



T H È S E

pour obtenir le titre de

Docteur en Sciences

de l'Université de Cergy-Pontoise
Spécialité : Mathématiques

Présentée par

Hayk NERSISYAN

Contrôlabilité et stabilisation des équations d'Euler incompressible et compressible

soutenance le 12 décembre 2011

Jury :

Rapporteur : Andrei AGRACHEV, SISSA, Trieste
Rapporteur : Olivier GLASS, Université Paris-Dauphine
Examineur : Sergio GUERRERO, Université Paris-6
Examineur : Jean-Pierre RAYMOND, Université de Toulouse
Directeur de thèse : Armen SHIRIKYAN, Université de Cergy-Pontoise
Examineur : Nikolay TZVETKOV, Université de Cergy-Pontoise

à ma famille

Remerciements

C'est bien sur Armen Shirikyan, mon directeur de thèse, que je voudrais remercier premièrement. Pendant ces années de thèse, j'ai eu la chance de profiter de ses visions profondes, de ses immenses connaissances mathématiques et de ses idées enrichissantes. Ses grandes qualités scientifiques et humaines ont été indispensables à l'élaboration de cette thèse. Il est un modèle à suivre pour moi.

Un grand merci à mes rapporteurs Olivier Glass et Andrei Agrachev, pour avoir accepté de lire et d'évaluer mon travail. Leur lecture attentive et leurs remarques judicieuses ont été précieuses. Merci également à Sergio Guerrero, Jean-Pierre Raymond et Nikolay Tzvetkov, qui m'ont fait l'honneur en participant à mon jury.

Je voudrais remercier du fond du coeur toute l'équipe du département de mathématiques de l'Université de Cergy-Pontoise. C'était un grand plaisir pour moi de travailler dans un environnement scientifique si riche et généreux. Je remercie les thésards actuels et anciens pour l'ambiance chaleureuse qui a régné tout au long de la préparation de cette thèse.

Je n'oublie pas mes professeurs de l'Université de l'État d'Erevan qui ont su m'insuffler la passion des mathématiques par leurs encouragements et leur disponibilité. Je les remercie tous chaleureusement.

Je profite de l'occasion pour adresser un merci à mes amis d'Arménie, pour leurs encouragements et tous ces bons moments partagés.

Mes pensées vont enfin à mes parents et à mon frère Vahagn (avec Anna et Aren), mais là je manque de mots pour dire merci pour tout l'amour, le soutien et l'encouragement que j'ai reçus de leur part.

Je remercie ma femme, Emma, pour son soutien infini, la patience dont elle a fait preuve durant toute la durée de cette thèse. Merci à mon miracle, Gor, qui m'a soutenu à sa manière (areuh-euhhhh!) pour mener cette thèse à son terme.

Table des matières

1	Introduction	1
1.1	Problème de Cauchy pour l'équation d'Euler	2
1.2	Contrôlabilité de l'équation d'Euler 3D incompressible	6
1.3	Contrôlabilité de l'équation d'Euler 3D compressible	12
1.4	Stabilisation de l'équation d'Euler 2D incompressible	17
2	Contrôlabilité de l'équation d'Euler 3D incompressible	23
2.1	Introduction	24
2.2	Perturbative result on solvability of the 3D Euler system	26
2.3	Controllability of the velocity	29
2.4	Controllability of finite-dimensional projections	32
2.5	Proof of Theorem 2.3.5	35
2.6	Proof of Theorem 2.5.1	36
2.7	Non controllability result	42
3	Contrôlabilité de l'équation d'Euler 3D compressible	45
3.1	Introduction	46
3.2	Preliminaries on 3D compressible Euler system	48
3.2.1	Symmetrizable hyperbolic systems	48
3.2.2	Well-posedness of the Euler equations	49
3.2.3	Continuity property of the resolving operator	52
3.3	Main results	54
3.3.1	Controllability of Euler system	54
3.3.2	Proof of Theorem 3.3.2	57
3.4	Proof of Theorem 3.3.4	58
3.4.1	Reduction to controllability with E_1 -valued controls	58
3.4.2	Proof of Proposition 3.4.1	60
4	Stabilisation de l'équation d'Euler 2D incompressible	67
4.1	Introduction	68
4.2	Preliminaries	70
4.2.1	Poisson equations in an unbounded strip	70
4.2.2	Euler equations in an unbounded strip	75
4.3	Main result	81
4.4	Construction of the particular solution	90
4.4.1	Proof of Proposition 4.3.3	90
4.4.2	Proof of Proposition 4.3.4	93
4.5	Appendix : proof of Lemma 4.2.3	95
	Références	99

CHAPITRE 1

Introduction

1.1 Problème de Cauchy pour l'équation d'Euler

Dans cette partie, nous présentons quelques résultats bien connus sur le problème de Cauchy pour les équations d'Euler incompressible et compressible en dimension deux et trois. L'écoulement d'un fluide incompressible non visqueux évoluant dans un domaine D est décrit par l'équation d'Euler :

$$\dot{\mathbf{u}} + \langle \mathbf{u}, \nabla \rangle \mathbf{u} + \nabla p = \mathbf{f}, \quad (1.1.1)$$

$$\operatorname{div} \mathbf{u} = 0, \quad (1.1.2)$$

$$\langle \mathbf{u} \cdot \mathbf{n} \rangle|_{\partial D} = 0, \quad (1.1.3)$$

$$\mathbf{u}(0, x) = \mathbf{u}_0(x), \quad (1.1.4)$$

où $\mathbf{u} = (u_1, \dots, u_d)$ est le champ de vitesse du fluide, p est sa pression, \mathbf{f} est la force extérieure, \mathbf{u}_0 est une donnée initiale et

$$\langle \mathbf{u}, \nabla \rangle \mathbf{v} = \sum_{i=1}^d u_i(t, x) \frac{\partial}{\partial x_i} \mathbf{v}, \quad \operatorname{div} \mathbf{u} = \sum_{i=1}^d \frac{\partial}{\partial x_i} u_i.$$

L'ensemble D est un ouvert simplement connexe borné et régulier de \mathbb{R}^d et n est la normale extérieure à la frontière de D . Nous utilisons des caractères gras pour indiquer les fonctions vectorielles.

L'équation (1.1.1) est la deuxième loi de Newton, indiquant que l'accélération du fluide est proportionnelle à la pression de la force. L'équation (1.1.2) est la condition d'incompressibilité, indiquant que le volume d'une partie du fluide ne change pas dans le flux.

Le tourbillon

Commençons d'abord par le cas $d = 2$. Le tourbillon w associé au champ de vitesses \mathbf{u} est défini par la formule

$$w = \partial_2 u_1 - \partial_1 u_2 := \operatorname{curl} \mathbf{u}.$$

En appliquant l'opérateur curl à (1.1.1), on obtient que le tourbillon vérifie l'équation de transport

$$\dot{w} + \langle \mathbf{u}, \nabla \rangle w = \operatorname{curl} \mathbf{f}, \quad w(x, 0) = \operatorname{curl} \mathbf{u}_0(x). \quad (1.1.5)$$

Maintenant supposons que (\mathbf{u}, w) est la solution de (1.1.5) et

$$\operatorname{curl} \mathbf{u} = w, \quad \operatorname{div} \mathbf{u} = 0, \quad \langle \mathbf{u} \cdot \mathbf{n} \rangle|_{\partial D} = 0. \quad (1.1.6)$$

Alors, on trouve une solution unique (\mathbf{u}, p) de (1.1.1)-(1.1.4). En effet, on a $\operatorname{curl}(\dot{\mathbf{u}} + \langle \mathbf{u}, \nabla \rangle \mathbf{u} - \mathbf{f}) = 0$ ce qui implique l'existence d'une fonction p telle que $\nabla p = \dot{\mathbf{u}} + \langle \mathbf{u}, \nabla \rangle \mathbf{u} - \mathbf{f}$. Dans le cas tridimensionnel, le tourbillon \mathbf{w} est défini par

$$\mathbf{w} = (\partial_2 u_3 - \partial_3 u_2, \partial_3 u_1 - \partial_1 u_3, \partial_1 u_2 - \partial_2 u_1) =: \operatorname{curl} \mathbf{u}$$

et il est la solution du système

$$\dot{\mathbf{w}} + \langle \mathbf{u}, \nabla \rangle \mathbf{w} = \langle \mathbf{w}, \nabla \rangle \mathbf{u} + \operatorname{curl} \mathbf{f}, \quad w(x, 0) = \operatorname{curl} \mathbf{u}_0(x). \quad (1.1.7)$$

Dans ce cas aussi le système d'Euler est équivalent au problème (1.1.6), (1.1.7).

Existence et unicité

L'unicité de la solution est une conséquence de l'inégalité de Gronwall. En dimension deux d'espace, l'existence globale d'une solution régulière a été établie par Wolibner [46] en 1933 (voir aussi [29]). Dans le cas tridimensionnel, en revanche, l'existence globale reste ouverte : on ne sait pas si une solution régulière existe pour tout temps ou si elle explose en temps fini. Néanmoins, l'existence locale de la solution a été établie dans [19], [32] et [43]. Pour tout $T > 0$, on note $J_T := [0, T]$. On a le théorème suivant.

Théorème 1.1.1. *Soient $k > \frac{5}{2}$ et $\Omega \subset \mathbb{R}^3$ un ouvert borné régulier. Alors pour tout $\mathbf{f} \in L^1(J_T, \mathbf{H}^k)$ et tout $\mathbf{u}_0 \in \mathbf{H}^k(\Omega)$ tel que*

$$\begin{aligned} \operatorname{div} \mathbf{u}_0(x) &= 0, \\ \mathbf{u}_0 \cdot \mathbf{n} &= 0 \quad \text{sur } \partial\Omega \times [0, T], \end{aligned}$$

il existe $T^ \leq T$ qui ne dépend que de $\|\mathbf{u}_0\|_k + \|\mathbf{f}\|_{L^1(J_T, \mathbf{H}^k)}$ et une solution unique $(\mathbf{u}, p) \in C(J_{T^*}, \mathbf{H}^k) \times C(J_{T^*}, \mathbf{H}^{k+1})$ de (1.1.1)-(1.1.4).*

Critère d'explosion

On dispose aussi d'un critère d'explosion, établi par Beale, Kato et Majda [7].

Théorème 1.1.2. *Soient $k > \frac{5}{2}$ et $\mathbf{u} \in C(J_T, \mathbf{H}^k)$ une solution du système d'Euler. Supposons qu'il existe un temps T_* tel que la solution ne peut pas être continuée jusqu'à T_* . Alors*

$$\int_0^{T_*} \|\operatorname{curl} \mathbf{u}(t)\|_{L^\infty} = \infty.$$

Résultat perturbatif

Il y a également un résultat perturbatif sur l'existence des solutions fortes. C'est-à-dire, si $\mathbf{u} \in C(J_T, \mathbf{H}_\sigma^k)$ est la solution du système (1.1.1)-(1.1.4), alors il existe aussi une solution unique du problème perturbé. De plus, cette solution appartient à \mathbf{H}^{k-1} -voisinage de \mathbf{u} . Rappelons que dans le cas du système de Navier–Stokes, la solution du système perturbé appartient à \mathbf{H}^k -voisinage de la solution (voir [39]). Ceci est une conséquence du fait que, contrairement au système d'Euler, le système de Navier–Stokes a un effet régularisant. Cette différence provoque certaines difficultés dans la démonstration des propriétés de contrôle.

Le cas d'un cylindre non borné

Dans cette thèse on étudiera aussi les propriétés de stabilisation du system d'Euler dans un cylindre non borné. Supposons que

$$D := \{(x_1, x_2) : x_1 \in \mathbb{R}, x_2 \in (-1, 1)\}.$$

Le problème est globalement bien posé dans $C([0, \infty), H^s(D)) \times C([0, \infty), \dot{H}^{s+1}(D))$. L'unicité est toujours une conséquence de l'inégalité de Gronwall. Décrivons d'abord une méthode de construction d'une solution locale du system d'Euler dans D . Pour montrer que le système d'Euler est équivalent à (1.1.5)-(1.1.6), il suffit de prouver dans D la propriété suivante : si $\mathbf{v} \in \mathbf{H}^k$ et $\text{curl } \mathbf{v} = 0$ alors il existe $p \in H^{k+1}$ tel que $\mathbf{v} = \nabla p$. Pour établir cette propriété, il suffit de montrer que

$$(\mathbf{v}, \boldsymbol{\psi})_{L^2} = 0 \text{ pour tout } \boldsymbol{\psi} \in \mathbf{L}^2 \text{ tel que } \text{div } \boldsymbol{\psi} = 0. \quad (1.1.8)$$

Comme $\text{curl } \boldsymbol{\psi} = 0$, on a $(\mathbf{v}, \nabla^\perp \boldsymbol{\psi})_{L^2} = 0$ pour tout $\boldsymbol{\psi} \in L^2$ tel que $\nabla^\perp \boldsymbol{\psi} \in L^2$. Posons $\varphi = \int_{-1}^{x_2} \psi_1(y, x_2) dx_2$ on obtient (1.1.8). Donc, on va construire une solution du système (1.1.1)-(1.1.4). Pour cela on définit deux suites \mathbf{u}_n et w_n par

$$\begin{aligned} \mathbf{u}^0 &= \mathbf{u}_0, \\ \partial_t w^{m+1} + \langle \mathbf{u}^m, \nabla \rangle w^{m+1} &= 0, \quad w^{m+1}(0) = \text{curl } \mathbf{u}_0^{m+1}, \\ \text{curl } \mathbf{u}^{m+1} &= w^{m+1}, \quad \text{div } \mathbf{u}^{m+1} = 0, \quad \langle \mathbf{u}^{m+1} \cdot \mathbf{n} \rangle|_{\partial D} = 0, \end{aligned}$$

où $\mathbf{u}_0^{m+1} \in \mathbf{H}^{k+1}(D)$ tel que $\mathbf{u}_0^{m+1} \rightarrow \mathbf{u}_0$ dans $\mathbf{H}^k(D)$. Alors $\{\mathbf{u}^m\}$ est une suite convergente et la limite est la solution du système d'Euler avec la donnée initiale \mathbf{u}_0 .

Problème de Cauchy pour l'équation d'Euler compressible

Le mouvement d'un fluide parfait compressible barotrope s'écrit

$$\rho(\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u}) + \nabla p(\rho) = \rho \mathbf{f}, \quad (1.1.9)$$

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0, \quad (1.1.10)$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \rho(0) = \rho_0, \quad (1.1.11)$$

où la pression p est une fonction donnée de la densité ρ .

Pour savoir si le fluide est compressible ou incompressible on calcule le nombre de Mach M : le rapport entre la vitesse locale dans l'écoulement v et la vitesse de propagation du son c ($M = v/c$). Il est admis que les effets de compression peuvent être négligés si $M < 0,3$.

Nous considérons le cas où il n'y a pas de vide, c'est-à-dire, la densité $\rho > 0$. Posons $g = \log \rho$ et $h(s) = p'(e^s)$, le système ci-dessus prend la forme équivalente

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + h(g) \nabla g = \mathbf{f}, \quad (1.1.12)$$

$$(\partial_t + \mathbf{u} \cdot \nabla) g + \nabla \cdot \mathbf{u} = 0, \quad (1.1.13)$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad g(0) = g_0. \quad (1.1.14)$$

Montrons que le problème (1.1.12)-(1.1.13) est un système hyperbolique quasi-linéaire symétrisable. Rappelons que le système quasi-linéaire

$$\partial_t \mathbf{v} + \sum_{i=1}^n \mathbf{A}_i(t, \mathbf{x}, \mathbf{v}) \partial_i \mathbf{v} + \mathbf{G}(t, \mathbf{x}, \mathbf{v}) = 0, \quad \mathbf{v}(0) = \mathbf{v}_0, \quad (1.1.15)$$

est dit hyperbolique symétrique si les matrices \mathbf{A}_i sont symétriques, c'est-à-dire, $\mathbf{A}_i = \mathbf{A}_i^*$. Si les fonctions \mathbf{A}_i, \mathbf{G} sont régulières, alors le système (1.1.15) est localement bien posé (voir [30] ou [42, Chapitre 16]). Considérons maintenant un cas plus général :

$$\partial_t \mathbf{u} + \sum_{i=1}^n \mathbf{B}_i(t, \mathbf{x}, \mathbf{u}) \partial_i \mathbf{u} + \mathbf{H}(t, \mathbf{x}, \mathbf{u}) = 0, \quad \mathbf{u}(0) = \mathbf{u}_0,$$

où \mathbf{B}_i sont tels qu'il existe une matrice définie positive \mathbf{B}_0 telle que $\mathbf{B}_0 \cdot \mathbf{B}_i$ sont symétriques. Dans ce cas, on dit que le système est symétrisable. Comme il est montré dans [42, Chapitre 16, p. 366], ces systèmes sont aussi localement bien posés. Montrons que (1.1.12)-(1.1.14) est un système hyperbolique quasi-linéaire symétrisable. Posons $V = \begin{pmatrix} \mathbf{u} \\ \rho \end{pmatrix}$, alors le système (1.1.12)-(1.1.14) s'écrit

$$\partial_t V + \sum_{i=1}^3 \mathbf{A}_i(t, \mathbf{x}, V) \partial_i V = 0, \quad V(0) = (\mathbf{u}_0, \rho_0), \quad (1.1.16)$$

où

$$\mathbf{A}_i = \begin{pmatrix} u_i & 0 & 0 & h(g)\delta_1^i \\ 0 & u_i & 0 & h(g)\delta_2^i \\ 0 & 0 & u_i & h(g)\delta_3^i \\ \delta_1^i & \delta_2^i & \delta_3^i & u_i \end{pmatrix}.$$

Le système (1.1.16) est symétrisable, car la matrice

$$A_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & h(g) \end{pmatrix}$$

est définie positive et $\mathbf{A}_0 \cdot \mathbf{A}_i, i = 1, 2, 3$ sont symétriques. Par conséquent, on a le théorème suivant.

Théorème 1.1.3. *Soient $k > \frac{5}{2}$ et $T > 0$. Alors pour tout $\mathbf{f} \in L^1(J_T, \mathbf{H}^k)$ et tout $(\mathbf{u}_0, \rho) \in \mathbf{H}^k \times H^k$ il existe $T^* \leq T$ qui ne dépend que de $\|\mathbf{u}_0\|_k + \|\rho_0\|_k + \|\mathbf{f}\|_{L^1(J_T, \mathbf{H}^k)}$ et une solution unique $(\mathbf{u}, \rho) \in C(J_{T^*}, \mathbf{H}^k) \times C(J_{T^*}, H^k)$ de (1.1.12)-(1.1.14).*

1.2 Contrôlabilité de l'équation d'Euler 3D incompressible par une force extérieure de dimension finie

Résumé

Nous considérons le système de contrôle associé à l'équation d'Euler incompressible :

$$\dot{\mathbf{u}} + \langle \mathbf{u}, \nabla \rangle \mathbf{u} + \nabla p = \mathbf{f} + \boldsymbol{\eta} \quad \text{sur } \Omega \times (0, T), \quad (1.2.1)$$

$$\operatorname{div} \mathbf{u} = 0, \quad (1.2.2)$$

$$\mathbf{u} \cdot \mathbf{n} = 0 \quad \text{sur } \partial\Omega \times [0, T], \quad (1.2.3)$$

$$\mathbf{u}(0, x) = \mathbf{u}_0(x), \quad (1.2.4)$$

où Ω est un domaine borné régulier, $\boldsymbol{\eta}$ est le contrôle à valeurs dans un espace \mathbf{E} de dimension finie. Dans ce chapitre, nous montrons que pour un choix approprié de \mathbf{E} , le problème est exactement contrôlable en projections, c'est-à-dire, pour tout sous-espace $(\mathbf{F}, G) \subset \mathbf{H}^k \times H^k$ de dimension finie et pour tout $\hat{\mathbf{u}} \in \mathbf{F}, \hat{p} \in G$ il y a un contrôle $\boldsymbol{\eta}$ à valeurs dans \mathbf{E} tel que le problème (1.2.1)-(1.2.4) a une solution régulière (\mathbf{u}, p) sur $[0, T]$ dont la projection sur $\mathbf{F} \times G$ coïncide avec $(\hat{\mathbf{u}}, \hat{p})$ au temps T . Nous prouvons aussi que la vitesse \mathbf{u} est approximativement contrôlable, c'est-à-dire, $\mathbf{u}(T)$ est arbitrairement proche de $\hat{\mathbf{u}}$. L'équation (1.2.1) implique que la pression peut être exprimée en termes de vitesse, donc on ne peut pas s'attendre à contrôler approximativement la pression et la vitesse en même temps. Dans ce chapitre, nous montrons aussi que le système en question n'est pas exactement contrôlable par une force extérieure de dimension finie.

Résultats antérieurs

Citons quelques résultats antérieurs sur la contrôlabilité des systèmes d'Euler et de Navier–Stokes. Coron [12] a introduit la méthode de retour pour résoudre un problème de stabilisation. Ensuite, en utilisant cette méthode, Coron [13] a montré la contrôlabilité exacte frontière du système d'Euler incompressible en dimension 2. Glass [23] a généralisé cette méthode pour le cas de dimension 3. La stabilisation asymptotique de l'équation Euler 2D est étudiée par Coron [15] et Glass [25]. La contrôlabilité exacte des équations d'Euler et de Navier–Stokes par des contrôles distribués dans un domaine donné a été étudiée par Coron et Fursikov [17], Fursikov et Imanuvilov [22], Imanuvilov [27] et Fernández–Cara et al. [21]. Agrachev et Sarychev [3, 4] ont étudié les propriétés de contrôlabilité de certaines équations aux dérivées partielles par une force extérieure finie-dimensionnelle. Ils ont prouvé la contrôlabilité des équations d'Euler et de Navier–Stokes 2D. Rodrigues [38] utilise la méthode de Agrachev–Sarychev pour établir la contrôlabilité de l'équation Navier–Stokes 2D sur un rectangle avec la condition aux limites de Lions–Navier. Shirikyan [39, 40] généralise cette méthode pour les systèmes pour lesquels on ne sait pas si le système est globalement bien posé. En particulier, la contrôlabilité de l'équation de

Navier–Stokes 3D est prouvée. Dans [41], Shirikyan montre que l'équation d'Euler 2D n'est pas exactement contrôlable par une force extérieure de dimension finie.

Dans ce chapitre, nous nous intéressons au problème du contrôle de l'équation Euler 3D par une force finie-dimensionnelle. L'une des difficultés principales vient du fait que on ne sait pas si le système est bien posé et que on n'a pas d'effet régularisant. Nous contrôlons aussi la pression et montrons que l'équation n'est pas contrôlable dans le sens exact. La preuve est basée sur la méthode introduite par Agrachev et Sarychev, qui est décrite ci-dessous.

Méthode de Agrachev–Sarychev

Décrivons la méthode de Agrachev–Sarychev en prenant comme modèle l'équation d'Euler. On pose

$$H = \{u \in L^2 : \operatorname{div} u = 0, \int_{\mathbb{T}^3} u(x) dx = 0\}.$$

Notons Π le projecteur orthogonal sur H dans L^2 . On définit $H_\sigma^k := H^k \cap H$. Supposons que le contrôle $\boldsymbol{\eta}$ prend ses valeurs dans un espace fini-dimensionnel $\mathbf{E} \subset \mathbf{H}_\sigma^k$. Le système (1.2.1)-(1.2.4) est équivalent au problème (voir [42, Chapitre 17])

$$\dot{\boldsymbol{v}} + B(\boldsymbol{v}) = \Pi \boldsymbol{f}(t) + \boldsymbol{\eta}, \quad (1.2.5)$$

$$\boldsymbol{v}(0, x) = \Pi \boldsymbol{u}_0(x), \quad (1.2.6)$$

où $\boldsymbol{v} = \Pi u$, $B(\boldsymbol{a}, \boldsymbol{b}) = \Pi\{\langle \boldsymbol{a}, \nabla \rangle \boldsymbol{b}\}$ et $B(\boldsymbol{a}) = B(\boldsymbol{a}, \boldsymbol{a})$. Avec (1.2.5) on considère aussi le système de contrôle

$$\dot{\boldsymbol{v}} + B(\boldsymbol{v} + \boldsymbol{\zeta}) = \Pi \boldsymbol{f}(t) + \boldsymbol{\eta}, \quad (1.2.7)$$

où $\boldsymbol{\eta}$ et $\boldsymbol{\zeta}$ sont des contrôles à valeurs dans \mathbf{E} . Nous avons la propriété suivante, qui est un analogue pour les EDP d'un résultat plus général établi dans [2].

