{\alpha-1}|\lambda_l|^2$$

with $\alpha = 1$, we get in the case $s \leq 1/2$ that $\beta(s) - 2s - \alpha = 1 + 4s - 2s - 1 = 2s$ and in the case $s \geq 1/2$ that $\beta(s) - 2s - \alpha = 2 + 2s - 2s - 1 = 1 \leq 2s$, hence

$$II.ii \leq C \sum_{l} |l|^{2s}|\lambda_l|^2 < \infty .$$

Therefore, in the case of BBM,

$$|G_n(t)| \leq C|t|\, |n|^{-\beta(s)} .$$

This completes the proof of Theorem 7.

Remark 6.5.1. In the case of the KP equations, formally, if λ_k do not depend on k and g_n are standard complex gaussians then $\partial_t G_n(\lambda, t) = 0$. This goes with the fact that the measure induced by

$$\sum_n g_n e^{inx}$$

should be formally invariant through the flow of KP, as it is a renormalization of the formal object

$$e^{-\|u\|_{L^2}^2}du$$

and the L^2 norm of the solution of KP does not depend on time. However the support of these measures in the case of the KP equation contains functions which are too singular for the available well-posedness theory. In the case of the KdV equation the measures is supported by $H^{-1/2-}$. This could be a motivation to try to lower the regularity assumption in our approach in the KdV case.

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