(P1) L'équation (1.2.5) est approximativement contrôlable par des contrôles à valeurs dans \mathbf{E} en temps $T > 0$ si et seulement si l'équation (1.2.7) est approximativement contrôlable par $\boldsymbol{\eta}, \boldsymbol{\zeta} \in \mathbf{E}$.

Pour tout sous-espace $\mathbf{E} \subset \mathbf{H}^k$ de dimension finie, on note $\mathcal{F}(\mathbf{E})$ le plus grand espace vectoriel $\mathbf{F} \subset \mathbf{H}^k$ tel que pour tout $\boldsymbol{\eta}_1 \in \mathbf{F}$ il y a des vecteurs $\boldsymbol{\eta}, \boldsymbol{\zeta}^1, \dots, \boldsymbol{\zeta}^n \in \mathbf{E}$ satisfaisant la relation

$$\boldsymbol{\eta}_1 = \boldsymbol{\eta} - \sum_{i=1}^n B(\boldsymbol{\zeta}^i). \quad (1.2.8)$$

L'espace $\mathcal{F}(\mathbf{E})$ est bien défini. En effet, comme \mathbf{E} est un sous-espace de dimension finie et B est un opérateur bilinéaire, alors $\mathcal{F}(\mathbf{E})$ est contenu dans un espace de dimension finie. Il est facile à voir que, si les sous-espaces \mathbf{G}_1 et \mathbf{G}_2 vérifient (1.2.8),

alors $\mathbf{G}_1 + \mathbf{G}_2$ aussi satisfait à (1.2.8). Donc, $\mathcal{F}(\mathbf{E})$ est bien défini. Évidemment, $\mathbf{E} \subset \mathcal{F}(\mathbf{E})$. On définit \mathbf{E}_k par la règle

$$\mathbf{E}_0 = \mathbf{E}, \quad \mathbf{E}_n = \mathcal{F}(\mathbf{E}_{n-1}) \quad \text{pour } n \geq 1, \quad \mathbf{E}_\infty = \bigcup_{n=1}^{\infty} \mathbf{E}_n. \quad (1.2.9)$$

Considérons le système de contrôle

$$\dot{\mathbf{v}} + B(\mathbf{v}) = \Pi \mathbf{f}(t) + \boldsymbol{\eta}_1, \quad (1.2.10)$$

où $\boldsymbol{\eta}_1$ est un contrôle à valeurs dans \mathbf{E}_1 . On a la propriété suivante, qu'on appelle le principe de convexification :

(P2) L'équation (1.2.5) est approximativement contrôlable par les contrôles à valeurs dans \mathbf{E} en temps $T > 0$ si et seulement si l'équation (1.2.10) est approximativement contrôlable par \mathbf{E}_1 .

Par itération des propriétés **(P1)** et **(P2)** on obtient que l'équation (1.2.5) est contrôlable par des contrôles à valeurs dans \mathbf{E} si et seulement si elle est contrôlable par des contrôles à valeurs dans \mathbf{E}_n , pour tout n . On montre aussi que, si \mathbf{E}_∞ est dense dans \mathbf{H}_σ^k , alors pour $n \in \mathbb{N}$ suffisamment grand, (1.2.5) est contrôlable par \mathbf{E}_n , donc aussi par \mathbf{E} .

Résultats principaux : contrôle de la vitesse

Soient T une constante positive et $\mathbf{X} \subset L^1(J_T, \mathbf{H}_\sigma^k)$ un sous-espace.

Définition 1.2.1. On dit que (1.2.5) est contrôlable par $\boldsymbol{\eta} \in \mathbf{X}$ en temps T si pour tout $\varepsilon > 0$, pour tout sous-espace de dimension finie $\mathbf{F} \subset \mathbf{H}_\sigma^k$, pour toute projection $P_{\mathbf{F}} : \mathbf{H}_\sigma^k \rightarrow \mathbf{H}_\sigma^k$ sur \mathbf{F} et pour toutes fonctions $\mathbf{u}_0 \in \mathbf{H}_\sigma^k$, $\hat{\mathbf{u}} \in \mathbf{H}_\sigma^k$ il existe un contrôle $\boldsymbol{\eta} \in \mathbf{X}$ tel que

$$\begin{aligned} P_{\mathbf{F}} \mathcal{R}_T(\mathbf{u}_0, \boldsymbol{\eta}) &= P_{\mathbf{F}} \hat{\mathbf{u}}, \\ \|\mathcal{R}_T(\mathbf{u}_0, \boldsymbol{\eta}) - \hat{\mathbf{u}}\|_k &< \varepsilon, \end{aligned}$$

où \mathcal{R} est l'opérateur résolvant le problème (1.2.5)-(1.2.6).

Notons que ce concept de contrôlabilité est plus fort que la contrôlabilité approchée et il est plus faible que la contrôlabilité exacte. L'un des résultats principaux de ce chapitre est le théorème suivant :

Théorème 1.2.2. Soit $\mathbf{f} \in C^\infty([0, \infty), \mathbf{H}_\sigma^{k+2})$ et $T > 0$. Si $\mathbf{E} \subset \mathbf{H}_\sigma^{k+2}$ est un sous-espace de dimension finie tel que \mathbf{E}_∞ est dense dans \mathbf{H}_σ^k , alors (1.2.5) est contrôlable par $\boldsymbol{\eta} \in C^\infty(J_T, \mathbf{E})$ en temps T .

Dans le cas de conditions aux bords périodiques, on construit un exemple d'un sous-espace \mathbf{E} pour lequel l'hypothèse du Théorème 1.2.2 est satisfait, c'est-à-dire, \mathbf{E}_∞ est dense dans \mathbf{H}_σ^k .

Exemple 1.2.3. Pour tout $m \in \mathbb{Z}^3$ on introduit les fonctions

$$\mathbf{c}_m(x) = \mathbf{l}(m) \cos\langle m, x \rangle, \quad \mathbf{s}_m(x) = \mathbf{l}(m) \sin\langle m, x \rangle,$$

où $\langle \cdot, \cdot \rangle$ désigne le produit scalaire de \mathbb{R}^3 et

$$\{\mathbf{l}(m), \mathbf{l}(-m)\} \text{ est une base orthonormale de } \mathbf{m}^\perp := \{x \in \mathbb{R}^3, \langle m, x \rangle = 0\}.$$

Il est démontré dans [39] que pour

$$\mathbf{E} = \text{span}\{\mathbf{c}_m, \mathbf{s}_m, |m| \leq 3\},$$

l'ensemble \mathbf{E}_∞ est dense dans \mathbf{H}_σ^k . Soulignons que l'espace \mathbf{E} ne dépend pas du choix de la base $\{\mathbf{l}(m), \mathbf{l}(-m)\}$.

La preuve du Théorème 1.2.2 est basée sur la contrôlabilité approximative uniforme du système d'Euler.

Définition 1.2.4. *L'équation (1.2.5) est dite uniformément approximativement contrôlable par $\boldsymbol{\eta} \in \mathbf{X}$ au temps T , si pour tous $\varepsilon > 0$, $\mathbf{u}_0 \in \mathbf{H}_\sigma^k$ et pour tout ensemble compact $\mathbf{K} \subset \mathbf{H}_\sigma^k$ il existe une fonction continue $\Psi : \mathbf{K} \rightarrow \mathbf{X}$ telle que*

$$\sup_{\hat{\mathbf{u}} \in \mathbf{K}} \|\mathcal{R}_T(\mathbf{u}_0, \Psi(\hat{\mathbf{u}})) - \hat{\mathbf{u}}\|_k < \varepsilon,$$

où \mathbf{X} est muni de la norme $L^1(J_T, \mathbf{H}^k)$.

En utilisant le théorème de Brouwer, on montre que la contrôlabilité approximative uniforme est plus forte que la contrôlabilité dans le sens de la Définition 1.2.1. Pour montrer que le système est uniformément approximativement contrôlable par $\boldsymbol{\eta} \in \mathbf{E}$ on utilise la méthode de Agrachev–Sarychev. L'une des difficultés principales vient du fait que l'opérateur résolvant le système n'est pas lipschitzien dans l'espace de phase. Pour surmonter cette difficulté on a combiné un résultat perturbatif pour le système d'Euler avec le critère de Beale-Kato-Majda.

Contrôle des projections de dimension finie de la vitesse et de la pression

Nous sommes intéressés aussi par les propriétés de contrôlabilité de la pression dans le système d'Euler. Nous considérons le problème (1.2.1)-(1.2.4). Si $\mathbf{u} \in C(J_T, \mathbf{H}^k)$ est la solution de (1.2.5), (1.2.6) alors (\mathbf{u}, p) sera la solution de (1.2.1)-(1.2.4), où

$$p = \Delta^{-1}(\text{div } \mathbf{f} - \sum_{i,j=1}^3 \partial_j u_i \partial_i u_j) := L(\mathbf{u}). \quad (1.2.11)$$

Ici la fonction p est définie à une constante additive près et Δ^{-1} est l'inverse de $\Delta : H_\sigma^k \rightarrow H_\sigma^{k-2}$. Notons $(\mathcal{R}(\mathbf{u}_0, \boldsymbol{\eta}), \mathcal{P}(\mathbf{u}_0, \boldsymbol{\eta}))$ la solution de (1.2.1)-(1.2.4) et $(\mathcal{R}_t(\mathbf{u}_0, \boldsymbol{\eta}), \mathcal{P}_t(\mathbf{u}_0, \boldsymbol{\eta}))$ sa restriction à l'instant t . L'équation (1.2.11) implique que (1.2.1)-(1.2.4) n'est pas approximativement contrôlable pour le couple (\mathbf{u}, p) , donc on s'intéresse à la contrôlabilité exacte en projections.

Définition 1.2.5. L'équation (1.2.1) avec $\eta \in X$ est exactement contrôlable en projections en temps T si pour tous sous-espace de dimension finie $\mathbf{F} \subset \mathbf{H}_\sigma^k$, $G \subset H^k$ et pour toutes fonctions $\mathbf{u}_0 \in \mathbf{H}_\sigma^k$, $\hat{\mathbf{u}} \in \mathbf{F}$ et $\hat{p} \in G$ il y a un contrôle $\eta \in X$ tel que

$$\begin{aligned} P_F \mathcal{R}_T(\mathbf{u}_0, \eta) &= \hat{\mathbf{u}}, \\ P_G \mathcal{P}_T(\mathbf{u}_0, \eta) &= \hat{p}. \end{aligned}$$

Théorème 1.2.6. Si $\mathbf{E} \subset \mathbf{H}_\sigma^{k+2}$ est un sous-espace de dimension finie tel que \mathbf{E}_∞ est dense dans \mathbf{H}_σ^k , alors l'eq. (1.2.1) est exactement contrôlable en projections par $\eta \in C^\infty(J_T, \mathbf{E})$ à tout moment $T > 0$.

La preuve de ce résultat est basée sur le Théorème 1.2.2 et sur le fait que pour tout $(\hat{\mathbf{u}}, \hat{p}) \in \mathbf{F} \times G$ il existe $\mathbf{v} \in \mathbf{F}^\perp$ tel que

$$\hat{p} = P_G L(\hat{\mathbf{u}} + \mathbf{v}),$$

où L est défini dans (1.2.11).

Résultat de non contrôlabilité

Notons $\mathbf{A}_T(\mathbf{u}_0, \mathbf{h}, \mathbf{E})$ l'ensemble d'atteignabilité au temps T de $\mathbf{u}_0 \in \mathbf{H}_\sigma^k$ par les contrôles à valeurs dans \mathbf{E} , c'est-à-dire,

$$\mathbf{A}_T(\mathbf{u}_0, \mathbf{h}, \mathbf{E}) = \{\hat{\mathbf{u}} \in \mathbf{H}_\sigma^k : \hat{\mathbf{u}} = \mathcal{R}_T(\mathbf{u}_0, \eta) \text{ pour un contrôle } \eta \in L^1(J_T, \mathbf{E})\}.$$

Nous montrons que les idées de [41] peuvent être généralisées pour prouver que $\mathbf{A}(\mathbf{u}_0, \mathbf{h}, \mathbf{E}) = \cup_{T \in [0, \infty)} \mathbf{A}_T(\mathbf{u}_0, \mathbf{h}, \mathbf{E})$ ne contient aucune boule de $\mathbf{H}_\sigma^{k+\gamma}$, $\gamma < 2$ dans le cas tridimensionnel. On a le théorème suivant

Théorème 1.2.7. Soient $k \geq 4$, $\mathbf{u}_0 \in \mathbf{H}_\sigma^k$, $\mathbf{h} \in C([0, \infty), \mathbf{H}_\sigma^k)$ et $\mathbf{E} \subset \mathbf{H}_\sigma^k$ un sous-espace de dimension finie. Alors pour tout $\gamma \in [0, 2)$ et toute boule $\mathbf{Q} \subset \mathbf{H}_\sigma^{k+\gamma}$, on a

$$\mathbf{A}^c(\mathbf{u}_0, \mathbf{h}, \mathbf{E}) \cap \mathbf{Q} \neq \emptyset,$$

où $\mathbf{A}^c(\mathbf{u}_0, \mathbf{h}, \mathbf{E})$ est le complément de $\mathbf{A}(\mathbf{u}_0, \mathbf{h}, \mathbf{E})$ dans l'espace \mathbf{H}_σ^k .

La preuve du Théorème 1.2.7 repose sur la comparaison des ε -entropies de Kolmogorov d'une boule \mathbf{B} dans l'espace des contrôles et de celle de la boule \mathbf{Q} dans l'espace de phase. Rappelons la définition de ε -entropie de Kolmogorov (voir [34]) : pour tout $\varepsilon > 0$, on note $N_\varepsilon(\mathbf{K})$ le nombre minimal d'ensembles de diamètre n'excédant pas 2ε qui sont nécessaires pour couvrir \mathbf{K} . L'entropie de \mathbf{K} est définie comme $H_\varepsilon(\mathbf{K}) = \ln N_\varepsilon(\mathbf{K})$. On montre que $H_\varepsilon(\mathbf{B})$ est tellement petit par rapport à $H_\varepsilon(\mathbf{Q})$ que la boule \mathbf{Q} ne peut pas être couverte par $\mathcal{R}(\mathbf{u}_0, \mathbf{B})$.

Cette méthode est assez générale et peut être appliquée à diverses EDP contrôlées. En particulier, dans [35] nous l'avons appliquée pour l'équation de Schrödinger, où le contrôle est multiplicatif. Plus précisément, nous considérons le système

$$\begin{aligned} i\dot{z} &= -\Delta z + V(x)z + \eta(t)Q(x)z, \\ z|_{\partial D} &= 0, \\ z(0, x) &= z_0(x). \end{aligned}$$

Ici la variable d'espace x appartient à un ouvert borné régulier $\Omega \subset \mathbb{R}^d, d \geq 1$, les fonctions $V, Q \in C^\infty(\overline{\Omega}, \mathbb{R})$ sont données, η est le contrôle, et z désigne l'état. Nous montrons que le système n'est pas exactement contrôlable en temps fini dans les espaces H^k avec $k \in (0, d)$. Ceci généralise les résultats des Ball, Marsden et Slemrod [5] et Turinici [45], qui ont montré que le problème n'est pas contrôlable dans l'espace H^2 .

1.3 Contrôlabilité de l'équation d'Euler 3D compressible

Résumé

Dans ce chapitre nous présentons les résultats obtenus dans [37] concernant la contrôlabilité de l'équation d'Euler 3D compressible :

$$\rho(\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u}) + \nabla p(\rho) = \rho(\mathbf{f} + \boldsymbol{\eta}), \quad (1.3.1)$$

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0, \quad (1.3.2)$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \rho(0) = \rho_0, \quad (1.3.3)$$

où $\boldsymbol{\eta}$ est le contrôle à valeurs dans un espace \mathbf{E} de dimension finie. On suppose que la variable d'espace $x = (x_1, x_2, x_3)$ appartient au tore $\mathbb{T}^3 = \mathbb{R}^3/2\pi\mathbb{Z}^3$. Le résultat principal de ce chapitre est le suivant : sous des conditions appropriées sur l'espace \mathbf{E} , le système (1.3.1)-(1.3.3) est contrôlable dans le sens suivant : pour toute donnée initiale (\mathbf{u}_0, ρ_0) et tout état final $(\hat{\mathbf{u}}, \hat{\rho})$ tels que

$$\int_{\mathbb{T}^3} \rho_0 d\mathbf{x} = \int_{\mathbb{T}^3} \hat{\rho} d\mathbf{x},$$

pour tout $T, \varepsilon > 0$ il existe un contrôle $\boldsymbol{\eta} : J_T \rightarrow \mathbf{E}$ tel que le système (1.3.1)-(1.3.3) a une solution unique régulière (\mathbf{u}, ρ) qui vérifie

$$\|(\mathbf{u}(T), \rho(T)) - (\hat{\mathbf{u}}, \hat{\rho})\|_{\mathbf{H}^k \times H^k} < \varepsilon.$$

De plus, on peut choisir $\boldsymbol{\eta}$ de sorte que pour toute fonction continue $\mathbf{F} : \mathbf{H}^k \times H^k \rightarrow \mathbb{R}^N$ admettant un inverse à droite on a

$$\mathbf{F}(\mathbf{u}(T), \rho(T)) = \mathbf{F}(\hat{\mathbf{u}}, \hat{\rho}).$$

Par exemple, on peut exiger que l'état final $(\mathbf{u}(T), \rho(T))$ ait exactement la même énergie totale que $(\hat{\mathbf{u}}, \hat{\rho})$.

Résultat principal

Comme on a dit dans le chapitre 1.1, le système (1.3.1)-(1.3.3) est équivalent à

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + h(g) \nabla g = \mathbf{f} + \boldsymbol{\eta}, \quad (1.3.4)$$

$$(\partial_t + \mathbf{u} \cdot \nabla) g + \nabla \cdot \mathbf{u} = 0, \quad (1.3.5)$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad g(0) = g_0. \quad (1.3.6)$$

Notons $\mathcal{R}(\mathbf{u}_0, g_0, \boldsymbol{\eta})$ la solution de ce système. Pour tout $\alpha > 0$ et $k \in \mathbb{N}$ on pose

$$G_\alpha^k = \{g \in H^k : \int e^{g(\mathbf{x})} d\mathbf{x} = \alpha\}.$$

Soient \mathbf{E}_n les espaces définis par (1.2.9).

Théorème 1.3.1. *Soit $\mathbf{f} \in C^\infty([0, \infty), \mathbf{H}^{k+2})$ et $T > 0$. Si $\mathbf{E} \subset \mathbf{H}^{k+2}$ est un sous-espace de dimension finie tel que \mathbf{E}_∞ est dense dans \mathbf{H}^k , alors pour toutes constantes positives $\varepsilon, \alpha \in \mathbb{R}$, pour toute fonction continue $\mathbf{F} : \mathbf{H}^k \times H^k \rightarrow \mathbb{R}^N$ admettant un inverse à droite et pour toutes fonctions $\mathbf{u}_0, \mathbf{u}_1 \in \mathbf{H}^k$ et $g_0, g_1 \in G_\alpha^k$ il existe un contrôle $\boldsymbol{\eta} \in \mathbf{E}$ tel que*

$$\begin{aligned} \|\mathcal{R}_T(\mathbf{u}_0, g_0, \boldsymbol{\eta}) - (\mathbf{u}_1, g_1)\|_{\mathbf{H}^k \times H^k} &< \varepsilon, \\ \mathbf{F}(\mathcal{R}_T(\mathbf{u}_0, g_0, \boldsymbol{\eta})) &= \mathbf{F}(\mathbf{u}_1, g_1). \end{aligned}$$

Avant de passer aux idées de la preuve, citons quelques résultats antérieurs sur la contrôlabilité de l'équation d'Euler compressible. Li et Rao [33] ont prouvé la contrôlabilité locale exacte frontière du système pour des équations 1D hyperboliques quasi-linéaires. La contrôlabilité exacte frontière pour les solutions faibles de l'équation d'Euler 1D compressible a été établi par Glass [24].

La preuve du Théorème 1.3.1 est basée sur la méthode de Agrachev–Sarychev. Nous fixons une constante $\varepsilon > 0$, un point initial $(\mathbf{u}_0, g_0) \in \mathbf{H}^k \times H^k$ et un ensemble compact $\mathbf{K} \subset \mathbf{H}^k \times H^k$.

Définition 1.3.2. *On dit que le système (1.3.4)-(1.3.6) est $(\varepsilon, \mathbf{u}_0, g_0, \mathbf{K})$ -contrôlable en temps $T > 0$ par un contrôle à valeurs dans \mathbf{X} s'il existe une application $\Psi : \mathbf{K} \rightarrow \mathbf{X}$ continue telle que*

$$\sup_{(\hat{\mathbf{u}}, \hat{g}) \in \mathbf{K}} \|\mathcal{R}_T(\mathbf{u}_0, g_0, \Psi(\hat{\mathbf{u}}, \hat{g})) - (\hat{\mathbf{u}}, \hat{g})\|_{\mathbf{H}^k \times H^k} < \varepsilon.$$

Pour prouver le Théorème 1.3.1, il suffit de montrer que pour tout $\varepsilon, \mathbf{u}_0, g_0$ et \mathbf{K} le système est $(\varepsilon, \mathbf{u}_0, g_0, \mathbf{K})$ -contrôlable.

Étape 1. Pour tout $\varepsilon > 0$, $(\mathbf{u}_0, g_0) \in \mathbf{H}^k \times G_\alpha^k$ et $\mathbf{K} \subset \mathbf{H}^k \times G_\alpha^k$ il existe un entier $N > 0$ tel que le système est $(\varepsilon, \mathbf{u}_0, g_0, \mathbf{K})$ -contrôlable par des contrôles à valeurs dans \mathbf{E}_N .

Pour cela, parallèlement à (1.3.4)-(1.3.6) on considère le système de contrôle

$$\partial_t \mathbf{u} + ((\mathbf{u} + \boldsymbol{\xi}) \cdot \nabla)(\mathbf{u} + \boldsymbol{\xi}) + h(g) \nabla g = \mathbf{f} + \boldsymbol{\eta}, \quad (1.3.7)$$

$$(\partial_t + (\mathbf{u} + \boldsymbol{\zeta}) \cdot \nabla)g + \nabla \cdot (\mathbf{u} + \boldsymbol{\zeta}) = 0, \quad (1.3.8)$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad g(0) = g_0, \quad (1.3.9)$$

où $\boldsymbol{\xi}, \boldsymbol{\zeta}$ sont aussi des contrôles. Soit $\mathcal{R}(\mathbf{u}_0, g_0, \boldsymbol{\xi}, \boldsymbol{\zeta}, \boldsymbol{\eta})$ la solution de ce système. On prend des fonctions $\tilde{\mathbf{u}}(t, \mathbf{u}_0, \hat{\mathbf{u}})$ et $\tilde{g}(t, g_0, \hat{g})$ telles que

$$\begin{aligned} \tilde{\mathbf{u}}(0, \hat{\mathbf{u}}) &= \mathbf{u}_0, \quad \tilde{\mathbf{u}}(T, \hat{\mathbf{u}}) = \hat{\mathbf{u}}, \\ \tilde{g}(0, \hat{g}) &= g_0, \quad \tilde{g}(T, \hat{g}) = \hat{g}, \\ \tilde{g}(t, \hat{g}) &\in G_\alpha^k \quad \text{pour tout } t \in J_T. \end{aligned} \quad (1.3.10)$$

Montrons d'abord qu'il existe des contrôles $\tilde{\boldsymbol{\eta}}, \tilde{\boldsymbol{\xi}} \in C^\infty(J_T, \mathbf{H}^k)$ tels que

$$\mathcal{R}(\mathbf{u}_0, g_0, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\eta}}) = (\tilde{\mathbf{u}}, \tilde{g}). \quad (1.3.11)$$

Soit $\tilde{\xi} \in C^\infty(J_T, \mathbf{H}^k)$ une solution de l'EDP linéaire

$$\partial_t \tilde{g} + ((\tilde{\mathbf{u}} + \tilde{\xi}) \cdot \nabla) \tilde{g} + \nabla \cdot (\tilde{\mathbf{u}} + \tilde{\xi}) = 0.$$

Cette équation implique que

$$\Delta(e^{\tilde{g}} \tilde{\xi}) = -\partial_t e^{\tilde{g}} - \nabla \cdot (e^{\tilde{g}} \tilde{\mathbf{u}}),$$

et la condition (1.3.10) donne l'existence de $\tilde{\xi}$. Si on prend

$$\tilde{\eta} = \partial_t \tilde{\mathbf{u}} + ((\tilde{\mathbf{u}} + \tilde{\xi}) \cdot \nabla)(\tilde{\mathbf{u}} + \tilde{\xi}) + h(\tilde{g}) \nabla \tilde{g} - \mathbf{f},$$

alors on obtient (1.3.11). Ensuite, en utilisant le fait que \mathbf{E}_∞ est dense dans \mathbf{H}^k , on trouve $\xi, \eta \in C^\infty(J_T, \mathbf{E}_N)$ tels que $\xi(0) = \xi(T) = 0$ et

$$\|\mathcal{R}_T(\mathbf{u}_0, g_0, \tilde{\xi}, \tilde{\xi}, \tilde{\eta}) - \mathcal{R}_T(\mathbf{u}_0, g_0, \xi, \xi, \eta)\|_{\mathbf{H}^k \times \mathbf{H}^k} < \varepsilon.$$

D'autre part, on a

$$\mathcal{R}_T(\mathbf{u}_0, g_0, \xi, \xi, \eta) = \mathcal{R}_T(\mathbf{u}_0, g_0, 0, 0, \eta - \partial_t \xi),$$

ce qui implique la contrôlabilité par des contrôles à valeurs dans \mathbf{E}_N .

Étape 2. $(\varepsilon, \mathbf{u}_0, g_0, \mathbf{K})$ -contrôlabilité du système par des contrôles à valeurs dans \mathbf{E}_1 est équivalente à la contrôlabilité par \mathbf{E} .

On montre que pour toute application $\Psi_1 : \mathbf{K} \rightarrow L^2(J_T, \mathbf{E}_1)$ il existe $\Psi : \mathbf{K} \rightarrow L^2(J_T, \mathbf{E})$ tel que

$$\sup_{(\hat{\mathbf{u}}, \hat{g}) \in \mathbf{K}} \|\mathcal{R}_T(\mathbf{u}_0, g_0, 0, 0, \Psi_1(\hat{\mathbf{u}}, \hat{g})) - \mathcal{R}_T(\mathbf{u}_0, g_0, 0, 0, \Psi(\hat{\mathbf{u}}, \hat{g}))\|_{\mathbf{H}^k \times \mathbf{H}^k} < \varepsilon.$$

En utilisant la propriété perturbative du système on peut supposer que le contrôle Ψ_1 est constante par morceaux en temps. Par itération on montre qu'il suffit de considérer le cas où le contrôle est constant. Donc, on suppose que

$$\Psi_1(\hat{\mathbf{u}}, \hat{g}) = \sum_{l=1}^m \varphi_l(\hat{\mathbf{u}}, \hat{g}) \boldsymbol{\eta}_1^l,$$

où $\boldsymbol{\eta}_1^l \in \mathbf{E}_1$ et $\varphi_l \in C(\mathbf{K})$. La construction de \mathbf{E}_1 implique l'existence des vecteurs $\zeta^{l,1}, \dots, \zeta^{l,2n}, \boldsymbol{\eta}^l \in \mathbf{E}$ et des constantes positives $\lambda_{l,1}, \dots, \lambda_{l,2n}$ dont la somme est égale à 1 telles que

$$\zeta^i = -\zeta^{i+n} \text{ pour } i = 1, \dots, n,$$

$$(\mathbf{u} \cdot \nabla) \mathbf{u} - \boldsymbol{\eta}_1^l = \sum_{j=1}^{2n} \lambda_{l,j} ((\mathbf{u} + \zeta^{l,j}) \cdot \nabla)(\mathbf{u} + \zeta^{l,j}) - \boldsymbol{\eta}^l \text{ pour tout } \mathbf{u} \in \mathbf{H}^1.$$

Soit $(\mathbf{u}_1, g_1) = \mathcal{R}(\mathbf{u}_0, g_0, 0, 0, \Psi_1(\hat{\mathbf{u}}, \hat{g}))$, alors

$$\dot{\mathbf{u}}_1 + \sum_{i=1}^q d_i(\hat{\mathbf{u}}, \hat{g}) ((\mathbf{u}_1 + \zeta^i) \cdot \nabla)(\mathbf{u}_1 + \zeta^i) + h(g_1) \nabla g_1 = \mathbf{f}(t) + \boldsymbol{\eta}(\hat{\mathbf{u}}, \hat{g}), \quad (1.3.12)$$

$$(\partial_t + \mathbf{u}_1 \cdot \nabla) g_1 + \nabla \cdot \mathbf{u}_1 = 0.$$

où $d_i \in C(\mathbf{K})$ et $\boldsymbol{\eta} \in C(\mathbf{K}, \mathbf{E})$. Pour tout $n \in \mathbb{N}$ on pose $\zeta_n(t, \hat{\mathbf{u}}, \hat{g}) = \zeta(\frac{nt}{T}, \hat{\mathbf{u}}, \hat{g})$, où $\zeta(t, \hat{\mathbf{u}}, \hat{g})$ est fonction 1-périodique telle que

$$\zeta(s, \hat{\mathbf{u}}, \hat{g}) = \zeta^j \text{ pour } 0 \leq s - (d_1(\hat{\mathbf{u}}, \hat{g}) + \dots + d_{j-1}(\hat{\mathbf{u}}, \hat{g})) < d_j(\hat{\mathbf{u}}, \hat{g}), \quad j = 1, \dots, q.$$

On a $\zeta(t, \hat{\mathbf{u}}, \hat{g}) = -\zeta(t - \frac{1}{2}, \hat{\mathbf{u}}, \hat{g})$ pour $t \in (\frac{1}{2}, 1)$. L'équation (1.3.12) est équivalente à

$$\begin{aligned} \dot{\mathbf{u}}_1 + ((\mathbf{u}_1 + \zeta_n(t, \hat{\mathbf{u}}, \hat{g}) + \int_0^t \mathbf{f}_n(s) ds) \cdot \nabla)(\mathbf{u}_1 + \zeta_n(t, \hat{\mathbf{u}}, \hat{g}) + \int_0^t \mathbf{f}_n(s) ds) \\ + h(g_1) \nabla g_1 = \mathbf{f}(t) + \boldsymbol{\eta}(\hat{\mathbf{u}}, \hat{g}), \\ (\partial_t + (\mathbf{u}_1 + \int_0^t \mathbf{f}_n(s) ds) \cdot \nabla) g_1 + \nabla \cdot (\mathbf{u}_1 + \int_0^t \mathbf{f}_n(s) ds) = 0, \end{aligned}$$

où

$$\begin{aligned} \mathbf{f}_n(t, \hat{\mathbf{u}}, \hat{g}) = & ((\mathbf{u}_1 + \zeta_n(t, \hat{\mathbf{u}}, \hat{g})) \cdot \nabla)(\mathbf{u}_1 + \zeta_n(t, \hat{\mathbf{u}}, \hat{g})) \\ & - \sum_{i=1}^{2q} d_i(\hat{\mathbf{u}}, \hat{g}) ((\mathbf{u}_1 + \zeta^i) \cdot \nabla)(\mathbf{u}_1 + \zeta^i). \end{aligned}$$

Alors

$$\sup_{(\hat{\mathbf{u}}, \hat{g}) \in \mathbf{K}} \left\| \int_0^t \mathbf{f}_n(s) ds \right\|_{C(J_T, \mathbf{H}^{k+1})} \rightarrow 0,$$

donc

$$\|(\mathbf{u}_1, g_1) - (\tilde{\mathbf{u}}_n, \tilde{g}_n)\|_{C(J_T, \mathbf{H}^k) \times C(J_T, H^k)} \rightarrow 0 \text{ quand } n \rightarrow \infty,$$

où $(\tilde{\mathbf{u}}_n, \tilde{g}_n)$ satisfait le problème

$$\begin{aligned} \partial_t \tilde{\mathbf{u}}_n + ((\tilde{\mathbf{u}}_n + \zeta_n(t, \hat{\mathbf{u}}, \hat{g})) \cdot \nabla)(\tilde{\mathbf{u}}_n + \zeta_n(t, \hat{\mathbf{u}}, \hat{g})) + h(\tilde{g}_n) \nabla \tilde{g}_n = \mathbf{f}(t) + \boldsymbol{\eta}(\hat{\mathbf{u}}, \hat{g}), \\ (\partial_t + \tilde{\mathbf{u}}_n \cdot \nabla) \tilde{g}_n + \nabla \cdot \tilde{\mathbf{u}}_n = 0, \\ \tilde{\mathbf{u}}_n(0) = \tilde{\mathbf{u}}_0, \quad \tilde{g}_n(0) = \tilde{g}_0. \end{aligned}$$

Pour compléter la preuve il suffit de montrer que

$$\sup_{(\hat{\mathbf{u}}, \hat{g}) \in \mathbf{K}} \|\mathcal{R}_T(\mathbf{u}_0, g_0, \zeta_n, \zeta_n, \boldsymbol{\eta}(\hat{\mathbf{u}}, \hat{g})) - \mathcal{R}_T(\mathbf{u}_0, g_0, 0, 0, \boldsymbol{\eta}(\hat{\mathbf{u}}, \hat{g}))\|_{\mathbf{H}^k \times H^k} < \varepsilon. \quad (1.3.13)$$

Comme la norme de ζ_n ne tend pas vers zero, on ne peut pas appliquer la propriété perturbative usuelle du système pour obtenir immédiatement (1.3.13). Néanmoins, on a le résultat suivant

Théorème 1.3.3. *Soit ζ_n et ξ_n deux suites bornées dans $C(J_T, \mathbf{H}^{k+2})$ et ζ_n est tel que*

$$\int_0^{t_0} \zeta_n(t) \cdot \chi_n(t) dt \rightarrow 0 \text{ dans } H^k \quad (1.3.14)$$

pour tout $t_0 \in J_T$ et pour toute suite $\chi_n : J_T \rightarrow \mathbf{H}^k$ uniformément équicontinue. Supposons que pour $\mathbf{U}_n = (\mathbf{u}_0, g_0, \boldsymbol{\xi}_n, \boldsymbol{\zeta}_n, \mathbf{f})$ le problème (1.3.7)-(1.3.9) a une solution $(\mathbf{u}_n, g_n) \in C(J_T, \mathbf{H}^{k+1}) \times C(J_T, H^{k+1})$. Alors pour $n \geq 1$ assez grand il existe une solution $\mathcal{R}(\mathbf{V}_n) \in C(J_T, \mathbf{H}^{k+1}) \times C(J_T, H^{k+1})$ avec $\mathbf{V}_n = (\mathbf{u}_0, g_0, \boldsymbol{\xi}_n, 0, \mathbf{f})$ qui vérifie

$$\mathcal{R}(\mathbf{U}_n) - \mathcal{R}(\mathbf{V}_n) \rightarrow 0 \text{ dans } C(J_T, \mathbf{H}^k) \times C(J_T, H^k).$$

La construction de $\boldsymbol{\zeta}_n$ implique (1.3.14), donc on a la contrôlabilité du système par des contrôles à valeurs dans \mathbf{E}_N .

1.4 Stabilisation de l'équation d'Euler 2D incompressible dans un cylindre infini

Résumé

Dans ce paragraphe, correspondant au chapitre 4, on s'intéresse à la stabilisation de l'équation d'Euler dans un cylindre infini. Nous montrons que pour toute solution stationnaire $(c, 0)$ du système d'Euler il existe un contrôle supporté dans une partie de la frontière du cylindre qui stabilise le système à $(c, 0)$. Pour obtenir ce résultat nous allons appliquer la *méthode du retour*.

Méthode du retour

Cette méthode a été introduite par Coron dans [13]. Il consiste à trouver une trajectoire particulière telle que le système linéarisé autour de cette trajectoire est contrôlable. Expliquons brièvement les idées principales de cette méthode dans le cas de l'équation d'Euler dans un domaine D borné et simplement connexe. Soit Γ_0 un ouvert non vide de frontière $\Gamma := \partial D$. On considère le problème

$$\dot{\mathbf{u}} + \langle \mathbf{u}, \nabla \rangle \mathbf{u} + \nabla p = 0, \quad (1.4.1)$$

$$\operatorname{div} \mathbf{u} = 0, \quad (1.4.2)$$

$$\langle \mathbf{u} \cdot \mathbf{n} \rangle|_{\Gamma \setminus \Gamma_0} = 0, \quad (1.4.3)$$

$$\mathbf{u}(0, x) = \mathbf{u}_0(x). \quad (1.4.4)$$

Coron [13] a montré le résultat suivant :

Théorème 1.4.1. *Pour tout temps $T > 0$ et pour toutes fonctions de divergence nulle \mathbf{u}_0 et \mathbf{u}_1 satisfaisant*

$$\mathbf{u}_0 \cdot \mathbf{n} = \mathbf{u}_1 \cdot \mathbf{n} = 0 \text{ sur } [0, T] \times \Gamma \setminus \Gamma_0$$

il existe une solution \mathbf{u} de (1.4.1)-(1.4.4) telle que $\mathbf{u}(0) = \mathbf{u}_0$ et $\mathbf{u}(T) = \mathbf{u}_1$.

Esquisse de la preuve. Par un changement d'échelle et par réversibilité en temps de l'équation on peut ramener le problème de la contrôlabilité globale à la contrôlabilité locale à zéro. En effet, supposons que pour une constante M suffisamment grande on a deux solutions $\mathbf{u}_{1,M}, \mathbf{u}_{2,M}$ telles que $\mathbf{u}_{1,M}(0) = \frac{\mathbf{u}_0}{M}, \mathbf{u}_{2,M}(0) = -\frac{\mathbf{u}_1}{M}$ et $\mathbf{u}_{1,M}(\frac{T}{2M}) = 0, \mathbf{u}_{2,M}(\frac{T}{2M}) = 0$. Alors la fonction

$$\mathbf{u}(t, x) = \begin{cases} M\mathbf{u}_{1,M}(x, \frac{t}{M}), & t \in [0, T/2], \\ -M\mathbf{u}_{2,M}(x, \frac{T-t}{M}), & x \in [T/2, T] \end{cases}$$

est une solution d'équation d'Euler avec $\mathbf{u}(0) = \mathbf{u}_0$ et $\mathbf{u}(T) = \mathbf{u}_1$. Donc il suffit de prouver la contrôlabilité locale à zéro. Pour cela, on construit une solution particulière (\bar{u}, \bar{p}) du système d'Euler et un recouvrement $\{B_i\}_{i=1}^N$ de \bar{D} de sorte que

(P1) Chaque boule B_i en suivant le flot de $\bar{\mathbf{u}}$ sort de \bar{D} par Γ_0 à un moment t_i .

Comme on l'a vu dans le premier paragraphe, l'équation d'Euler est équivalente au système (1.1.5), (1.1.6). On considère l'équation linéarisée de (1.1.5) autour de $\bar{\mathbf{u}}$:

$$\dot{w} + \langle \bar{\mathbf{u}}, \nabla \rangle w = 0, \quad w(x, 0) = \text{curl } \mathbf{u}_0(x). \quad (1.4.5)$$

La solution de ce problème est donnée par

$$w(\phi^{\bar{\mathbf{u}}}(x, t)) = w_0(x),$$

où $\phi^{\bar{\mathbf{u}}} : D \times \mathbb{R}_+ \rightarrow D$ est le flot de $\bar{\mathbf{u}}$. En utilisant la propriété (P1), on construit une solution de (1.4.5) telle que $w(x, T) = 0$. De plus, comme le domaine D est borné, on montre la contrôlabilité de (1.4.5) lorsque l'on remplace $\bar{\mathbf{u}}$ par $\tilde{\mathbf{u}}$ qui est assez proche. Pour montrer la contrôlabilité du problème non linéaire on construit une application F définie sur un voisinage de $\bar{\mathbf{u}}$ telle que $F(\tilde{\mathbf{u}}) =: \mathbf{v}$ est la solution de

$$\text{curl } \mathbf{v} = w, \quad (1.4.6)$$

$$\text{div } \mathbf{v} = 0, \quad (1.4.7)$$

$$\mathbf{v} \cdot \mathbf{n}|_{\Gamma} = 0. \quad (1.4.8)$$

Le point fixe de F sera une solution de (1.4.1)-(1.4.4) qui s'annule en T . \square

Résultats principaux

Soit

$$D := \{(x_1, x_2) : x_1 \in \mathbb{R}, x_2 \in (-1, 1)\}.$$

Prenons deux intervalles $(a, b), (a + d, b + d) \subset \mathbb{R}$. On note

$$\Gamma_0 = (a, b) \times \{1\} \cup (a + d, b + d) \times \{-1\}.$$

Pour tout entier $s > 0$ on définit $\mathcal{H}^s(D)$ comme l'espace de distributions u dans D avec $\nabla u \in H^{s-1}(D)$. Nous équipons $\mathcal{H}^s(D)$ avec la semi-norme

$$\|\mathbf{u}\|_{\mathcal{H}^s(D)} := \|\nabla \mathbf{u}\|_{s-1}.$$

On note $\dot{H}^s(D)$ l'espace quotient $\mathcal{H}^s(D)/\mathbb{R}$. Le théorème suivant est le résultat principal de ce chapitre.

Théorème 1.4.2. *Soit $s \geq 4$. Pour toute constante $c \in \mathbb{R}$, pour toute donnée initiale $\mathbf{u}_0 \in H^s(D)$ qui décroît de façon sur-exponentielle à l'infini et vérifie*

$$\text{div } \mathbf{u}_0 = 0, \quad \mathbf{u}_0 \cdot \mathbf{n} = 0 \text{ sur } \Gamma \setminus \Gamma_0$$

il existe une solution $(\mathbf{u}, p) \in C(\mathbb{R}_+, C(\bar{D}) \cap \dot{H}^s(D)) \times C(\mathbb{R}_+, H^{s+1}(D))$ de (1.4.1)-(1.4.4) telle que

$$\lim_{t \rightarrow \infty} (\|\mathbf{u}(\cdot, t) - (c, 0)\|_{L^\infty} + \|\nabla \mathbf{u}(\cdot, t)\|_{H^{s-1}} + \|\nabla p(\cdot, t)\|_{H^{s-1}}) = 0.$$

Dans cette formulation, le contrôle n'est pas donné explicitement, mais on peut supposer que le contrôle agit sur le système comme une condition aux bords.

Ici aussi par changement d'échelle le problème est ramené à la stabilisation locale. Pour montrer la stabilisation locale, nous allons utiliser la méthode du retour. Dans notre cas, comme le domaine D est non borné, le nombre de boules B_l est infini, donc nous ne pouvons pas construire une fonction bornée $\bar{\mathbf{u}}$, dont le flot déplace toutes les boules à l'extérieur de D en un temps fini. Néanmoins, nous pouvons trouver une solution particulière $\bar{\mathbf{u}}$ telle qu'on a la propriété **(P1)** pour un temps infini. De plus on a $\bar{\mathbf{u}}(0) = 0$ et $\bar{\mathbf{u}}(t)$ tend vers $(c, 0)$ quand $t \rightarrow \infty$.

Pour construire une solution de (1.4.5) telle que $\lim_{t \rightarrow \infty} w(t) = 0$, on considère le problème

$$\begin{aligned} \dot{w}^l + \langle \tilde{\mathbf{u}}, \nabla \rangle w^l &= 0, \\ w^l(0) &= \kappa^l \operatorname{curl}(\pi \mathbf{u}_0), \end{aligned}$$

où κ^l est une partition de l'unité subordonnée à B_l . Alors on pose

$$w(\cdot, t) = \sum_{l=i+1}^{\infty} w^l(\cdot, t) \quad \text{pour } t \in [t_{i-1/2}, t_{i+1/2}], \quad (1.4.9)$$

où $t_{-1/2} := 0$. Pour montrer que la somme (1.4.9) est finie on a besoin du fait que u_0 décroît de façon sur-exponentielle à l'infini et on a la propriété suivante

(P2) Il existe une constante N telle que pour tout $k \in \mathbb{Z}$ le nombre des boules B_i qui rencontre le rectangle $[k, k+1] \times [-1, 1]$ est plus petit que N .

Pour montrer la stabilisation du système non linéaire, il faut que

(P3) La fonction \bar{u} décroît à l'infini plus vite que $1/x_1^2$.

On définit l'application F sur un voisinage de $\bar{\mathbf{u}}$ telle que $F(\tilde{\mathbf{u}}) = \mathbf{v}$ est la solution de (1.4.6)-(1.4.8). Le point fixe de F sera une solution de (1.4.1)-(1.4.4) qui tend vers $(c, 0)$ quand $t \rightarrow \infty$.

Construction de $\bar{\mathbf{u}}$

Décrivons la construction de la fonction $\bar{\mathbf{u}}$ et des boules B_l telles que les propriétés **(P1)** – **(P3)** sont vérifiées. On cherche $\bar{\mathbf{u}}$ sous la forme $\bar{\mathbf{u}} = \nabla \theta(x, t)$ où

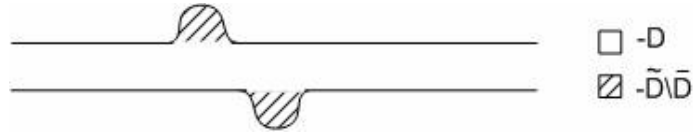
$$\Delta \theta = 0, \quad \partial_n \theta|_{\Gamma \setminus \Gamma_0} = 0.$$

On montre que pour tout $x_0 \in D$ on a

$$\mathbb{R}^2 = \{ \nabla \xi(x_0) : \Delta \xi = 0, \quad \partial_n \xi|_{\Gamma \setminus \Gamma_0} = 0 \text{ and } \|x_1^2 \xi\|_{s,D} < \infty \}. \quad (1.4.10)$$

Pour montrer (1.4.10), nous suivons les idées de [14, Lemma A.1] : on suppose qu'il existe $\mathbf{V} \in \mathbb{R}^2$ tel que

$$\mathbf{V} \cdot \nabla \xi(x_0) = 0.$$

FIGURE 1.1 – Domaine \tilde{D}

Soit $\gamma \in C_0^\infty(\mathbb{R})$ une fonction non négative telle que $\text{supp } \gamma = [a, b]$ et $\gamma \neq 0$ dans (a, b) . Posons

$$\tilde{D} := \{(x_1, x_2) : x_1 \in \mathbb{R}, x_2 \in (-1 - \gamma(x_1 - d), 1 + \gamma(x_1))\}$$

(voir figure 1).

Soit $a \in \tilde{D} \setminus D$ et G_a la solution du problème

$$\begin{aligned} \Delta G_a &= \partial_1 \delta_a \quad \text{in } \tilde{D}, \\ \frac{\partial G_a}{\partial n} &= 0 \quad \text{on } \partial \tilde{D}. \end{aligned}$$

Alors on a

$$V \cdot \nabla G_a(x_0) = 0 \tag{1.4.11}$$

pour tout $a \in \tilde{D} \setminus D$. L'analyticité de $G_a(x_0)$ par rapport à la variable $a \in \tilde{D} \setminus \{x_0\}$ implique (1.4.11) pour tout $a \in \tilde{D} \setminus \{x_0\}$. D'autre part, on a

$$\nabla G_a(x) = -\frac{1}{2\pi} \left(\frac{|x - a|^2 - 2(x_1 - a_1)^2}{|x - a|^4}, \frac{-2(x_1 - a_1)(x_2 - a_2)}{|x - a|^4} \right) + O(1), \tag{1.4.12}$$

quand $a \rightarrow x$, ce qui contredit (1.4.11). En utilisant (1.4.10), on montre qu'il existe des fonctions $\tilde{\theta}_i$ et un recouvrement $\{B_i\}_{i=1}^N$ de $[0, 1] \times [-1, 1]$ tels que

$$\phi^{\nabla \tilde{\theta}_i + (c, 0)}(B_i, 1) \in \tilde{D} \setminus \bar{D}.$$

Ensuite, on pose $B^{2kN+j} := B(x_j, r_j) + (k, 0)$ et $B^{(2k+1)N+j} := B(x_j, r_j) - (k+1, 0)$, $j = 1, \dots, N$. Alors $\{B_i\}_{i=1}^\infty$ sera un recouvrement de D . Soit $h \in C^\infty([0, 1])$ tel que

$$\begin{aligned} h(t) &= 0 & \text{si } t \in [0, 1/4], \\ h(t) &= 1 & \text{si } t \in [3/4, 1], \\ |h(t)| &\leq 1 & \text{si } t \in [0, 1]. \end{aligned}$$

Pour tout $x = (x_1, x_2) \in D$ et $c \in \mathbb{R}$ on définit

$$\tilde{\theta}^{2kN+j}(x, t) = \begin{cases} (-k - c)x_1 h'(t) & \text{si } t \in [0, 1], \\ \tilde{\theta}^j(x, t - 1) & \text{si } t \in [1, 2]. \end{cases}$$

Alors, pour tout $i \geq N$ on a

$$\phi^{\nabla\tilde{\theta}^i+(c,0)}(B_i, 2) \in \tilde{D} \setminus \bar{D}. \quad (1.4.13)$$

On pose

$$\theta^i(x, t) := \frac{2\tilde{\theta}^i(x, \frac{2t}{\tau_i})}{\tau_i}, \quad \tau_i := i \sup_{t \in [0,1]} \|\nabla\tilde{\theta}^i(\cdot, t)\|_{s, \hat{D}},$$

$$t_i := 2 \sum_{j=1}^i \tau_j, \quad t_{i+1/2} := \frac{t_i + t_{i+1}}{2}.$$

On définit la solution $\bar{\mathbf{u}}$ par

$$\begin{aligned} \bar{\mathbf{u}}(x, t) &= \nabla\theta^i(x, t - t_{i-1}) + (c, 0) && \text{pour } t \in [t_{i-1}, t_{i-1/2}], \\ \bar{\mathbf{u}}(x, t) &= -\nabla\theta^i(x, t_i - t) + (c, 0) && \text{pour } t \in [t_{i-1/2}, t_i]. \end{aligned}$$

Alors la construction des boules B_i et les relations (1.4.12), (1.4.13) impliquent que les propriétés **(P1)** – **(P3)** sont vérifiées.

Contrôlabilité de l'équation
d'Euler 3D incompressible par une
force extérieure de dimension finie

Controllability of 3D incompressible Euler equations by a finite-dimensional external force

Abstract. In this paper, we study the control system associated with the incompressible 3D Euler system. We show that the velocity field and pressure of the fluid are exactly controllable in projections by the same finite-dimensional control. Moreover, the velocity is approximately controllable. We also prove that 3D Euler system is not exactly controllable by a finite-dimensional external force.

2.1 Introduction

Let us consider the controlled incompressible 3D Euler system :

$$\dot{u} + \langle u, \nabla \rangle u + \nabla p = h + \eta, \quad \operatorname{div} u = 0, \quad (2.1.1)$$

$$u(0, x) = u_0(x). \quad (2.1.2)$$

where $u = (u_1, u_2, u_3)$ and p are unknown velocity field and pressure of the fluid, h is a given function, u_0 is an initial condition, η is the control taking values in a finite-dimensional space E , and

$$\langle u, \nabla \rangle v = \sum_{i=1}^3 u_i(t, x) \frac{\partial}{\partial x_i} v.$$

We assume that space variable $x = (x_1, x_2, x_3)$ belongs to the 3D torus $\mathbb{T}^3 = \mathbb{R}^3/2\pi\mathbb{Z}^3$.

The question of global well-posedness of 3D Euler system continues to be one of the most challenging problems of fluid mechanics. However, the local existence of solutions is well known (e.g., see [42, 43]). Moreover, Beale, Kato and A. Majda [7] proved that under the condition

$$\int_0^T \|\operatorname{curl} u(t)\|_{L^\infty} dt < \infty$$

the smooth solution exists up to time T .

In this paper, we show that for an appropriate choice of E , the problem is exactly controllable in projections, i.e., for any finite-dimensional subspaces $F, G \subset H^k$ and for any $\hat{u} \in F, \hat{p} \in G$ there is an E -valued control η such that problem (2.1.1), (2.1.2) has a solution (u, p) on $[0, T]$ whose projection onto $F \times G$ coincides with (\hat{u}, \hat{p}) at time T . We also prove that the velocity u is approximately controllable, i.e., $u(T)$ is arbitrarily close to \hat{u} . From Eq. (2.1.1) it follows that the pressure can be expressed in terms of the velocity, so we can not expect to control approximately

the pressure and the velocity simultaneously. The proofs of these results are based on a development of some ideas from [3, 4, 39, 40].

Let us mention some earlier results on the controllability of the Euler and Navier–Stokes systems. The exact controllability of Euler and Navier–Stokes systems with control supported by a given domain was studied by Coron [13], Fursikov and Imanuvilov [22], Glass [23], and Fernández-Cara et al. [21]. Agrachev and Sarychev [3, 4] were first to study controllability properties of some PDE’s of fluid dynamics by finite-dimensional external force. They proved the controllability of 2D Navier–Stokes and 2D Euler equations. Rodrigues [38] used Agrachev–Sarychev method to prove controllability of 2D Navier–Stokes equation on the rectangle with Lions boundary condition. Later Shirikyan [40] generalized this method to the case of not well-posed equations. In particular, the controllability of 3D Navier–Stokes equation is proved.

Notice that the above papers concern the problem of controllability of the velocity. In this paper, we first develop the ideas of these works to get the controllability of the velocity of 3D Euler system. One of the main difficulties comes from the fact that the resolving operator of the system is not Lipschitz continues in the phase space. We next deduce the controllability of the pressure from that of the velocity with the help of an appropriate correction of the control function.

We also treat the question of exact controllability of 3D Euler equation. In [41], Shirikyan shows that the set of attainability $A_T(u_0)$ of 2D Euler equation from initial data $u_0 \in C^s$ at time $T > 0$ cannot contain a ball of C^s . We show that the ideas of [41] can be generalized to prove that the set $A(u_0) = \cup_T A_T(u_0)$ also does not contain a ball in 3D case. In particular, 3D Euler equation is not exactly controllable.

The paper is organized as follows. In Section 2.2, we give a perturbative result for 3D Euler system. In Sections 2.3 and 2.4, we formulate the main results of this paper, which are proved in Sections 2.5 and 2.6. Section 2.7 is devoted to the problem of exact controllability.

Acknowledgments. I want to thank Armen Shirikyan for many fruitful suggestions and discussions.

Notation. We set

$$H = \{u \in L^2 : \operatorname{div} u = 0, \int_{\mathbb{T}^3} u(x) dx = 0\}.$$

Let us denote by Π the orthogonal projection onto H in L^2 . Let H^k be the space of vector functions $u = (u_1, u_2, u_3)$ with components in the Sobolev space of order k , and let $\|\cdot\|_k$ be the corresponding norm. Define $H_\sigma^k := H^k \cap H$. The Stokes operator is denoted by $L := -\Pi\Delta$, $D(L) = H_\sigma^2$. For any vector $n = (n_1, n_2, n_3) \in \mathbb{R}^3$ we denote $|n| := |n_1| + |n_2| + |n_3|$.

Let $J_T := [0, T]$ and X be a Banach space endowed with the norm $\|\cdot\|_X$. For $1 \leq p < \infty$ let $L^p(J_T, X)$ be the space of measurable functions $u : J_T \rightarrow X$ such that

$$\|u\|_{L^p(J_T, X)} := \left(\int_0^T \|u\|_X^p ds \right)^{\frac{1}{p}} < \infty.$$

The space of continuous functions $u : J_T \rightarrow X$ is denoted by $C(J_T, X)$.

2.2 Perturbative result on solvability of the 3D Euler system

Let us consider the Cauchy problem for Euler system on the 3D torus :

$$\dot{u} + \langle u, \nabla \rangle u + \nabla p = f(t), \quad \operatorname{div} u = 0, \quad (2.2.1)$$

$$u(0, x) = u_0(x). \quad (2.2.2)$$

System (2.2.1), (2.2.2) is equivalent to the problem (see [42, Chapter 17])

$$\dot{v} + B(v) = \Pi f(t),$$

$$v(0, x) = \Pi u_0(x),$$

where $v = \Pi u$, $B(a, b) = \Pi\{\langle a, \nabla \rangle b\}$ and $B(a) = B(a, a)$. We shall need the following standard estimates for the bilinear form B :

$$\|B(a, b)\|_k \leq C \|a\|_k \|b\|_{k+1} \quad \text{for } k \geq 2, \quad (2.2.3)$$

$$|(B(a, b), L^k b)| \leq C \|a\|_k \|b\|_k^2 \quad \text{for } k \geq 3, \quad (2.2.4)$$

for any $a \in H_\sigma^k$ and $b \in H_\sigma^{k+1}$ (see [11]).

Let us consider the problem

$$\dot{u} + B(u + \zeta) = f(t), \quad (2.2.5)$$

$$u(0, x) = u_0(x). \quad (2.2.6)$$

Theorem 2.2.1. *Let $T > 0$ and $k \geq 4$. Suppose that for some functions $v_0 \in H_\sigma^k$, $\xi \in L^2(J_T, H_\sigma^{k+1})$ and $g \in L^1(J_T, H_\sigma^k)$ problem (2.2.5), (2.2.6) with $u_0 = v_0$, $\zeta = \xi$ and $f = g$ has a solution $v \in C(J_T, H_\sigma^k)$. Then there are positive constants δ and C depending only on the quantity*

$$\|v\|_{C(J_T, H^k)} + \|\xi\|_{L^2(J_T, H^k)}$$

such that the following statements hold.

(i) *If $u_0 \in H_\sigma^k$, $\zeta \in L^2(J_T, H_\sigma^{k+1})$ and $f \in L^1(J_T, H_\sigma^k)$ satisfy the inequalities*

$$\|v_0 - u_0\|_{k-1} < \delta, \quad \|\zeta - \xi\|_{L^2(J_T, H^k)} < \delta, \quad \|f - g\|_{L^1(J_T, H^{k-1})} < \delta, \quad (2.2.7)$$

then problem (2.2.5), (2.2.6) has a unique solution $u \in C(J_T, H_\sigma^k)$.

(ii) Let

$$\mathcal{R} : H_\sigma^k \times L^2(J_T, H_\sigma^{k+1}) \times L^1(J_T, H_\sigma^k) \rightarrow C(J_T, H_\sigma^k)$$

be the operator that takes each triple (u_0, ζ, f) satisfying (2.2.7) to the solution u of (2.2.5), (2.2.6). Then

$$\begin{aligned} \|\mathcal{R}(u_0, \zeta, f) - \mathcal{R}(v_0, \xi, g)\|_{C(J_T, H^{k-1})} &\leq C(\|v_0 - u_0\|_{k-1} \\ &+ \|\zeta - \xi\|_{L^2(J_T, H^k)} + \|f - g\|_{L^1(J_T, H^{k-1})}). \end{aligned}$$

(iii) Let $\zeta \in C(J_T, H^k)$ and $f \in C(J_T, H^{k-1})$, and let \mathcal{R}_t be the restriction of \mathcal{R} to the time t . Then \mathcal{R} is Lipschitz-continuous in time, i.e.,

$$\|\mathcal{R}_t(u_0, \zeta, f) - \mathcal{R}_s(u_0, \zeta, f)\|_{k-1} \leq M|t - s|,$$

where M depends on $\|\mathcal{R}(u_0, \zeta, f)\|_{C(J_T, H^k)}$, $\|\zeta\|_{C(J_T, H^k)}$ and $\|f\|_{C(J_T, H^{k-1})}$.

Démonstration. We seek a solution of (2.2.5), (2.2.6) in the form $u = v + w$. Substituting this into (2.2.5), (2.2.6) and performing some transformations, we obtain the following problem for w :

$$\dot{w} + B(w + \eta, v + \xi) + B(v + \xi, w + \eta) + B(w + \eta) = q(t, x), \quad (2.2.8)$$

$$w(0, x) = w_0(x), \quad (2.2.9)$$

where $w_0 = u_0 - v_0$, $\eta = \zeta - \xi$ and $q = f - g$. By bilinearity of B , (2.2.8) is equivalent to the equation

$$\dot{w} + B(w) + \tilde{B}(w, \eta) + \tilde{B}(w, v) + \tilde{B}(w, \xi) = q(t, x) - (B(\eta) + \tilde{B}(v, \eta) + \tilde{B}(\xi, \eta)), \quad (2.2.10)$$

where $\tilde{B}(u, v) = B(u, v) + B(v, u)$. It follows from (2.2.7) that we can choose $\delta > 0$ such that the right-hand side of (2.2.10) and initial data w_0 are small in $L^1(J_T, H_\sigma^{k-1})$ and H_σ^{k-1} , respectively. Hence, by the standard theorem of existence (see [42], [43]), system (2.2.10), (2.2.9) has a unique solution $w \in C(J_T, H_\sigma^{k-1})$. From the embedding $H_\sigma^2 \hookrightarrow L^\infty$ we deduce that

$$\sup_{t \in [0, T]} \|\operatorname{curl} u(t, \cdot)\|_{L^\infty} < \infty. \quad (2.2.11)$$

In view of $u_0 \in H_\sigma^k$, $\zeta \in L^2(J_T, H_\sigma^{k+1})$, $f \in L^1(J_T, H_\sigma^k)$ and (2.2.11), the Beale–Kato–Majda theorem (see [7]) implies $u \in C(J_T, H_\sigma^k)$.

To prove (ii), let us get an a priori estimate for w . Multiplying (2.2.10) by $L^{k-1}w$ and using (2.2.3), (2.2.4), we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|w\|_{k-1}^2 &\leq C \left(\|w\|_{k-1}^3 + \|w\|_{k-1}^2 (\|\eta\|_k + \|v\|_k + \|\xi\|_k) \right. \\ &\quad \left. + \|w\|_{k-1} (\|q\|_{k-1} + \|\eta\|_k (\|\eta\|_{k-1} + \|v\|_k + \|\xi\|_k)) \right). \end{aligned} \quad (2.2.12)$$

Integrating (2.2.12), we obtain for any $t \in J_T$

$$\begin{aligned} \|w(t)\|_{k-1}^2 &\leq C\|w\|_{C(J_t, H^{k-1})} \\ &\quad \times \left[\int_0^t \left(\|w\|_{k-1}^2 + \|w\|_{k-1}(\|\eta\|_k + \|v\|_k + \|\xi\|_k) \right) ds + A \right], \end{aligned} \quad (2.2.13)$$

where $A = \|w_0\|_{k-1} + \int_0^T [\|q\|_{k-1} + \|\eta\|_k(\|\eta\|_{k-1} + \|v\|_k + \|\xi\|_k)] ds$. This implies that for any $\tau \in J_t$ we can estimate $\|w(\tau)\|_{k-1}^2$ by the right hand side of (2.2.13). Thus, we get

$$\begin{aligned} \|w\|_{C(J_t, H^{k-1})}^2 &\leq C\|w\|_{C(J_t, H^{k-1})} \\ &\quad \times \left[\int_0^t \left(\|w\|_{k-1}^2 + \|w\|_{k-1}(\|\eta\|_k + \|v\|_k + \|\xi\|_k) \right) ds + A \right], \end{aligned} \quad (2.2.14)$$

Dividing (2.2.14) by $\|w\|_{C(J_t, H^{k-1})}$, we get

$$\begin{aligned} \|w(t)\|_{k-1} &\leq \|w\|_{C(J_t, H^{k-1})} \\ &\leq C \left[\int_0^t \left(\|w\|_{k-1}^2 + \|w\|_{k-1}(\|\eta\|_k + \|v\|_k + \|\xi\|_k) \right) ds + A \right]. \end{aligned}$$

The Gronwall inequality implies

$$\|w(t)\|_{k-1} \leq A_1 + C_1 \int_0^t \|w(s, \cdot)\|_{k-1}^2 ds,$$

where $A_1 = C_1 A$ and C_1 is a constant depending on $\|v\|_{C(J_T, H^k)} + \|\xi\|_{L^2(J_T, H^k)}$. Another application of Gronwall inequality gives that

$$\|w(t)\|_{k-1} \leq \frac{A_1}{1 - C_1 A_1 t} \leq 2A_1 \text{ for any } t \leq \frac{1}{2C_1 A_1}. \quad (2.2.15)$$

We can choose $\delta > 0$ such that $\frac{1}{2C_1 A_1} \geq T$. From the definition of A_1 and (2.2.15) we deduce that

$$\|w\|_{C(J_T, H^{k-1})} \leq C(\|w_0\|_{H^{k-1}} + \|\eta\|_{L^2(J_T, H^k)} + \|q\|_{L^1(J_T, H^{k-1})}). \quad (2.2.16)$$

Statement (ii) is a straightforward consequence of (2.2.16).

Let us prove (iii). Integrating (2.2.5) over (s, t) and using (2.2.3), we get

$$\|u(t) - u(s)\|_{k-1} \leq \int_s^t \|f(\tau) - B(u(\tau) + \zeta(\tau))\|_{k-1} d\tau \leq M|t - s|.$$

This completes the proof of Theorem 2.2.1. \square

2.3 Controllability of the velocity

Let us consider the controlled Euler system :

$$\dot{u} + B(u) = h(t) + \eta(t), \quad (2.3.1)$$

$$u(0, x) = u_0(x), \quad (2.3.2)$$

where $h \in C^\infty([0, \infty), H_\sigma^{k+2})$ and $u_0 \in H_\sigma^k$ are given functions, and η is the control taking values in a finite-dimensional subspace $E \subset H_\sigma^{k+2}$. We denote by $\Theta(h, u_0)$ the set of functions $\eta \in L^1(J_T, H_\sigma^k)$ for which (2.3.1), (2.3.2) has a unique solution in $C(J_T, H_\sigma^k)$. By Theorem 2.2.1, $\Theta(h, u_0)$ is an open subset of $L^1(J_T, H_\sigma^k)$. To simplify the notation, we write $\mathcal{R}(\cdot, 0, \cdot) = \mathcal{R}(\cdot, \cdot)$. Let us recall the definition of controllability. Suppose $X \subset L^1(J_T, H_\sigma^k)$ is an arbitrary vector space.

Définition 1. *Eq. (2.3.1) with $\eta \in X$ is said to be controllable at time T if for any $\varepsilon > 0$, for any finite-dimensional subspace $F \subset H_\sigma^k$, for any projection $P_F : H_\sigma^k \rightarrow H_\sigma^k$ onto F and for any functions $u_0 \in H_\sigma^k$, $\hat{u} \in H_\sigma^k$ there is a control $\eta \in \Theta(h, u_0) \cap X$ such that*

$$\begin{aligned} P_F \mathcal{R}_T(u_0, \eta) &= P_F \hat{u}, \\ \|\mathcal{R}_T(u_0, \eta) - \hat{u}\|_k &< \varepsilon. \end{aligned}$$

Let us recall some notation introduced in [3], [4] and [39]. For any finite-dimensional subspace $E \subset H_\sigma^{k+2}$, we denote by $\mathcal{F}(E)$ the largest vector space $\mathcal{F} \subset H_\sigma^{k+2}$ such that for any $\eta_1 \in \mathcal{F}$ there are vectors $\eta, \zeta^1, \dots, \zeta^n \in E$ and positive constants $\alpha_1, \dots, \alpha_n$ satisfying the relation

$$\eta_1 = \eta - \sum_{i=1}^n \alpha_i B(\zeta^i). \quad (2.3.3)$$

The space $\mathcal{F}(E)$ is well defined. Indeed, as E is a finite-dimensional subspace and B is a bilinear operator, then $\mathcal{F}(E)$ is contained in a finite-dimensional space. It is easy to see that if subspaces G_1 and G_2 satisfy (2.3.3), then so does $G_1 + G_2$. Thus, $\mathcal{F}(E)$ is well defined. Obviously, $E \subset \mathcal{F}(E)$. We define E_k by the rule

$$E_0 = E, \quad E_n = \mathcal{F}(E_{n-1}) \quad \text{for } n \geq 1, \quad E_\infty = \bigcup_{n=1}^{\infty} E_n.$$

The following theorem is the main result of this section.

Theorem 2.3.1. *Let $h \in C^\infty([0, \infty), H_\sigma^{k+2})$. If $E \subset H_\sigma^{k+2}$ is a finite-dimensional subspace such that E_∞ is dense in H_σ^k , then Eq. (2.3.1) with $\eta \in C^\infty(J_T, E)$ is controllable at any time T .*

Example 2.3.2. Let us introduce the functions

$$c_m(x) = l(m) \cos\langle m, x \rangle, \quad s_m(x) = l(m) \sin\langle m, x \rangle, \quad (2.3.4)$$

where $m \in \mathbb{Z}^3$ and

$$\{l(m), l(-m)\} \text{ is an orthonormal basis in } m^\perp := \{x \in \mathbb{R}^3, \langle x, m \rangle = 0\}. \quad (2.3.5)$$

It is shown in [39] that if

$$E = \text{span}\{c_m, s_m, |m| \leq 3\},$$

then E_∞ is dense in H_σ^k . We emphasise for what follows that the space E does not depend on the choice of the basis $\{l(m), l(-m)\}$.

The proof of Theorem 2.3.1 is based on the uniform approximate controllability of the Euler system.

Définition 2. *Eq. (2.3.1) with $\eta \in X$ is said to be uniformly approximately controllable at time T if for any $\varepsilon > 0$, any $u_0 \in H_\sigma^k$ and any compact set $K \subset H_\sigma^k$ there is a continuous function $\Psi : K \rightarrow \Theta(h, u_0) \cap X$ such that*

$$\sup_{\hat{u} \in K} \|\mathcal{R}_T(u_0, \Psi(\hat{u})) - \hat{u}\|_k < \varepsilon, \quad (2.3.6)$$

where $\Theta(h, u_0) \cap X$ is endowed with the norm of $L^1(J_T, H^k)$.

The following lemma shows that to prove uniform approximate controllability of Eq. (2.3.1) it suffices to consider the case in which the target set $K \subset H_\sigma^k$ consists of sufficiently smooth functions.

Lemma 2.3.3. *Suppose that for compact subset $K \subset H_\sigma^{k+1}$ there is a continuous function $\Psi : K \rightarrow \Theta(h, u_0) \cap X$ such that (2.3.6) holds. Then Eq. (2.3.1) with $\eta \in X$ is uniformly approximately controllable at time T .*

Démonstration. For any compact set $K \subset H_\sigma^k$ there is a small constant $\delta > 0$ such that

$$\sup_{\hat{u} \in K} \|e^{-\delta L} \hat{u} - \hat{u}\|_k < \frac{\varepsilon}{2}.$$

As $K_1 := e^{-\delta L} K$ is compact in H_σ^{k+1} , by assumption, there is a continuous mapping $\Psi : K_1 \rightarrow \Theta(h, u_0) \cap C^\infty(J_T, X)$ such that

$$\sup_{\hat{u} \in K_1} \|\mathcal{R}_T(u_0, \Psi(\hat{u})) - \hat{u}\|_k < \frac{\varepsilon}{2}.$$

Therefore the continuous mapping $\Phi : K \rightarrow \Theta(h, u_0) \cap X$, $\hat{u} \rightarrow \Psi(e^{-\delta L} \hat{u})$ satisfies the inequality

$$\sup_{\hat{u} \in K} \|\mathcal{R}_T(u_0, \Phi(\hat{u})) - \hat{u}\|_k < \varepsilon.$$

□

The following lemma shows that the uniform approximate controllability is stronger than controllability.

Lemma 2.3.4. *If Eq. (2.3.1) with $\eta \in X$ is uniformly approximately controllable at time T , then it is also controllable.*

Démonstration. Suppose $F \subset H_\sigma^k$ is a finite-dimensional subspace and P_F is a projection onto F , $u_0 \in H_\sigma^k$ and $\hat{u} \in F$. Let $B_F(R)$ be the closed ball in F of radius R centred at origin with $R > M\varepsilon$, where M is the norm of P_F and $\varepsilon > 0$ is an arbitrary constant. Since $B_F(R)$ is a compact subset of H_σ^k , there is a continuous mapping $\Psi : B_F(R) \rightarrow \Theta(h, u_0) \cap X$ such that

$$\sup_{\hat{u} \in B_F(R)} \|\mathcal{R}_T(u_0, \Psi(\hat{u})) - \hat{u}\|_k < \varepsilon. \quad (2.3.7)$$

Therefore the continuous mapping $\Phi : B_F(R) \rightarrow F$, $\hat{u} \rightarrow P_F \mathcal{R}_T(u_0, \Psi(\hat{u}))$ satisfies the inequality

$$\sup_{\hat{u} \in B_F(R)} \|\Phi(\hat{u}) - \hat{u}\|_k < M\varepsilon.$$

Fixing $v \in B_F(R - M\varepsilon)$ and applying the Brouwer theorem to the mapping $u \rightarrow v + u - \Phi(u) : B_F(R) \rightarrow B_F(R)$, we get

$$B_F(R - M\varepsilon) \subset \Phi(B_F(R)). \quad (2.3.8)$$

Let $\hat{u} \in F$. By (2.3.8), for sufficiently large R there is a function $u_1 \in B_F(R)$ such that

$$P_F \mathcal{R}_T(u_0, \Psi(u_1)) = \hat{u}. \quad (2.3.9)$$

Using (2.3.7) and (2.3.9), we obtain

$$\begin{aligned} \|\mathcal{R}_T(u_0, \Psi(u_1)) - \hat{u}\|_k &\leq \|\mathcal{R}_T(u_0, \Psi(u_1)) - u_1\|_k \\ &\quad + \|u_1 - P_F \mathcal{R}_T(u_0, \Psi(u_1))\|_k < \varepsilon + M\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, this completes the proof. \square

Lemma 2.3.4 implies that Theorem 2.3.1 is an immediate consequence of the following result, which will be proved in Sections 5 and 6.

Theorem 2.3.5. *Let $h \in C^\infty([0, \infty), H_\sigma^{k+2})$. If $E \subset H_\sigma^{k+2}$ is a finite-dimensional subspace such that E_∞ is dense in H_σ^k , then Eq. (2.3.1) with $\eta \in C^\infty(J_T, E)$ is uniformly approximately controllable at any time T .*

2.4 Controllability of finite-dimensional projections of the velocity and pressure

In this section, we are interested in controllability properties of pressure in Euler system. We consider the problem (2.1.1), (2.1.2). If $u \in C(J_T, H^k)$ is a solution of (2.3.1), (2.3.2), then (u, p) will be the solution of (2.1.1), (2.1.2), where

$$p = \Delta^{-1}(\operatorname{div} h - \sum_{i,j=1}^3 \partial_j u_i \partial_i u_j). \quad (2.4.1)$$

Here the function p is defined up to the an additive constant and Δ^{-1} is the inverse of $\Delta : H_\sigma^k \rightarrow H_\sigma^{k-2}$. In what follows we normalise p by the condition that its mean value on \mathbb{T}^3 is zero. Denote by $(\mathcal{R}(u_0, \eta), \mathcal{P}(u_0, \eta))$ the solution of (2.1.1), (2.1.2) and by $(\mathcal{R}_t(u_0, \eta), \mathcal{P}_t(u_0, \eta))$ its restriction to the time t . Eq. (2.4.1) implies that (2.1.1), (2.1.2) is not approximately controllable, so we will be interested in exact controllability in projections.

Définition 3. *Eq. (2.1.1) with $\eta \in X$ is said to be exactly controllable in projections at time T if for any finite-dimensional subspaces $F \subset H_\sigma^k$, $G \subset H^k$ and for any functions $u_0 \in H_\sigma^k$, $\hat{u} \in F$ and $\hat{p} \in G$ there is a control $\eta \in \Theta(h, u_0) \cap X$ such that*

$$\begin{aligned} P_F \mathcal{R}_T(u_0, \eta) &= \hat{u}, \\ P_G \mathcal{P}_T(u_0, \eta) &= \hat{p}. \end{aligned}$$

Theorem 2.4.1. *If $E \subset H_\sigma^{k+2}$ is a finite-dimensional subspace such that E_∞ is dense in H_σ^k , then Eq. (2.1.1) with $\eta \in C^\infty(J_T, E)$ is exactly controllable in projections at any time $T > 0$.*

Démonstration. To simplify the proof, we shall assume that $h = 0$. The proof remains literally the same in the case $h \neq 0$. An argument similar to that used in the proof of Lemma 2.3.4 shows that it suffices to establish the following property : for any compact set $K \subset H_\sigma^k \times H^k$ and for any constant $\varepsilon > 0$ there is a continuous function $\Psi : K \rightarrow \Theta(h, u_0) \cap X$ such that

$$\begin{aligned} \sup_{(\hat{u}, \hat{p}) \in K} \|\mathcal{R}_T(u_0, \Psi(\hat{u}, \hat{p})) - \hat{u}\|_k &< \varepsilon, \\ \sup_{(\hat{u}, \hat{p}) \in K} \|P_G \mathcal{P}_T(u_0, \Psi(\hat{u}, \hat{p})) - \hat{p}\|_k &< \varepsilon. \end{aligned}$$

We introduce the spaces

$$\begin{aligned} F_m &:= \operatorname{span}\{c_n, s_n, |n| \leq m, n \in \mathbb{Z}_*^3\}, \\ G_m &:= \operatorname{span}\{\sin\langle n, x \rangle, \cos\langle n, x \rangle, |n| \leq m, n \in \mathbb{Z}_*^3\}, \end{aligned}$$

where the functions c_n, s_n are defined in (2.3.4), (2.3.5). By an approximation argument, it suffices to construct Ψ for any compact set $K \subset F_m \times G_m$. For an integer

$m \geq 1$, we introduce the symmetric quadratic form

$$A(u, v) = -P_{G_m} \Delta^{-1} \sum_{i,j=1}^3 \partial_j u_i \partial_i v_j$$

and set $A(u) = A(u, u)$. Clearly, we have the following inequality

$$\|A(u) - A(v)\|_k \leq C \|u - v\|_k, \quad (2.4.2)$$

where $u, v \in H_\sigma^k$ and C is constant depending on $\|u\|_k + \|v\|_k$. Eq. (2.4.1) implies

$$P_{G_m} \mathcal{P}_t(u_0, \eta) = A(\mathcal{R}_t(u_0, \eta)). \quad (2.4.3)$$

We admit for the moment the following lemma.

Lemma 2.4.2. *For any $\hat{u} \in F_m$ and $\hat{p} \in G_m$ there is $v \in F_m^\perp \cap H_\sigma^k$ such that*

$$\hat{p} = A(\hat{u} + v), \quad (2.4.4)$$

where F_m^\perp is the orthogonal complement of F_m in the space H . Moreover, the mapping $(\hat{u}, \hat{p}) \rightarrow v$ is continuous from $F_m \times G_m$ to F_m^\perp , where F_m, G_m and F_m^\perp are endowed with the norm of H^k .

By Theorem 2.3.5, there is a continuous mapping Ψ such that

$$\sup_{(\hat{u}, \hat{p}) \in K} \|\mathcal{R}_T(u_0, \Psi(\hat{u}, \hat{p})) - (\hat{u} + v)\|_k < \varepsilon,$$

where v satisfies (2.4.4). From (2.4.2), (2.4.3) and (2.4.4), we have

$$\begin{aligned} \sup_{(\hat{u}, \hat{p}) \in K} \|P_{G_m} \mathcal{P}_T(u_0, \Psi(\hat{u}, \hat{p})) - \hat{p}\|_k &\leq \sup_{(\hat{u}, \hat{p}) \in K} \|A(\mathcal{R}_T(u_0, \Psi(\hat{u}, \hat{p}))) - A(\hat{u} + v)\|_k \\ &\leq C \sup_{(\hat{u}, \hat{p}) \in K} \|\mathcal{R}_T(u_0, \Psi(\hat{u}, \hat{p})) - (\hat{u} + v)\|_k. \end{aligned}$$

This completes the proof of Theorem 2.4.1. □

Proof of Lemma 2.4.2. It is easy to see that (2.4.4) is equivalent to

$$A(v) + 2A(\hat{u}, v) = \hat{p} - A(\hat{u}) =: \sum_{|n| \leq m} (C_n \sin \langle n, x \rangle + D_n \cos \langle n, x \rangle). \quad (2.4.5)$$

For all $n \in \mathbb{Z}_*^3$, $|n| \leq m$ let us take $\{k_n^1\}$, $\{k_n^2\}$, $\{k_n^3\}$ and $\{k_n^4\}$ in \mathbb{Z}_*^3 such that $|k_n^i| > 2m$ and

- (a) $k_n^2 - k_n^1 = k_n^4 - k_n^3 = n$,
- (b) $\min\{|k_n^i + k_n^j|, |k_n^i \pm k_n^j|, |k_n^3 - k_n^d|, |k_n^4 - k_n^d|\} > m$,
- (c) k_n^1 and k_n^3 are not parallel to k_n^2 and k_n^4 , respectively,

34 Chapitre 2. Contrôlabilité de l'équation d'Euler 3D incompressible

for all $i, j = 1, 2, 3, 4$, $d = 1, 2$, $|r| < m$ and $n \neq r$. This choice is possible. Indeed, let $\phi : \mathbb{Z}_*^3 \rightarrow \mathbb{N}_*$ be an injection and let

$$\begin{aligned} k_n^1 &= 8\phi(n)\mathbf{m}(n), & k_n^3 &= (8\phi(n) + 4)\mathbf{m}(n), \\ k_n^2 &= 8\phi(n)\mathbf{m}(n) + n, & k_n^4 &= (8\phi(n) + 4)\mathbf{m}(n) + n, \end{aligned} \quad (2.4.6)$$

where $\mathbf{m}(n) \in \mathbb{Z}_*^3$ is not parallel to n and $|\mathbf{m}(n)| = m$. It is easy to see that $\{k_n^j\}$ satisfy (a) – (c). We seek v in the form

$$v = \sum_{|n| \leq m} (C_{k_n^1} s_{k_n^1} + D_{k_n^2} c_{k_n^2} + C_{k_n^3} s_{k_n^3} + C_{k_n^4} s_{k_n^4}).$$

Substituting this expression of v into (2.4.5) and using the construction of k_n^i , we obtain

$$\begin{aligned} & \sum_{|n| \leq m} \left(A(C_{k_n^1} s_{k_n^1} + D_{k_n^2} c_{k_n^2}) + A(C_{k_n^3} s_{k_n^3} + C_{k_n^4} s_{k_n^4}) \right) \\ &= \sum_{|n| \leq m} (C_n \sin\langle n, x \rangle + D_n \cos\langle n, x \rangle). \end{aligned}$$

On the other hand,

$$\begin{aligned} A(C_{k_n^1} s_{k_n^1} + D_{k_n^2} c_{k_n^2}) &= \Delta^{-1} \sum_{i,j=1}^3 l_i(k_n^1)(k_n^1)_j l_j(k_n^2)(k_n^2)_i C_{k_n^1} D_{k_n^2} \sin\langle n, x \rangle \\ &= -\frac{C_{k_n^1} D_{k_n^2}}{n_1^2 + n_2^2 + n_3^2} \langle l(k_n^1), k_n^2 \rangle \langle l(k_n^2), k_n^1 \rangle \sin\langle n, x \rangle, \end{aligned}$$

where $l_j(k_n^i)$ and $(k_n^i)_j$ are j -th coordinates of $l(k_n^i)$ and k_n^i , respectively. As k_n^1 is not parallel to k_n^2 , we can choose $l(k_n^1)$ and $l(k_n^2)$ not perpendicular to k_n^2 and k_n^1 , respectively, i.e.,

$$\langle l(k_n^1), k_n^2 \rangle \langle l(k_n^2), k_n^1 \rangle \neq 0.$$

Hence, there are constants $C_{k_n^1}, D_{k_n^2}$ continuously depending on C_n , and therefore on (\hat{u}, \hat{p}) , such that

$$A(C_{k_n^1} s_{k_n^1} + D_{k_n^2} c_{k_n^2}) = C_n \sin\langle n, x \rangle.$$

In the same way, we can choose $C_{k_n^3}, C_{k_n^4}$ such that

$$A(C_{k_n^3} s_{k_n^3} + C_{k_n^4} s_{k_n^4}) = D_n \cos\langle n, x \rangle.$$

Thus we have (2.4.4). □

2.5 Proof of Theorem 2.3.5

Let us fix a constant $\varepsilon > 0$, an initial point $u_0 \in H_\sigma^k$, a compact set $K \subset H_\sigma^k$ and a vector subspace $X \subset L^1(J_T, H_\sigma^k)$. Eq. (2.3.1) with $\eta \in X$ is said to be uniformly (ε, u_0, K) -controllable at time $T > 0$ if there is a continuous mapping

$$\Psi : K \rightarrow \Theta(h, u_0) \cap X$$

such that

$$\sup_{\hat{u} \in K} \|\mathcal{R}_T(u_0, \Psi(\hat{u})) - \hat{u}\|_k < \varepsilon,$$

where $\Theta(h, u_0) \cap X$ is endowed with the norm of $L^1(J_T, H_\sigma^k)$.

Theorem 2.3.5 is deduced from the following result, which is established in next section.

Theorem 2.5.1. *Let $E \subset H_\sigma^{k+2}$ be a finite-dimensional subspace. If Eq. (2.3.1) with $\eta \in C^\infty(J_T, \mathcal{F}(E))$ is uniformly (ε, u_0, K) -controllable, then it is also (ε, u_0, K) -controllable with $\eta \in C^\infty(J_T, E)$.*

Proof of Theorem 2.3.5. We first prove that there is an integer $N \geq 1$ depending only on ε, u_0 and K such that Eq. (2.3.1) with $\eta \in C(J_T, E_N)$ is uniformly (ε, u_0, K) -controllable at time T . Let us define a continuous operator defined on $J_T \times K$ by

$$u_{\mu, \delta}(t, \hat{u}) = T^{-1}(te^{\mu L}\hat{u} + (T-t)e^{\delta L}u_0).$$

It is easy to see that $u_{\mu, \delta}$ satisfies Eq. (2.3.1) with

$$\eta_{\mu, \delta} = \dot{u}_{\mu, \delta} + B(u_{\mu, \delta}) - h(t).$$

As K is a compact set in H_σ^k , we have

$$\begin{aligned} \sup_{\hat{u} \in K} \|u_{\mu, \delta}(T, \hat{u}) - \hat{u}\|_k &\rightarrow 0 \text{ as } \mu \rightarrow 0, \\ \sup_{\hat{u} \in K} \|u_{\mu, \delta}(0, \hat{u}) - u_0\|_k &\rightarrow 0 \text{ as } \delta \rightarrow 0. \end{aligned}$$

The fact that E_∞ is dense in H_σ^k implies

$$\|P_{E_N}\eta_{\mu, \delta} - \eta_{\mu, \delta}\|_{L^1(J_T, H^k)} \rightarrow 0 \text{ as } N \rightarrow \infty.$$

By Theorem 2.2.1, we can chose N, μ and δ such that

$$\sup_{\hat{u} \in K} \|\mathcal{R}(u_0, P_{E_N}\eta_{\mu, \delta}(\hat{u})) - \hat{u}\|_k < \varepsilon.$$

We note that the mapping $P_{E_N}\eta_{\mu, \delta}(\cdot, \cdot) : \hat{u} \rightarrow P_{E_N}\eta_{\mu, \delta}(\cdot, \hat{u})$ is continuous from K to $C(J_T, H_\sigma^k)$. Hence Eq. (2.3.1) is uniformly (ε, u_0, K) -controllable with $\eta \in C(J_T, E_N)$. Applying N times Theorem 2.5.1, we complete the proof of Theorem 2.3.5. \square

2.6 Proof of Theorem 2.5.1

The proof of Theorem 2.5.1 is inspired by ideas from [3, 4, 39, 40]. Let us consider the following control system :

$$\dot{u} + B(u + \zeta) = h + \eta, \quad (2.6.1)$$

where η, ζ are E -valued controls. Let $\hat{\Theta}(u_0, h)$ be the set of pairs $(\eta, \zeta) \in L^1(J_T, H_\sigma^k) \times L^2(J_T, H_\sigma^{k+1})$ for which problem (2.6.1), (2.3.2) has a unique solution in $C(J_T, H_\sigma^k)$. Eq (2.6.1) with $(\eta, \zeta) \in \hat{X} \subset L^1(J_T, H_\sigma^k) \times L^2(J_T, H_\sigma^{k+1})$ is said to be uniformly (ε, u_0, K) -controllable if there is a continuous mapping

$$\hat{\Psi} : K \rightarrow \hat{\Theta}(h, u_0) \cap \hat{X}$$

such that

$$\sup_{\hat{u} \in K} \|\mathcal{R}_T(u_0, \hat{\Psi}(\hat{u})) - \hat{u}\|_k < \varepsilon, \quad (2.6.2)$$

where $\hat{\Theta}(h, u_0) \cap \hat{X}$ is endowed with the norm of $L^1(J_T, H_\sigma^k) \times L^2(J_T, H_\sigma^{k+1})$.

We claim that, when proving Theorem 2.5.1, it suffices to assume $u_0 \in H_\sigma^{k+2}$. Suppose that for any $v_0 \in H_\sigma^{k+2}$ and for any continuous mapping $\Phi : K \rightarrow \Theta(h, v_0) \cap C^\infty(J_T, E_1)$ there is a continuous mapping

$$\hat{\Phi} : K \rightarrow \Theta(h, v_0) \cap C^\infty(J_T, E)$$

such that

$$\sup_{\hat{u} \in K} \|\mathcal{R}_T(v_0, \Phi(\hat{u})) - \mathcal{R}_T(v_0, \hat{\Phi}(\hat{u}))\|_k < \frac{\varepsilon}{3}.$$

Let us show that for any $u_0 \in H_\sigma^k$ and for any continuous mapping $\Psi : K \rightarrow \Theta(h, v_0) \cap C^\infty(J_T, E_1)$ there is a continuous mapping $\hat{\Phi} : K \rightarrow \Theta(h, v_0) \cap C^\infty(J_T, E)$ such that

$$\sup_{\hat{u} \in K} \|\mathcal{R}_T(u_0, \Psi(\hat{u})) - \mathcal{R}_T(u_0, \hat{\Psi}(\hat{u}))\|_k < \varepsilon.$$

By Theorem 2.2.1, there is $v_0 \in H_\sigma^{k+2}$ such that

$$\sup_{\hat{u} \in K} \|\mathcal{R}(u_0, \Psi(\hat{u})) - \mathcal{R}(v_0, \Psi(\hat{u}))\|_{C(J_T, H^k)} < \frac{\varepsilon}{3}. \quad (2.6.3)$$

By our assumption, as $v_0 \in H_\sigma^{k+2}$, there is a continuous mapping

$$\hat{\Psi}_\varepsilon : K \rightarrow \Theta(h, u_0) \cap C^\infty(J_T, E)$$

such that

$$\sup_{\hat{u} \in K} \|\mathcal{R}_T(v_0, \Psi(\hat{u})) - \mathcal{R}_T(v_0, \hat{\Psi}_\varepsilon(\hat{u}))\|_k < \frac{\varepsilon}{3}. \quad (2.6.4)$$

By Theorem 2.2.1, we have

$$\|\mathcal{R}_T(v_0, \hat{\Psi}_\varepsilon(\hat{u})) - \mathcal{R}_T(u_0, \hat{\Psi}_\varepsilon(\hat{u}))\|_k \leq C\|v_0 - u_0\|, \quad (2.6.5)$$

where C is a constant not depending on ε . Choosing v_0 sufficiently close to u_0 and using inequalities (2.6.3), (2.6.4) and (2.6.5), we get

$$\|\mathcal{R}_T(u_0, \Psi(\hat{u})) - \mathcal{R}_T(u_0, \hat{\Psi}_\varepsilon(\hat{u}))\|_k < \varepsilon.$$

From now on, we assume that $u_0 \in H_\sigma^{k+2}$. In this case, Theorem 2.5.1 is deduced from the following two propositions.

Proposition 2.6.1. *Eq. (2.3.1) with $\eta \in C^\infty(J_T, E)$ is uniformly (ε, u_0, K) -controllable if and only if so is Eq. (2.6.1) with $(\eta, \zeta) \in C^\infty(J_T, E \times E)$.*

Proposition 2.6.2. *Eq. (2.6.1) with $(\eta, \zeta) \in C^\infty(J_T, E \times E)$ is uniformly (ε, u_0, K) -controllable if and only if so is Eq. (2.3.1) with $\eta_1 \in C^\infty(J_T, E_1)$.*

Proof of Proposition 2.6.1. We show that if (2.6.1) with $(\eta, \zeta) \in C^\infty(J_T, E \times E)$ is uniformly (ε, u_0, K) -controllable, then so is (2.3.1) with $\eta \in C^\infty(J_T, E)$. Let

$$\hat{\Psi} : K \rightarrow \hat{\Theta}(h, u_0) \cap C^\infty(J_T, E \times E), \quad \hat{\Psi}(\hat{u}) = (\eta(t, \hat{u}), \zeta(t, \hat{u}))$$

be such that

$$\hat{\varepsilon} := \sup_{\hat{u} \in K} \|\mathcal{R}_T(u_0, \hat{\Psi}(\hat{u})) - \hat{u}\|_k < \varepsilon. \quad (2.6.6)$$

Let us choose $\zeta_n(\cdot, \hat{u}) \in C^\infty(J_T, E)$ such that $\zeta_n(0) = \zeta_n(T) = 0$, the mapping $\zeta_n(\cdot, \cdot) : \hat{u} \rightarrow \zeta_n(\cdot, \hat{u})$ from K to $C^1(J_T, H_\sigma^{k+1})$ is continuous and

$$\|\zeta_n - \zeta\|_{L^2(J_T, H^{k+1})} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

By Theorem 2.2.1, for sufficiently large n we have

$$\sup_{\hat{u} \in K} \|\mathcal{R}_T(u_0, \zeta_n(\hat{u}), \eta) - \mathcal{R}_T(u_0, \hat{\Psi}(\hat{u}))\|_k < \varepsilon - \hat{\varepsilon}. \quad (2.6.7)$$

Define $\Psi_n(t, \hat{u}) = \eta(t, \hat{u}) - \dot{\zeta}_n(t, \hat{u})$. It is easy to see that $\Psi_n(\cdot, \cdot) : \hat{u} \rightarrow \Psi_n(\cdot, \hat{u})$ is a continuous mapping from K to $L^1(J_T, H_\sigma^k)$. Clearly,

$$\mathcal{R}(u_0, \zeta_n(\hat{u}), \eta) = \mathcal{R}(u_0, \Psi_n(\hat{u})) - \zeta_n(\hat{u}).$$

Using the fact that $\zeta_n(T) = 0$, (2.6.7) and (2.6.6), we derive

$$\sup_{\hat{u} \in K} \|\mathcal{R}_T(u_0, \Psi_n(\hat{u}), \eta) - \hat{u}\|_k < \varepsilon - \hat{\varepsilon} + \sup_{\hat{u} \in K} \|\mathcal{R}_T(u_0, \hat{\Psi}(\hat{u})) - \hat{u}\|_k < \varepsilon,$$

which completes the proof of Proposition 2.6.1. \square

Proof of Proposition 2.6.2. By Proposition 2.6.1 and the fact $E \subset E_1$, if Eq. (2.6.1) is uniformly (ε, u_0, K) -controllable, then so is Eq. (2.3.1) with $\eta \in C^\infty(J_T, E_1)$. We need to prove the converse assertion. We assume that there is a continuous mapping

$$\Psi_1 : K \rightarrow \Theta(h, u_0) \cap L^1(J_T, E_1)$$

such that

$$\hat{\varepsilon} := \sup_{\hat{u} \in K} \|\mathcal{R}_T(u_0, \Psi_1(\hat{u})) - \hat{u}\|_k < \varepsilon.$$

We approximate $\mathcal{R}_T(u_0, \Psi_1(\hat{u}))$ by a solution $u(t, \hat{u})$ of problem (2.6.1), (2.3.2) with some $\eta(t, \hat{u}), \zeta(t, \hat{u}) \in C^\infty(J_T, E)$ such that $(\eta(t, \hat{u}), \zeta(t, \hat{u}))$ depends continuously on $\hat{u} \in K$.

Step 1. We first approximate $\Psi_1(\hat{u})$ by a family of piecewise constant controls. Let us introduce a finite set $A = \{\eta_1^l \in E_1, l = 1, \dots, m\}$. For any integer s , we denote by $P_s(J_T, A)$ the set of functions

$$\eta_1(t) = \sum_{l=1}^m \varphi_l(t) \eta_1^l \quad \text{for } t \in [0, T],$$

where φ_l are non-negative functions such that $\sum_{l=1}^m \varphi_l(t) = 1$,

$$\varphi_l(t) = \sum_{r=0}^{s-1} c_{l,r} I_{r,s}(t) \quad \text{for } t \in [0, T],$$

and $I_{r,s}$ is the indicator function of the interval $[t_r, t_{r+1})$ with $t_r = rT/s$.

We define a metric in $P_s(J_T, A)$ by

$$d_P(\eta_1, \zeta_1) = \sum_{l=1}^m \|\varphi_l - \psi_l\|_{L^\infty(J_T)}, \quad \eta_1, \zeta_1 \in P_s(J_T, A),$$

where $\{\varphi_l\}$ and $\{\psi_l\}$ are the functions corresponding to η_1 and ζ_1 , respectively. We shall need the following lemmas, which are proved at the end of this section.

Lemma 2.6.3. *If Eq. (2.3.1) with $\eta \in C^\infty(J_T, E_1)$ is uniformly (ε, u_0, K) -controllable, then there is a finite set $A = \{\eta_1^l, l = 1, \dots, m\} \subset E_1$, an integer $s \geq 1$ and a mapping $\Psi_s : K \rightarrow P_s(J_T, A)$ continuous with respect to the metric of $P_s(J_T, A)$ such that $\Psi_s(K) \subset \Theta(u_0, h)$ and*

$$\sup_{\hat{u} \in K} \|\mathcal{R}_T(u_0, \Psi_s(\hat{u})) - \hat{u}\|_k < \varepsilon.$$

Lemma 2.6.4. *Let $E \subset H_\sigma^{k+2}$ be a finite-dimensional space and $E_1 = \mathcal{F}(E)$. Then for any $\eta_1 \in E_1$ there are vectors $\zeta^1, \dots, \zeta^p, \eta \in E$ and positive constants $\lambda_1, \dots, \lambda_p$ whose sum is equal to 1 such that*

$$B(u) - \eta_1 = \sum_{j=1}^p \lambda_j B(u + \zeta^j) - \eta \quad \text{for any } u \in H^1.$$

Let Ψ_s be the function constructed in Lemma 2.6.3 :

$$\Psi_s(\hat{u}) = \sum_{l=1}^m \varphi_l(t, \hat{u}) \eta_1^l.$$

As $\eta_1^l \in E_1$, by Lemma 2.6.4, there are vectors $\zeta^{l,1}, \dots, \zeta^{l,p}, \eta^l \in E$ and positive constants $\lambda_{l,1}, \dots, \lambda_{l,p}$ whose sum is equal to 1 such that

$$B(u) - \eta_1^l = \sum_{j=1}^p \lambda_{l,j} B(u + \zeta^{l,j}) - \eta^l \quad \text{for any } u \in H^1. \quad (2.6.8)$$

Let $u_1 = \mathcal{R}(u_0, \Psi_s(\hat{u}))$. It follow from (2.6.8) that u_1 satisfies the equation

$$\dot{u}_1 + \sum_{j=1}^p \sum_{l=1}^m \lambda_{l,j} \varphi_l(t, \hat{u}) B(u_1 + \zeta^{l,j}) = h(t) + \sum_{l=1}^m \varphi_l(t, \hat{u}) \eta^l. \quad (2.6.9)$$

We can rewrite Eq. (2.6.9) in the form

$$\dot{u}_1 + \sum_{i=1}^q \psi_i(t, \hat{u}) B(u_1 + \zeta^i) = h(t) + \eta(t, \hat{u}), \quad (2.6.10)$$

where $\zeta^i \in E$ for $i = 1, \dots, q$, $\eta(t, \hat{u}) = \sum_{l=1}^m \varphi_l(t, \hat{u}) \eta^l$ such that

$$\psi_i(t, \hat{u}) = \sum_{r=0}^{s-1} d_{i,r}(\hat{u}) I_{r,s}(t), \quad \sum_{i=1}^q d_{i,r} = 1$$

for some non-negative functions $d_{i,r} \in C(K)$.

Step 2. We approximate u_1 by a solution of problem (2.6.1), (2.3.2). First we assume $s = 1$. In this case (2.6.10) becomes

$$\dot{u}_1 + \sum_{i=1}^q d_i(\hat{u}) B(u_1 + \zeta^i) = h(t) + \eta(\hat{u}), \quad (2.6.11)$$

where $d_i \in C(K)$ and $\eta \in C(K, E)$. Let $\zeta_n(t, \hat{u}) = \zeta(\frac{nt}{T}, \hat{u})$, where $\zeta(t, \hat{u})$ is a 1-periodic function such that

$$\zeta(s, \hat{u}) = \zeta^j \text{ for } 0 \leq s - (d_1(\hat{u}) + \dots + d_{j-1}(\hat{u})) < d_j(\hat{u}), \quad j = 1, \dots, q,$$

where $d_0(\hat{u}) = 0$. Eq. (2.6.11) is equivalent to the equation

$$\dot{u}_1 + B(u_1 + \zeta_n(t, \hat{u})) = h(t) + \eta(t, \hat{u}) + f_n(t, \hat{u}),$$

where

$$f_n(t, \hat{u}) = B(u_1 + \zeta_n(t, \hat{u})) - \sum_{i=1}^q d_i(\hat{u}) B(u_1 + \zeta^i). \quad (2.6.12)$$

Let us define

$$\mathcal{K}g(t) = \int_0^t g(s)ds.$$

Then $v_n = u_1 - \mathcal{K}f_n$ is a solution of the problem

$$\begin{aligned} \dot{v}_n + B(v_n + \zeta_n(t, \hat{u}) + \mathcal{K}f_n(t, \hat{u})) &= h(t) + \eta(t, \hat{u}), \\ v_n &= u_0. \end{aligned}$$

Suppose we have shown that

$$\sup_{\hat{u} \in K} \|\mathcal{K}f_n(t, \hat{u})\|_{C(J_T, H^{k+1})} \rightarrow 0. \quad (2.6.13)$$

Then v_n satisfies

$$\|v_n - u_1\|_{C(J_T, H^{k+1})} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

There is an integer $n_0 \geq 1$ such that if $n \geq n_0$

$$\sup_{\hat{u} \in K} \|\mathcal{R}(u_0, \zeta_n(\hat{u}), \eta(\hat{u})) - u_1(\cdot, \hat{u})\|_{C(J_T, H^k)} < \varepsilon - \hat{\varepsilon}.$$

Then the operator

$$\hat{\Psi}_n : K \rightarrow L^1(J_T, E) \times L^2(J_T, E), \quad \hat{u} \rightarrow (\eta(\hat{u}), \zeta_n(\hat{u}))$$

satisfies (2.6.2).

To finish the proof of Proposition 2.6.2 in the case $s = 1$, it suffices to prove (2.6.13). Suppose we have shown that

$$\|\mathcal{K}f_n(t, \hat{u})\|_{C(J_T, H^{k+1})} \rightarrow 0 \text{ for any } \hat{u} \in K. \quad (2.6.14)$$

To prove (2.6.13), by the Arzelà–Ascoli theorem, it suffices to show that the family $\{\hat{u} \rightarrow \mathcal{K}f_n(\cdot, \hat{u})\}$ is uniformly equicontinuous from K to $C(J_T, H_\sigma^{k+1})$. By (2.6.12), it suffices to show that so is $\hat{u} \rightarrow \zeta_n(\hat{u})$ from K to $L^1(J_T, H_\sigma^{k+2})$. The definition of ζ_n implies

$$\begin{aligned} \|\zeta_n(\cdot, \hat{u}_1) - \zeta_n(\cdot, \hat{u}_2)\|_{L^2(J_T, H^{k+2})}^2 &\leq \int_0^T \|\zeta(\frac{nt}{T}, \hat{u}_1) - \zeta(\frac{nt}{T}, \hat{u}_2)\|_{k+2}^2 dt \\ &= \frac{T}{n} \int_0^n \|\zeta(t, \hat{u}_1) - \zeta(t, \hat{u}_2)\|_{k+2}^2 dt \leq C \sum_{i=1}^q |d_i(\hat{u}_1) - d_i(\hat{u}_2)|. \end{aligned}$$

The uniform continuity of d_i over K gives us the required result.

Step 3. To complete the proof of Proposition 2.6.2 in the case $s = 1$, it remains to prove (2.6.14). If we show that for any piecewise constant H_σ^{k+2} -valued function u_1 on J_T , the sequence $\{\mathcal{K}f_n\}$ converges to zero in the space $C(J_T, H_\sigma^{k+1})$, then an approximation argument shows (2.6.14) for any $u_1 \in C(J_T, H_\sigma^{k+2})$.

The family $\{\mathcal{K}f_n\}$ is relatively compact in the space $C(J_T, H_\sigma^{k+1})$ for any piecewise constant function u_1 . Indeed, the set $f_n(t), t \in J_T$ is contained in a finite subset of H_σ^{k+1} not depending on n . Thus, there is a compact set $G \subset H_\sigma^{k+1}$ such that

$$\mathcal{K}f_n(t) \in G \text{ for all } t \in J_T, n \geq 1.$$

As

$$\sup_{n \geq 1} \|f_n\|_{C(J_T, H^{k+1})} < \infty,$$

the family $\{\mathcal{K}f_n\}$ is uniformly equicontinuous on J_T . Thus, by the Arzelà–Ascoli theorem, $\{\mathcal{K}f_n\}$ is relatively compact. Therefore convergence (2.6.14) will be established if we show that

$$\mathcal{K}f_n(t) \rightarrow 0 \text{ in } H_\sigma^{k+1} \text{ for any } t \in J_T. \quad (2.6.15)$$

To prove (2.6.15), we first assume that $u(t) = b \in H_\sigma^{k+2}$ for all $t \in J_T$. Let $t = t_l + \tau$, where $t_l = \frac{lT}{n}$, $l \in \mathbb{N}$ and $\tau \in [0, \frac{T}{n})$. From the definition of $\zeta_n(t)$ we have

$$\int_0^{\frac{lT}{n}} f_n(s) ds = \int_0^{\frac{lT}{n}} \left(B(b + \zeta_n(t)) \right) ds - \frac{lT}{n} \sum_{j=1}^p \lambda_j B(b + \zeta^j) = 0,$$

so

$$\mathcal{K}f_n(t) = -\tau \sum_{j=1}^p \lambda_j B(b + \zeta^j) - \int_0^\tau B(b + \zeta_n(s)) ds.$$

Since $\tau \rightarrow 0$ as $n \rightarrow \infty$, we arrive at (2.6.15). In the same way, we can show that (2.6.15) holds for any piecewise constant function u .

The case $s \geq 2$ is deduced from the case $s = 1$ exactly in the same way as in [40, section 3.3]. \square

Proof of Lemma 2.6.3. Let $\{e_1, \dots, e_d\}$ be an orthonormal basis in E_1 with respect to scalar product $\langle \cdot, \cdot \rangle$ and $\xi_l(t, \hat{u}) := \langle \Psi_1(t, \hat{u}), e_l \rangle$ for $l = 1, \dots, d$. Let us define for $M = \max_{l,t,\hat{u}} |\xi_l(t, \hat{u})|$ and $m = 2d$ the vectors

$$\eta_1^l = dM e_l \text{ for } l = 1, \dots, d, \eta_1^l = -dM e_l \text{ for } l = d+1, \dots, m.$$

We can see that the functions

$$\tilde{\xi}_l(t, \hat{u}) = \frac{1}{2d} \left(1 + \frac{\xi_l(t, \hat{u})}{M} \right), \tilde{\xi}_{l+d}(t, \hat{u}) = \frac{1}{2d} \left(1 - \frac{\xi_l(t, \hat{u})}{M} \right) \text{ for } l = 1, \dots, d$$

are non-negative, their sum is equal to 1, and they satisfy the relation

$$\Psi_1(t, \hat{u}) = \sum_{l=1}^m \tilde{\xi}_l(t, \hat{u}) \eta_1^l.$$

Let us define an operator $\Psi_s : K \rightarrow P_s(J_T, A)$ with $A = \{\eta_1^l, l = 1, \dots, m\}$ as

$$\Psi_s(t, \hat{u}) = \sum_{l=1}^m \tilde{\xi}_l\left(\frac{rT}{s}, \hat{u}\right) \eta_1^l \text{ for } t \in \left[\frac{rT}{s}, \frac{(r+1)T}{s} \right).$$

Since $\tilde{\xi}_l \in C(J_T \times K)$ and $K \subset H_\sigma^{k+2}$ is compact, we have

$$\sup_{\hat{u} \in K} \|\Psi_1(t, \hat{u}) - \Psi_s(t, \hat{u})\|_{k+2} = \sup_{\hat{u} \in K} \left\| \sum_{l=1}^m (\tilde{\xi}_l(t, \hat{u}) - \tilde{\xi}_l(\frac{r^T}{s}, \hat{u})) \eta_1^l \right\|_{k+2} \rightarrow 0 \text{ as } s \rightarrow \infty.$$

Thus for sufficiently large s , we have $\Psi_s(K) \subset \Theta(h, u_0)$, and

$$\sup_{\hat{u} \in K} \|\mathcal{R}_T(u_0, \Psi_s(\hat{u})) - \mathcal{R}_T(u_0, \Psi_1(\hat{u}))\|_k < \varepsilon.$$

Hence (2.6.1) is uniformly (ε, u_0, K) -controllable with $\eta \in P_s(J_T, A)$. □

Proof of Lemma 2.6.4. By the definition of $\mathcal{F}(E)$, for any $\eta_1 \in \mathcal{F}(E)$ there are $\xi^1, \dots, \xi^n, \eta \in E$ and positive constants $\alpha_1, \dots, \alpha_n$ such that

$$\eta_1 = \eta - \sum_{i=1}^n \alpha_i B(\xi^i).$$

Let us set $p = 2n$, $\alpha = \alpha_1 + \dots + \alpha_n$,

$$\lambda_i = \lambda_{i+n} = \frac{\alpha_i}{2\alpha}, \quad \zeta^i = -\zeta^{i+n} = \sqrt{\alpha} \xi^i, \quad i = 1, \dots, n.$$

Then we have

$$B(u) - \eta_1 = \sum_{j=1}^p \lambda_j B(u + \zeta^j) - \eta \quad \text{for any } u \in H_\sigma^1.$$

□

2.7 Non controllability result

Let us denote by $A_T(u_0, h, E)$ the set of attainability at time T from $u_0 \in H_\sigma^k$ by E -valued controls, i.e.,

$$A_T(u_0, h, E) = \{\hat{u} \in H_\sigma^k : \hat{u} = \mathcal{R}_T(u_0, \eta) \text{ for some } \eta \in \Theta(u_0, h)\}.$$

In this section, we show that the ideas of [41] can be generalized to prove that the set $A(u_0, h, E) = \cup_{T \in [0, \infty)} A_T(u_0, h, E)$ does not contain a ball of $H_\sigma^{k+\gamma}$, $\gamma < 2$ in the three-dimensional case.

Let us recall the definition of Kolmogorov ε -entropy (see [34]). For any $\varepsilon > 0$, we denote by $N_\varepsilon(K)$ the minimal number of sets of diameters not exceeding 2ε that are needed to cover K . The Kolmogorov ε -entropy of K is defined as $H_\varepsilon(K) = \ln N_\varepsilon(K)$.

Let us consider the equation

$$\dot{v} + B(v + z) = h. \tag{2.7.1}$$

We fix an integer $k \geq 4$ and denote by $\Theta_t(h, u_0)$ the set of functions $\eta \in L^1(J_t, H_\sigma^k)$ for which (2.7.1), (2.3.2) with $z(t) = \int_0^t \eta(s) ds$ has a unique solution $v \in C(J_t, H_\sigma^k)$. We note that

$$\mathcal{R}_t(u_0, \eta) = v(t) + z(t),$$

where $z(t) = \int_0^t \eta(s) ds$. The following theorem is the main result of this section.

Theorem 2.7.1. *Let $k \geq 4$, $u_0 \in H_\sigma^k$, $h \in C([0, \infty), H_\sigma^k)$ and $E \subset H_\sigma^k$ be any finite-dimensional subspace. For any $\gamma \in [0, 2)$ and any ball $Q \subset H_\sigma^{k+\gamma}$, we have*

$$A^c(u_0, h, E) \cap Q \neq \emptyset,$$

where $A^c(u_0, h, E)$ is the complement of $A(u_0, h, E)$ in the space H_σ^k .

Démonstration. We argue by contradiction. Suppose that $A(u_0, h, E)$ contains a closed ball $Q \subset H_\sigma^{k+\gamma}$. Let $\{t_l\}$ be a dense sequence in $[0, \infty)$ and let

$$\begin{aligned} D_{l,n} &:= \{(z, y) \in W^{1,1}(J_{t_l}, H_\sigma^k) \cap \Theta_{t_l}(u_0, h) \times E : \|z\|_{W^{1,1}(J_{t_l}, H_\sigma^k)} \leq n, \|y\|_k \leq n\}, \\ B_{l,n} &:= \{\hat{u} \in H_\sigma^k : \hat{u} = \mathcal{R}_t(u_0, z, h) + y \text{ for some } (z, y) \in D_{l,n}, t \in [0, t_l]\}. \end{aligned}$$

It is easy to see that $\bigcup_{l,n} B_{l,n} \supset A(u_0, h, E)$. By the Baire theorem, there are integers p and m such that $B_{p,m}$ is dense in a ball \hat{Q} with respect to the metric of $H_\sigma^{k+\gamma}$. Let us denote by $K : [0, \infty) \times L^1([0, \infty), E) \times E \rightarrow H_\sigma^{k-1}$ the continuous operator that takes the triple $(t, z, y) \in J_{t_p} \times D_{p,m}$ to $\mathcal{R}_t(u_0, z, h) + y$. As $K(J_{t_p} \times D_{p,m}) \subset B_{p,m}$ is closed in $H_\sigma^{k+\gamma} \cap B_{p,m}$, then $\hat{Q} \subset B_{p,m}$. We have from [20]

$$H_\varepsilon(Q, L^2) \sim \left(\frac{1}{\varepsilon}\right)^{\frac{3}{k}}, \quad (2.7.2)$$

where Q is a ball in H^k . To obtain (2.7.2) for any $Q \subset H_\sigma^k$, we follow the ideas of [41, Proposition 2.2]. Let us denote by Σ^k the closure in H^k of the set of functions $u = (\partial_2 v, -\partial_1 v, 0) \in H^k$, where $v \in H^{k+1}$ is a scalar function. Since Σ^k is closed subspace of H_σ^k , it suffices to prove (2.7.2) any ball $Q \subset \Sigma^k$. Let us introduce the set of scalar functions

$$\dot{H}^k(\mathbb{T}^3) := \{u \in H^k(\mathbb{T}^3) : \int_0^{2\pi} u(x_1, x') dx_1 = 0 \text{ for any } x' \in \mathbb{T}^2\}.$$

As

$$H^k(\mathbb{T}^3) = \dot{H}^k(\mathbb{T}^3) \dot{+} H^k(\mathbb{T}^2),$$

we get (2.7.2) for any ball $Q \subset \dot{H}^k(\mathbb{T}^3)$. Finally, if Π_2 is the projection $\Pi_2(u_1, u_2, u_3) \rightarrow u_2$, then $\Pi_2 \Sigma^k = \dot{H}^k(\mathbb{T}^3)$. Thus (2.7.2) holds for any $Q \subset H_\sigma^k$. Hence

$$H_\varepsilon(Q, H^{k-1}) \sim \left(\frac{1}{\varepsilon}\right)^\alpha, \quad (2.7.3)$$

where Q is a ball in $H_\sigma^{k+\gamma}$ and $\alpha = \frac{3}{1+\gamma} > 1$. On the other hand, from [41, (3.10)] it follows that

$$H_\varepsilon\left(J_{t_p} \times D_{p,m}, \mathbb{R} \times L^1(J, E) \times E\right) \prec \frac{1}{\varepsilon} \ln \frac{1}{\varepsilon}. \quad (2.7.4)$$

As $h \in C([0, \infty), H_\sigma^k)$, by Theorem 2.2.1, the operator $K : J_{t_p} \times D_{p,m} \rightarrow H^{k-1}$ is Lipschitz-continuous. Then (2.7.4) implies

$$H_\varepsilon(B_{p,m}, H^{k-1}) \prec \frac{1}{\varepsilon} \ln \frac{1}{\varepsilon}. \quad (2.7.5)$$

Combining this with relation (2.7.3), we see that

$$H_\varepsilon(\hat{Q}, H^{k-1}) \succ \varepsilon^\nu H_\varepsilon(B_{p,m}, H^{k-1}),$$

where $\nu > 0$, which contradicts the inclusion $\hat{Q} \subset B_{p,m}$. □

CHAPITRE 3

Contrôlabilité de l'équation d'Euler 3D compressible

Controllability of the 3D compressible Euler system

Abstract. The paper is devoted to the controllability problem for 3D compressible Euler system. The control is a finite-dimensional external force acting only on the velocity equation. We show that the velocity and density of the fluid are simultaneously controllable. In particular, the system is approximately controllable and exactly controllable in projections.

3.1 Introduction

The time evolution of an isentropic ideal gas is described by the compressible Euler system

$$\rho(\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u}) + \nabla p(\rho) = \rho \mathbf{f}, \quad (3.1.1)$$

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0, \quad (3.1.2)$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \rho(0) = \rho_0, \quad (3.1.3)$$

where $\mathbf{u} = (u_1, u_2, u_3)$ and ρ are unknown velocity field and density of the gas, p is the pressure and \mathbf{f} is the external force, \mathbf{u}_0 and ρ_0 are the initial conditions. We assume that the space variable $\mathbf{x} = (x_1, x_2, x_3)$ belongs to the 3D torus $\mathbb{T}^3 = \mathbb{R}^3/2\pi\mathbb{Z}^3$.

Problem (3.1.1)-(3.1.3) can be reduced by a simple change of variables to a quasi-linear symmetrizable hyperbolic system. Thus local-in-time existence and uniqueness of a smooth solution is well known (for instance, see [30, 42]). Moreover, a blow-up criterion holds for the compressible Euler equation (see [42, Section 16, Proposition 2.4]).

The aim of this paper is the study of some controllability issues for system (3.1.1)-(3.1.3). We suppose that the external force is of the form $\mathbf{f} = \tilde{\mathbf{f}} + \boldsymbol{\eta}$, where $\tilde{\mathbf{f}}$ is any given function and $\boldsymbol{\eta}$ is the control taking values in a finite-dimensional space. Let H^k be the Sobolev space of order k on \mathbb{T}^3 and let \mathbf{H}^k be the space of vector functions $\mathbf{u} = (u_1, u_2, u_3)$ with components in H^k . For both spaces, we denote by $\|\cdot\|_k$ the corresponding norms. We denote $J_T := [0, T]$. The following theorem is our main result.

Main theorem. Let $T > 0$, $k \geq 4$ and $\tilde{\mathbf{f}} \in C^\infty(J_T, \mathbf{H}^{k+2})$. There is a finite-dimensional space $\mathbf{E} \subset \mathbf{H}^k$ with $\dim \mathbf{E} = 45$ such that for any constant $\varepsilon > 0$, for any continuous function $\mathbf{F} : \mathbf{H}^k \times H^k \rightarrow \mathbb{R}^N$ admitting a right inverse, for any functions $\mathbf{u}_0, \hat{\mathbf{u}} \in \mathbf{H}^k$ and $\rho_0, \hat{\rho} \in H^k$ with

$$\int_{\mathbb{T}^3} \rho_0 d\mathbf{x} = \int_{\mathbb{T}^3} \hat{\rho} d\mathbf{x} \quad (3.1.4)$$

there is a smooth control $\boldsymbol{\eta} : J_T \rightarrow \mathbf{E}$ such that system (3.1.1)-(3.1.3) has a unique regular solution (\mathbf{u}, ρ) , which verifies

$$\begin{aligned} \|(\mathbf{u}(T), \rho(T)) - (\hat{\mathbf{u}}, \hat{\rho})\|_{\mathbf{H}^k \times H^k} &< \varepsilon, \\ \mathbf{F}(\mathbf{u}(T), \rho(T)) &= \mathbf{F}(\hat{\mathbf{u}}, \hat{\rho}). \end{aligned}$$

See Subsection 3.3.1 for the exact formulation. We stress that condition (3.1.4) is essential, because integrating (3.1.2), we get $\int \rho(\cdot, \mathbf{x}) d\mathbf{x} = \text{const}$.

Before turning to the ideas of the proof, let us describe in a few words some previous results on the controllability of Euler and Navier–Stokes systems. Li and Rao [33] proved a local exact boundary controllability property for general 1D first-order quasi-linear hyperbolic equations. Exact boundary controllability problems for weak entropy solutions of 1D compressible Euler system has been established by Glass [24]. Controllability of incompressible Euler and Navier–Stokes systems has been studied by several authors. Coron [13] introduced the return method to show exact boundary controllability of 2D incompressible Euler system. Glass [23] generalized this result for 3D Euler system. Exact controllability of Navier–Stokes systems with control supported by a given domain was studied by Coron and Fursikov [17], Fursikov and Imanuvilov [22], Imanuvilov [27], Fernández-Cara et al. [21]. Agrachev and Sarychev [3, 4] proved controllability of 2D Navier–Stokes and 2D Euler equations with finite-dimensional external control. Rodrigues [38] used Agrachev–Sarychev method to prove controllability of the 2D Navier–Stokes equation on the rectangle with Lions boundary condition. Shirikyan [39, 40] generalized this method to the case of 3D Navier–Stokes equation. Furthermore, he shows [41] that 2D Euler equation is not exactly controllable by a finite-dimensional external force. In [36], we show that in the case of 3D Euler equation, the velocity and pressure are exactly controllable in projections.

One of the main difficulties of the proof of Main theorem is the fact that the control $\boldsymbol{\eta}$ acts only on the first equation. We combine the Agrachev–Sarychev method with a perturbative result for compressible Euler equations and a property of the transport equation to prove that the velocity \mathbf{u} and the density ρ can be controlled simultaneously with the help of a finite-dimensional external force $\boldsymbol{\eta}$. The Agrachev–Sarychev method is based on construction of an increasing sequence of finite-dimensional spaces $\mathbf{E}_n \subset \mathbf{H}^k, n \geq 0$ such that

- (i) The system is controllable with \mathbf{E}_N -valued controls for some $N \geq 1$.
- (ii) Controllability of the system with $\boldsymbol{\eta} \in \mathbf{E}_n$ is equivalent to that with $\boldsymbol{\eta} \in \mathbf{E}_{n+1}$.

See Definition 5 for the notion of the controllability, which we use in this paper. As in the case of incompressible Euler and Navier–Stokes systems, the proof of property (i) is deduced from the hypothesis that $\mathbf{E}_\infty := \cup_{n=0}^\infty \mathbf{E}_n$ is dense in \mathbf{H}^k and from the fact that for any functions $\mathbf{V}_0, \mathbf{V}_1$ there is a control (not necessarily \mathbf{E} -valued) which steers the system from \mathbf{V}_0 to \mathbf{V}_1 . As the control acts only on the first equation,

along with (3.1.1)-(3.1.2) we need to consider the control system

$$\rho(\partial_t \mathbf{u} + ((\mathbf{u} + \boldsymbol{\xi}) \cdot \nabla)(\mathbf{u} + \boldsymbol{\xi})) + \nabla p(\rho) = \rho(\tilde{\mathbf{f}} + \boldsymbol{\eta}), \quad (3.1.5)$$

$$\partial_t \rho + \nabla \cdot (\rho(\mathbf{u} + \boldsymbol{\xi})) = 0. \quad (3.1.6)$$

For any \mathbf{V}_0 and \mathbf{V}_1 we find controls $\boldsymbol{\xi}, \boldsymbol{\eta}$ such that the solution of (3.1.5)-(3.1.6) links \mathbf{V}_0 and \mathbf{V}_1 . Now to prove (i), it suffices to show that the control systems (3.1.1)-(3.1.2) and (3.1.5)-(3.1.6) are equivalent. This can be done by a simple change of the variable $\mathbf{v} = \mathbf{u} + \boldsymbol{\xi}$. To establish property (ii), we first show that the controllability of (3.1.1)-(3.1.2) with $\boldsymbol{\eta} \in \mathbf{E}_{n+1}$ is equivalent to that of the system

$$\rho(\partial_t \mathbf{u} + ((\mathbf{u} + \boldsymbol{\xi}) \cdot \nabla)(\mathbf{u} + \boldsymbol{\xi})) + \nabla p(\rho) = \rho(\tilde{\mathbf{f}} + \boldsymbol{\eta}), \quad (3.1.7)$$

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0 \quad (3.1.8)$$

with $\boldsymbol{\eta} \in \mathbf{E}_n$ and $\boldsymbol{\xi} \in \mathbf{E}_n$. Here we use the ideas from [3, 4, 39, 40, 36]. Then using a continuity property of the resolving operator of compressible Euler system (see Theorem 3.2.3), we show that control systems (3.1.7)-(3.1.8) and (3.1.5)-(3.1.6) are also equivalent. We refer the reader to Section 3.4.2 for a detailed proof of this property.

Acknowledgments. The author would like to express deep gratitude to Armen Shirikyan for drawing his attention to this problem and for many valuable discussions and to the referees for their detailed comments and suggestions which have helped to improve the paper.

Notation. We use bold characters to denote vector functions. Let X be a separable Banach space endowed with the norm $\|\cdot\|_X$. For $1 \leq p < \infty$ let $L^p(J_T, X)$ be the space of measurable functions $u : J_T \rightarrow X$ such that

$$\|u\|_{L^p(J_T, X)} := \left(\int_0^T \|u\|_X^p ds \right)^{\frac{1}{p}} < \infty.$$

The space of continuous functions $u : J_T \rightarrow X$ is denoted by $C(J_T, X)$. We denote by C a constant whose value may change from line to line. We write $\int f(x)dx$ instead of $\int_{\mathbb{T}^3} f(x)dx$. Let $\delta_{i,j}$ be the Kronecker delta, i.e, $\delta_{i,j} = 0$ if $i \neq j$ and $\delta_{i,i} = 1$.

3.2 Preliminaries on 3D compressible Euler system

3.2.1 Symmetrizable hyperbolic systems

In this subsection, we recall some results on local existence of symmetrizable hyperbolic systems. Let us consider the system

$$\partial_t \mathbf{v} + \sum_{i=1}^n \mathbf{A}_i(t, \mathbf{x}, \mathbf{v}) \partial_i \mathbf{v} + \mathbf{G}(t, \mathbf{x}, \mathbf{v}) = 0, \quad \mathbf{v}(0) = \mathbf{v}_0. \quad (3.2.1)$$

We say that (3.2.1) is a quasi-linear symmetric hyperbolic system if matrices A_i are symmetric, i.e., $A_i = A_i^*$. If functions \mathbf{A}_i, \mathbf{G} are smooth and system (3.2.1) is symmetric hyperbolic, then for any $\mathbf{v}_0 \in \mathbf{H}^k$, $k > n/2 + 1$ there exists $T > 0$ such that system (3.2.1) has a solution $\mathbf{v} \in C(J_T, \mathbf{H}^k)$ (see [30] or [42, Chapter 16] for an exact statement). Now consider a more general case :

$$\partial_t \mathbf{u} + \sum_{i=1}^n \mathbf{B}_i(t, \mathbf{x}, \mathbf{u}) \partial_i \mathbf{u} + \mathbf{H}(t, \mathbf{x}, \mathbf{u}) = 0, \quad \mathbf{u}(0) = \mathbf{u}_0, \quad (3.2.2)$$

where \mathbf{B}_i are such that there exists a positive definite matrix \mathbf{B}_0 such that $\mathbf{B}_0 \cdot \mathbf{B}_i$ are symmetric. These systems are called quasi-linear symmetrizable hyperbolic systems. As it is remarked in [42, Chapter 16, p. 366], we have the following local well-posedness of this system.

Theorem 3.2.1. *Let $\mathbf{u}_0 \in \mathbf{H}^k$, $k > n/2 + 1$ and $\mathbf{B}_i, \mathbf{H} \in L^2(J_T, \mathbf{H}^k \times \mathbf{H}^k)$. Then there exists $T_0 > 0$, which depends on*

$$\|\mathbf{u}_0\|_k + \|\mathbf{B}_i\|_{L^2(J_T, \mathbf{H}^k \times \mathbf{H}^k)} + \|\mathbf{H}\|_{L^2(J_T, \mathbf{H}^k \times \mathbf{H}^k)},$$

such that system (3.2.1) has a unique solution $\mathbf{u} \in C(J_{T_0}, \mathbf{H}^k)$.

3.2.2 Well-posedness of the Euler equations

Let us consider the compressible Euler system

$$\begin{aligned} \rho(\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u}) + \nabla p(\rho) &= \rho \mathbf{f}, \\ \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) &= 0, \\ \mathbf{u}(0) = \mathbf{u}_0, \quad \rho(0) &= \rho_0. \end{aligned}$$

We study the case in which there is no vacuum, so that the initial density is separated from zero. Let us show that in this case the above problem can be reduced to a quasi-linear symmetrizable hyperbolic system. Setting $g = \log \rho$ and $h(s) = p'(e^s)$, the above system takes the equivalent form

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + h(g) \nabla g = \mathbf{f}, \quad (3.2.3)$$

$$(\partial_t + \mathbf{u} \cdot \nabla) g + \nabla \cdot \mathbf{u} = 0, \quad (3.2.4)$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad g(0) = g_0. \quad (3.2.5)$$

In what follows, we shall deal with the more general system

$$\partial_t \mathbf{u} + ((\mathbf{u} + \zeta) \cdot \nabla) (\mathbf{u} + \zeta) + h(g) \nabla g = \mathbf{f}, \quad (3.2.6)$$

$$(\partial_t + (\mathbf{u} + \xi) \cdot \nabla) g + \nabla \cdot (\mathbf{u} + \xi) = 0, \quad (3.2.7)$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad g(0) = g_0. \quad (3.2.8)$$

We set $\mathbf{U} = (\mathbf{u}_0, g_0, \zeta, \xi, \mathbf{f})$,

$$\mathbf{Y}^k = C(J_T, \mathbf{H}^k) \times C(J_T, H^k),$$

$$\mathbf{X}^k = \mathbf{H}^k \times H^k \times L^2(J_T, \mathbf{H}^{k+1}) \times L^2(J_T, \mathbf{H}^{k+1}) \times L^2(J_T, \mathbf{H}^k),$$

and endow these spaces with natural norms. Standard arguments show that if $k \geq 4$, then for any $\mathbf{U} \in \mathbf{X}^k$ problem (3.2.6)-(3.2.8) has at most one solution $(\mathbf{u}, g) \in \mathbf{Y}^k$. The following theorem establishes a perturbative result on the existence of solution and some continuity properties of the resolving operator.

Theorem 3.2.2. *Let $T > 0$, $k \geq 4$ and $h \in C^k(\mathbb{R})$ be such that $0 < h(s)$ for any $s \in \mathbb{R}$. Suppose that for some function $\mathbf{U}_1 \in \mathbf{X}^k$ problem (3.2.6)-(3.2.8) has a solution $(\mathbf{u}_1, g_1) \in \mathbf{Y}^k$. Then there are constants $\delta > 0$ and $C > 0$ depending only on h and $\|\mathbf{U}_1\|_{\mathbf{X}^k}$ such that the following assertions hold.*

(i) *If $\mathbf{U}_2 \in \mathbf{X}^k$ satisfies the inequality*

$$\|\mathbf{U}_1 - \mathbf{U}_2\|_{\mathbf{X}^k} < \delta, \quad (3.2.9)$$

then problem (3.2.6)-(3.2.8) has a unique solution $(\mathbf{u}_2, g_2) \in \mathbf{Y}^k$.

(ii) *Let*

$$\mathcal{R} : \mathbf{X}^k \rightarrow \mathbf{Y}^k$$

be the operator that takes a function \mathbf{U}_2 satisfying (3.2.9) to the solution $(\mathbf{u}_2, g_2) \in \mathbf{Y}^k$ of problem (3.2.6)-(3.2.8). Then

$$\|\mathcal{R}(\mathbf{U}_1) - \mathcal{R}(\mathbf{U}_2)\|_{\mathbf{Y}^{k-1}} \leq C\|\mathbf{U}_1 - \mathbf{U}_2\|_{\mathbf{X}^{k-1}}.$$

(iii) *The operator $\mathcal{R} : \mathbf{X}^k \rightarrow \mathbf{Y}^k$ is continuous at \mathbf{U}_1 .*

We emphasize the fact that the constants δ and C depend only on the norm of \mathbf{U}_1 . This observation will be important in Section 3.4, where we construct a solution of (3.2.6)-(3.2.8) with the help of a perturbative argument.

Démonstration. We seek a solution of (3.2.6)-(3.2.8) in the form $(\mathbf{u}_2, g_2) := (\mathbf{u}_1, g_1) + (\mathbf{w}, \varphi)$. Substituting this into (3.2.6)-(3.2.8) and performing some transformations, we obtain the following problem :

$$\begin{aligned} \partial_t \mathbf{w} + ((\mathbf{u}_1 + \boldsymbol{\zeta}_1) \cdot \nabla)(\mathbf{w} + \boldsymbol{\eta}) + ((\mathbf{w} + \boldsymbol{\eta}) \cdot \nabla)(\mathbf{u}_1 + \boldsymbol{\zeta}_1) \\ + ((\mathbf{w} + \boldsymbol{\eta}) \cdot \nabla)(\mathbf{w} + \boldsymbol{\eta}) + h(g_1 + \varphi)\nabla(g_1 + \varphi) - h(g_1)\nabla g_1 = \mathbf{q}, \end{aligned} \quad (3.2.10)$$

$$\begin{aligned} \partial_t \varphi + ((\mathbf{u}_1 + \boldsymbol{\xi}_1) \cdot \nabla)\varphi + ((\mathbf{w} + \boldsymbol{\sigma}) \cdot \nabla)g_1 + ((\mathbf{w} + \boldsymbol{\sigma}) \cdot \nabla)\varphi \\ + \nabla \cdot (\mathbf{w} + \boldsymbol{\sigma}) = 0, \end{aligned} \quad (3.2.11)$$

$$(\mathbf{w}, \varphi)(0) = (\mathbf{w}_0, \varphi_0), \quad (3.2.12)$$

where $\boldsymbol{\eta} = \boldsymbol{\zeta}_2 - \boldsymbol{\zeta}_1$, $\boldsymbol{\sigma} = \boldsymbol{\xi}_2 - \boldsymbol{\xi}_1$, $\mathbf{q} = \mathbf{f}_2 - \mathbf{f}_1$, $\mathbf{w}_0 = \mathbf{u}_{20} - \mathbf{u}_{10}$ and $\varphi_0 = g_{20} - g_{10}$. Problem (3.2.10)-(3.2.12) is a quasi-linear symmetrizable hyperbolic system. Indeed, setting $V = \begin{pmatrix} \mathbf{w} \\ \varphi \end{pmatrix}$ and $a_{i,j} = h(g_1 + \varphi)\delta_{i,j}$, system (3.2.10)-(3.2.12) can be rewritten in the form

$$\partial_t V + \sum_{i=1}^3 \mathbf{A}_i(t, \mathbf{x}, V) \partial_i V + \mathbf{G}(t, \mathbf{x}, V) = 0, \quad V(0) = (\mathbf{w}_0, \varphi_0), \quad (3.2.13)$$

where

$$\mathbf{A}_i = \begin{pmatrix} (\mathbf{u}_1 + \boldsymbol{\zeta}_1 + \mathbf{w} + \boldsymbol{\eta})_i & 0 & 0 & a_{1,i} \\ 0 & (\mathbf{u}_1 + \boldsymbol{\zeta}_1 + \mathbf{w} + \boldsymbol{\eta})_i & 0 & a_{2,i} \\ 0 & 0 & (\mathbf{u}_1 + \boldsymbol{\zeta}_1 + \mathbf{w} + \boldsymbol{\eta})_i & a_{3,i} \\ \delta_{1,i} & \delta_{2,i} & \delta_{3,i} & (\mathbf{u}_1 + \boldsymbol{\zeta}_1 + \mathbf{w} + \boldsymbol{\sigma})_i \end{pmatrix},$$

$\mathbf{G}(t, \mathbf{x}, \mathbf{V})$

$$= \begin{pmatrix} ((\mathbf{u}_1 + \boldsymbol{\zeta}_1) \cdot \nabla) \boldsymbol{\eta} + ((\mathbf{w} + \boldsymbol{\eta}) \cdot \nabla)(\mathbf{u}_1 + \boldsymbol{\zeta}_1) + (h(g_1 + \varphi) - h(g_1)) \nabla g_1 - \mathbf{q} \\ ((\mathbf{w} + \boldsymbol{\sigma}) \cdot \nabla) g_1 + \nabla \boldsymbol{\sigma} \end{pmatrix}.$$

Now note that (3.2.13) is symmetrizable hyperbolic system, since

$$\mathbf{A}_0(t, \mathbf{x}, \mathbf{V}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & h(g_1 + \varphi) \end{pmatrix} \quad (3.2.14)$$

is positive definite and $\mathbf{A}_0 \cdot \mathbf{A}_i$, $i = 1, 2, 3$ are symmetric. By Theorem 3.2.1, there is a maximal solution $\mathbf{V} \in C(J_{T_0}, \mathbf{H}^k) \times C(J_{T_0}, H^k)$ of (3.2.13) for some $T_0 \leq T$. Now we prove that $T_0 = T$. First, let us rewrite system (3.2.10), (3.2.11) in the form

$$\begin{aligned} \partial_t \mathbf{w} + ((\mathbf{u}_1 + \boldsymbol{\zeta}_1) \cdot \nabla)(\mathbf{w} + \boldsymbol{\eta}) + ((\mathbf{w} + \boldsymbol{\eta}) \cdot \nabla)(\mathbf{u}_2 + \boldsymbol{\zeta}_2) \\ + h(g_1) \nabla \varphi + (h(g_2) - h(g_1)) \nabla g_2 = \mathbf{q}, \end{aligned} \quad (3.2.15)$$

$$\partial_t \varphi + ((\mathbf{u}_1 + \boldsymbol{\xi}_1) \cdot \nabla) \varphi + ((\mathbf{w} + \boldsymbol{\sigma}) \cdot \nabla) g_2 + \nabla \cdot (\mathbf{w} + \boldsymbol{\sigma}) = 0. \quad (3.2.16)$$

Taking the $\partial^\alpha := \frac{\partial^\alpha}{\partial x^\alpha}$, $|\alpha| \leq k-1$ derivative of (3.2.15) and multiplying the resulting equation by $\partial^\alpha \mathbf{w}$, we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\partial^\alpha \mathbf{w}\|_0^2 \\ + \int \partial^\alpha ((\mathbf{u}_1 + \boldsymbol{\zeta}_1) \cdot \nabla) \mathbf{w} \cdot \partial^\alpha \mathbf{w} dx + \int \partial^\alpha ((\mathbf{w} \cdot \nabla)(\mathbf{u}_2 + \boldsymbol{\zeta}_2)) \cdot \partial^\alpha \mathbf{w} dx \\ + \int \partial^\alpha (h(g_1) \nabla \varphi) \cdot \partial^\alpha \mathbf{w} dx + \int \partial^\alpha ((h(g_2) - h(g_1)) \nabla g_2) \cdot \partial^\alpha \mathbf{w} dx \\ \leq C \|\mathbf{w}\|_{k-1} (\|\boldsymbol{\eta}\|_k + \|\mathbf{q}\|_{k-1}). \end{aligned} \quad (3.2.17)$$

Integrating by parts, we see that

$$\begin{aligned} \int \partial^\alpha ((\mathbf{u}_1 + \boldsymbol{\zeta}_1) \cdot \nabla) \mathbf{w} \cdot \partial^\alpha \mathbf{w} dx &\leq \int ((\mathbf{u}_1 + \boldsymbol{\zeta}_1) \cdot \nabla) \partial^\alpha \mathbf{w} \cdot \partial^\alpha \mathbf{w} dx + C \|\mathbf{w}\|_{k-1}^2 \\ &= -\frac{1}{2} \int (\nabla \cdot (\mathbf{u}_1 + \boldsymbol{\zeta}_1)) |\partial^\alpha \mathbf{w}|^2 dx + C \|\mathbf{w}\|_{k-1}^2. \end{aligned} \quad (3.2.18)$$

Inequalities (3.2.17), (3.2.18) and the fact that $\mathbf{H}^k \hookrightarrow \mathbf{L}^\infty$ for $k > \frac{3}{2}$ imply that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\partial^\alpha \mathbf{w}\|_0^2 + \int h(g_1) \nabla \partial^\alpha \varphi \cdot \partial^\alpha \mathbf{w} d\mathbf{x} \leq C \|\mathbf{w}\|_{k-1} (\|\mathbf{w}\|_{k-1} + \|\varphi\|_{k-1} \\ + \|\boldsymbol{\eta}\|_k + \|\mathbf{q}\|_{k-1}). \end{aligned} \quad (3.2.19)$$

On the other hand, applying ∂^α to (3.2.16), multiplying the resulting equation by $h(g_1) \partial^\alpha \varphi$ and integrating over \mathbb{T}^3 , we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int h(g_1) (\partial^\alpha \varphi)^2 d\mathbf{x} - \frac{1}{2} \int \partial_t h(g_1) (\partial^\alpha \varphi)^2 d\mathbf{x} \\ + \int \partial^\alpha ((\mathbf{u}_1 + \boldsymbol{\xi}_1) \cdot \nabla \varphi) h(g_1) \partial^\alpha \varphi d\mathbf{x} + \int \partial^\alpha (\mathbf{w} \cdot \nabla g_2) h(g_1) \partial^\alpha \varphi d\mathbf{x} \\ + \int \partial^\alpha (\nabla \cdot \mathbf{w}) h(g_1) \partial^\alpha \varphi d\mathbf{x} \leq C \|\varphi\|_{k-1} \|\boldsymbol{\sigma}\|_k. \end{aligned}$$

As $g_1 \in C(J_T, H^k)$ and $h \in C^k(\mathbb{R})$, integration by parts in the third term on the left-hand side implies (cf. (3.2.18))

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int h(g_1) (\partial^\alpha \varphi)^2 d\mathbf{x} + \int \partial^\alpha (\nabla \cdot \mathbf{w}) h(g_1) \partial^\alpha \varphi d\mathbf{x} \leq C \|\varphi\|_{k-1} (\|\varphi\|_{k-1} \\ + \|\mathbf{w}\|_{k-1} + \|\boldsymbol{\sigma}\|_k). \end{aligned} \quad (3.2.20)$$

Adding (3.2.19) and (3.2.20) and using the facts that $h(s) > 0$ for any $s \in \mathbb{R}$

$$\int \partial^\alpha (\nabla \cdot \mathbf{w}) h(g_1) \partial^\alpha \varphi d\mathbf{x} + \int h(g_1) \nabla \partial^\alpha \varphi \cdot \partial^\alpha \mathbf{w} d\mathbf{x} = - \int \partial^\alpha \varphi (\nabla h(g_1)) \cdot \partial^\alpha \mathbf{w} d\mathbf{x},$$

we get

$$\begin{aligned} \frac{d}{dt} \|\partial^\alpha \mathbf{w}\|_0^2 + \frac{d}{dt} \int (\partial^\alpha \varphi)^2 d\mathbf{x} \leq C (\|\mathbf{w}\|_{k-1}^2 + \|\varphi\|_{k-1}^2 + \|\boldsymbol{\sigma}\|_k^2 + \|\boldsymbol{\eta}\|_k^2 \\ + \|\mathbf{q}\|_{k-1}^2). \end{aligned}$$

Taking the sum over all α , $|\alpha| \leq k-1$ and applying the Gronwall inequality, we obtain

$$\|\mathbf{w}\|_{k-1}^2 + \|\varphi\|_{k-1}^2 \leq C (\|\boldsymbol{\sigma}\|_{L^2(J_T, H^k)}^2 + \|\boldsymbol{\eta}\|_{L^2(J_T, H^k)}^2 + \|\mathbf{q}\|_{L^2(J_T, H^{k-1})}^2). \quad (3.2.21)$$

Thus we have that $T_0 = T$. Moreover, (3.2.21) completes also the proof of (ii).

Assertion (iii) can be proved by repeating the arguments of the proof of Theorem 1.4 in [8] for Sobolev spaces \mathbf{H}^k . \square

3.2.3 Continuity property of the resolving operator

In this subsection, we establish another property of resolving operator, which will play an essential role in Section 3.4.2.

Theorem 3.2.3. *Let ζ_n and ξ_n be bounded sequences in $C(J_T, \mathbf{H}^{k+2})$ and χ_n be such that*

$$\int_0^{t_0} \xi_n(t) \cdot \chi_n(t) dt \rightarrow 0 \text{ in } H^k \quad (3.2.22)$$

for any $t_0 \in J_T$ and for any uniformly equicontinuous sequence $\chi_n : J_T \rightarrow \mathbf{H}^k$. Suppose that for $\mathbf{U}_n = (\mathbf{u}_0, g_0, \zeta_n, \xi_n, \mathbf{f}) \in \mathbf{X}^{k+1}$ problem (3.2.6)-(3.2.8) has a solution $(\mathbf{u}_n, g_n) \in \mathbf{Y}^{k+1}$. Then for sufficiently large $n \geq 1$ there exists a solution $\mathcal{R}(\mathbf{V}_n) \in \mathbf{Y}^{k+1}$ with $\mathbf{V}_n = (\mathbf{u}_0, g_0, \zeta_n, 0, \mathbf{f})$, which verifies

$$\mathcal{R}(\mathbf{U}_n) - \mathcal{R}(\mathbf{V}_n) \rightarrow 0 \text{ in } \mathbf{Y}^k.$$

Démonstration. As $\mathbf{V}_n \in \mathbf{X}^{k+1}$, a blow-up criterion for quasi-linear symmetrizable hyperbolic systems [42, Section 16, Proposition 2.4] implies that, if we have the existence of $\mathcal{R}(\mathbf{V}_n)$ in \mathbf{Y}^k , then $\mathcal{R}(\mathbf{V}_n) \in \mathbf{Y}^{k+1}$. We seek the solution $\mathcal{R}(\mathbf{V}_n)$ in the form $(\mathbf{w}_n + \mathbf{u}_n, \varphi_n + g_n)$. For $(\mathbf{w}_n, \varphi_n)$ we have the following problem (cf. (3.2.10)-(3.2.12))

$$\begin{aligned} \partial_t \mathbf{w}_n + ((\mathbf{u}_n + \zeta_n) \cdot \nabla) \mathbf{w}_n + (\mathbf{w}_n \cdot \nabla)(\mathbf{u}_n + \zeta_n) \\ + (\mathbf{w}_n \cdot \nabla) \mathbf{w}_n + h(g_n + \varphi_n) \nabla(g_n + \varphi_n) - h(g_n) \nabla g_n = 0, \end{aligned} \quad (3.2.23)$$

$$\begin{aligned} \partial_t \varphi_n + (\mathbf{u}_n \cdot \nabla) \varphi_n + (\mathbf{w}_n \cdot \nabla) g_n + (\mathbf{w}_n \cdot \nabla) \varphi_n - \xi_n \cdot \nabla g_n \\ + \nabla \cdot (\mathbf{w}_n - \xi_n) = 0, \end{aligned} \quad (3.2.24)$$

$$(\mathbf{w}_n, \varphi_n)(0) = (0, 0). \quad (3.2.25)$$

As $\|\xi_n \cdot \nabla g_n\|_k + \|\nabla \cdot \xi_n\|_k$ is not necessarily small, we cannot immediately conclude the existence of a solution $(\mathbf{w}_n, \varphi_n) \in Y^k$. However, from the theory of the local existence of solutions for quasi-linear symmetrizable hyperbolic systems we have that for any constant $\nu > 0$ there is a time $T_{0,n} > 0$ such that if $\|\tilde{\mathbf{w}}_n(0)\|_k + \|\tilde{\varphi}_n(0)\|_k < \nu$, then problem (3.2.23)-(3.2.24) with initial data $(\tilde{\mathbf{w}}_n(0), \tilde{\varphi}_n(0))$ has a solution $(\mathbf{w}_n, \varphi_n) \in Y^k$ on the interval $[0, T_{0,n}]$. Here time $T_{0,n} > 0$ depends only on $\|\mathcal{R}(\mathbf{U}_n)\|_{\mathbf{Y}^k}$ and ν . Using estimate (3.2.21) and the fact that ζ_n and ξ_n are bounded sequences in $C(J_T, \mathbf{H}^{k+1})$, we get

$$\|\mathbf{u}_n\|_k^2 + \|g_n\|_k^2 \leq C(\|\zeta_n\|_{L^2(J_T, H^{k+1})}^2 + \|\xi_n\|_{L^2(J_T, H^{k+1})}^2 + \|\mathbf{f}\|_{L^2(J_T, H^k)}^2) \leq C_1.$$

Thus $\|\mathcal{R}(\mathbf{U}_n)\|_{\mathbf{Y}^k}$ is bounded and solutions $(\mathbf{w}_n, \varphi_n)$ are defined on the same interval J_{T_0} . A simple iterative argument shows that, to complete the proof, it suffices to prove that $\|\mathbf{w}_n\|_{C(T_0, H^k)} + \|\varphi_n\|_{C(T_0, H^k)} < \nu$ for sufficiently large n . To this end, let us argue as in the proof of Theorem 3.2.2. Taking the ∂^α , $|\alpha| \leq k$ derivative of (3.2.23) and multiplying the resulting equation by $\partial^\alpha \mathbf{w}_n$ in L^2 , we get (cf. (3.2.19))

$$\frac{1}{2} \frac{d}{dt} \|\partial^\alpha \mathbf{w}_n\|_0^2 + \int h(g_n) \nabla \partial^\alpha \varphi_n \cdot \partial^\alpha \mathbf{w}_n d\mathbf{x} \leq C \|\mathbf{w}_n\|_k (\|\mathbf{w}_n\|_k + \|\varphi_n\|_k). \quad (3.2.26)$$

Then, applying ∂^α , $|\alpha| \leq k$ to (3.2.24) and multiplying the obtained equation by $h(g_n)\partial^\alpha\varphi_n$, we derive

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int h(g_n)(\partial^\alpha\varphi_n)^2 d\mathbf{x} + \int \partial^\alpha(\nabla \cdot \mathbf{w}_n)h(g_n)\partial^\alpha\varphi_n d\mathbf{x} \\ & \leq \int h(g_n)\partial^\alpha\varphi_n\partial^\alpha(\boldsymbol{\xi}_n \cdot \nabla g_n + \nabla \cdot \boldsymbol{\xi}_n) d\mathbf{x} + C\|\varphi_n\|_k(\|\varphi_n\|_k + \|\mathbf{w}_n\|_k). \end{aligned} \quad (3.2.27)$$

Combining (3.2.26), (3.2.27) and the fact that

$$\int_0^{T_0} h(g_n)\partial^\alpha\varphi_n\partial^\alpha(\boldsymbol{\xi}_n \cdot \nabla g_n + \nabla \cdot \boldsymbol{\xi}_n) ds \rightarrow 0 \text{ in } L^2(\mathbb{T}^3),$$

we get that $\|\mathbf{w}_n\|_{C(T_0, H^k)} + \|\varphi_n\|_{C(T_0, H^k)} < \nu$ for sufficiently large n . Thus $\mathcal{R}(\mathbf{V}_n) \in Y^k$ and

$$\|\mathcal{R}(\mathbf{U}_n) - \mathcal{R}(\mathbf{V}_n)\|_{Y^k} \rightarrow 0.$$

□

3.3 Main results

3.3.1 Controllability of Euler system

Let us consider the controlled system associated with the compressible Euler problem :

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + h(g) \nabla g = \mathbf{f} + \boldsymbol{\eta}, \quad (3.3.1)$$

$$(\partial_t + \mathbf{u} \cdot \nabla) g + \nabla \cdot \mathbf{u} = 0, \quad (3.3.2)$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad g(0) = g_0, \quad (3.3.3)$$

where $\mathbf{f} \in C^\infty([0, \infty), \mathbf{H}^{k+2})$, $\mathbf{u}_0 \in \mathbf{H}^k$ and $g_0 \in H^k$ are given functions, and $\boldsymbol{\eta}$ is the control taking values in a finite-dimensional subspace $\mathbf{E} \subset \mathbf{H}^{k+2}$. We denote by $\Theta(\mathbf{u}_0, g_0, \mathbf{f})$ the set of functions $\boldsymbol{\eta} \in L^2(J_T, \mathbf{H}^k)$ for which problem (3.3.1)-(3.3.3) has a solution in \mathbf{Y}^k . For any $\alpha > 0$ and $k \in \mathbb{N}$ let us define the set

$$G_\alpha^k = \{g \in H^k : \int e^{g(\mathbf{x})} d\mathbf{x} = \alpha\}.$$

Recall that \mathcal{R} is the resolving operator of (3.2.6)-(3.2.8). We denote by $\mathcal{R}_t(\cdot)$ the restriction of $\mathcal{R}(\cdot)$ to the time t . Let $\mathbf{X} \subset L^2(J_T, \mathbf{H}^k)$ be an arbitrary vector subspace. We endow G_α^k with the metric defined by the norm of H^k and \mathbf{X} by the norm of $L^2(J_T, \mathbf{H}^k)$. Recall that for a function $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$ a point $x \in \mathbb{R}^m$ is said to be regular point if the differential $Df(x)$ is surjective. Then, by the inverse function theorem, there exists a neighborhood of $f(x)$ such that a right inverse of f is well defined. Now we give a generalization of the notion of a regular point for a continuous function $\mathbf{F} : \mathbf{H}^k \times G_\alpha^k \rightarrow \mathbb{R}^N$.

Définition 4. A point (\mathbf{u}_1, g_1) is said to be regular for \mathbf{F} if there is a non-degenerate closed ball $\mathbf{B} \subset \mathbb{R}^N$ centred at $\mathbf{y}_1 = \mathbf{F}(\mathbf{u}_1, g_1)$ and a continuous function $\mathbf{G} : \mathbf{B} \rightarrow \mathbf{H}^k \times G_\alpha^k$ such that $\mathbf{G}(\mathbf{y}_1) = (\mathbf{u}_1, g_1)$ and $\mathbf{F}(\mathbf{G}(\mathbf{y})) = \mathbf{y}$ for any $\mathbf{y} \in \mathbf{B}$.

Définition 5. System (3.3.1), (3.3.2) with $\boldsymbol{\eta} \in \mathbf{X}$ is said to be controllable at time $T > 0$ if for any constants $\varepsilon, \alpha > 0$, for any continuous function $\mathbf{F} : \mathbf{H}^k \times G_\alpha^k \rightarrow \mathbb{R}^N$, for any initial data $(\mathbf{u}_0, g_0) \in \mathbf{H}^k \times G_\alpha^k$ and for any regular point (\mathbf{u}_1, g_1) for \mathbf{F} there is a control $\boldsymbol{\eta} \in \Theta(\mathbf{u}_0, g_0, \mathbf{f}) \cap \mathbf{X}$ such that

$$\begin{aligned} \|\mathcal{R}_T(\mathbf{u}_0, g_0, 0, 0, \boldsymbol{\eta}) - (\mathbf{u}_1, g_1)\|_{\mathbf{H}^k \times \mathbf{H}^k} &< \varepsilon, \\ \mathbf{F}(\mathcal{R}_T(\mathbf{u}_0, g_0, 0, 0, \boldsymbol{\eta})) &= \mathbf{F}(\mathbf{u}_1, g_1). \end{aligned}$$

Let us note that this concept of controllability is stronger than the approximate controllability and is weaker than the exact controllability. In the following example the constructed function admits a right inverse.

Example 3.3.1. For any function $\mathbf{z} \in \mathbf{H}^k \times G_\alpha^k$ we set

$$\mathbf{F}(\mathbf{z}) := \int |z(x)|^2 dx.$$

Then for any nonzero elements $z_1 \in \mathbf{H}^k$ and $z_2 \in G_\alpha^k$ the point $\mathbf{z} = (z_1, z_2)$ is regular for \mathbf{F} .

For any finite-dimensional subspace $\mathbf{E} \subset \mathbf{H}^{k+2}$, we denote by $\mathcal{F}(\mathbf{E})$ the largest vector space $\mathbf{F} \subset \mathbf{H}^{k+2}$ such that for any $\boldsymbol{\eta}_1 \in \mathbf{F}$ there are vectors $\boldsymbol{\eta}, \boldsymbol{\zeta}^1, \dots, \boldsymbol{\zeta}^n \in \mathbf{E}$ satisfying the relation

$$\boldsymbol{\eta}_1 = \boldsymbol{\eta} - \sum_{i=1}^n (\boldsymbol{\zeta}^i \cdot \nabla) \boldsymbol{\zeta}^i.$$

We define \mathbf{E}_k by the rule

$$\mathbf{E}_0 = \mathbf{E}, \quad \mathbf{E}_n = \mathcal{F}(\mathbf{E}_{n-1}) \quad \text{for } n \geq 1, \quad \mathbf{E}_\infty = \bigcup_{n=1}^{\infty} \mathbf{E}_n.$$

The following theorem is the main result of this section.

Theorem 3.3.2. Suppose $\mathbf{f} \in C^\infty([0, \infty), \mathbf{H}^{k+2})$. If $\mathbf{E} \subset \mathbf{H}^{k+2}$ is a finite-dimensional subspace such that \mathbf{E}_∞ is dense in \mathbf{H}^{k+1} , then system (3.3.1), (3.3.2) with $\boldsymbol{\eta} \in C^\infty(J_T, \mathbf{E})$ is controllable at time $T > 0$ in the sense of Definition 5.

This theorem will be established in Section 3.3.2. We now construct an example of a subspace \mathbf{E} for which the hypothesis of Theorem 3.3.2 is satisfied.

Let us introduce the functions

$$c_{\mathbf{m}}^i(\mathbf{x}) = \mathbf{e}_i \cos\langle \mathbf{m}, \mathbf{x} \rangle, \quad s_{\mathbf{m}}^i(\mathbf{x}) = \mathbf{e}_i \sin\langle \mathbf{m}, \mathbf{x} \rangle, \quad i = 1, 2, 3,$$

where $\mathbf{m} \in \mathbb{Z}^3$ and $\{\mathbf{e}_i\}$ is the standard basis in \mathbb{R}^3 .

Lemma 3.3.3. *If $\mathbf{E} = \text{span}\{c_{\mathbf{m}}^i, s_{\mathbf{m}}^i, 0 \leq m_j \leq 1, i, j = 1, 2, 3\}$, then the vector space \mathbf{E}_∞ is dense in \mathbf{H}^k for any $k \geq 0$.*

It is straightforward to see that $\dim \mathbf{E} = 45$.

Proof of Lemma 3.3.3. It suffices to show that

$$\text{span}\{c_{\mathbf{m}}^i, s_{\mathbf{m}}^i, |\mathbf{m}| \leq 2^j\} \subset \mathbf{E}_{j+1} \quad \text{for all } j \geq 0, \quad (3.3.4)$$

where $|\mathbf{m}| = |m_1| + |m_2| + |m_3|$. We prove (3.3.4) by induction. The case $j = 0$ is clear. We shall prove (3.3.4) for $j \geq 1$ assuming that it is true for any $j' < j$. If $n_i \neq 0$, then it is easy to see

$$\begin{aligned} s_{2\mathbf{n}}^i(\mathbf{x}) &= -\frac{2}{n_i} c_{\mathbf{n}}^i(\mathbf{x}) \cdot \nabla c_{\mathbf{n}}^i(\mathbf{x}), \\ -s_{2\mathbf{n}}^i(\mathbf{x}) &= -\frac{2}{n_i} s_{\mathbf{n}}^i(\mathbf{x}) \cdot \nabla s_{\mathbf{n}}^i(\mathbf{x}), \\ c_{2\mathbf{n}}^i(\mathbf{x}) &= -\frac{1}{n_i} (s_{\mathbf{n}}^i(\mathbf{x}) - c_{\mathbf{n}}^i(\mathbf{x})) \cdot \nabla (s_{\mathbf{n}}^i(\mathbf{x}) - c_{\mathbf{n}}^i(\mathbf{x})), \\ -c_{2\mathbf{n}}^i(\mathbf{x}) &= -\frac{1}{n_i} (s_{\mathbf{n}}^i(\mathbf{x}) + c_{\mathbf{n}}^i(\mathbf{x})) \cdot \nabla (s_{\mathbf{n}}^i(\mathbf{x}) + c_{\mathbf{n}}^i(\mathbf{x})). \end{aligned}$$

Thus $s_{2\mathbf{n}}^i(\mathbf{x}), c_{2\mathbf{n}}^i(\mathbf{x}) \in \mathbf{E}_{j+1}$ for any $|\mathbf{n}| \leq 2^{j-1}$, $n_i \neq 0$. If $n_i = 0$, without loss of generality, we can assume $n_1 \neq 0$, then

$$s_{2\mathbf{n}}^1(\mathbf{x}) + s_{2\mathbf{n}}^i(\mathbf{x}) = -\frac{2}{n_1} (c_{\mathbf{n}}^1(\mathbf{x}) + c_{\mathbf{n}}^i(\mathbf{x})) \cdot \nabla (c_{\mathbf{n}}^1(\mathbf{x}) + c_{\mathbf{n}}^i(\mathbf{x})), \quad (3.3.5)$$

$$-s_{2\mathbf{n}}^1(\mathbf{x}) - s_{2\mathbf{n}}^i(\mathbf{x}) = -\frac{2}{n_1} (s_{\mathbf{n}}^1(\mathbf{x}) + s_{\mathbf{n}}^i(\mathbf{x})) \cdot \nabla (s_{\mathbf{n}}^1(\mathbf{x}) + s_{\mathbf{n}}^i(\mathbf{x})). \quad (3.3.6)$$

As $\pm s_{2\mathbf{n}}^1(\mathbf{x}) \in \mathbf{E}_{j+1}$ and the right-hand sides of (3.3.5), (3.3.6) are in \mathbf{E}_{j+1} , we get $s_{2\mathbf{n}}^i(\mathbf{x}) \in \mathbf{E}_{j+1}$ for any $|\mathbf{n}| \leq 2^{j-1}$, $i = 1, 2, 3$. In the same way, we can show that $c_{2\mathbf{n}}^i(\mathbf{x}) \in \mathbf{E}_{j+1}$. Now take $\mathbf{l} \in \mathbb{Z}^3$, $|\mathbf{l}| \leq 2^j$ and let us choose $\mathbf{n} \in \mathbb{Z}^3$, $|\mathbf{n}| \leq 2^{j-1}$ and $\mathbf{m} \in \mathbb{Z}^3$, $|\mathbf{m}| \leq 2^{j-1}$ such that

$$\mathbf{l} = \mathbf{n} + \mathbf{m} \quad \text{and} \quad c_{\mathbf{n}-\mathbf{m}}^i, s_{\mathbf{n}-\mathbf{m}}^i \in \mathbf{E}_1.$$

For example, if $\mathbf{l} = (l_1, l_2, l_3)$ and l_1 is even, we can take

$$\mathbf{n} = \left(\frac{l_1}{2}, \left[\frac{l_2}{2} \right], l_3 - \left[\frac{l_3}{2} \right] \right) \quad \text{and} \quad \mathbf{m} = \left(\frac{l_1}{2}, l_2 - \left[\frac{l_2}{2} \right], \left[\frac{l_3}{2} \right] \right).$$

A similar representation holds if l_2 or l_3 is even. On the other hand, if all l_i are odd, then necessarily $l_i \leq 2^j - 1$, and we can take

$$\mathbf{n} = \left(l_1 - \left[\frac{l_1}{2} \right], \left[\frac{l_2}{2} \right], l_3 - \left[\frac{l_3}{2} \right] \right) \quad \text{and} \quad \mathbf{m} = \left(\left[\frac{l_1}{2} \right], l_2 - \left[\frac{l_2}{2} \right], \left[\frac{l_3}{2} \right] \right).$$

Using the identities

$$\begin{aligned} (s_n^i(\mathbf{x}) \pm s_m^i(\mathbf{x})) \cdot \nabla(s_n^i(\mathbf{x}) \pm s_m^i(\mathbf{x})) &= \frac{n_i}{2} s_{2n}^i(\mathbf{x}) + \frac{m_i}{2} s_{2m}^i(\mathbf{x}) \pm \frac{l_i}{2} s_l^i(\mathbf{x}) \\ &\quad \pm \frac{n_i - m_i}{2} s_{n-m}^i(\mathbf{x}), \\ (s_n^i(\mathbf{x}) \pm c_m^i(\mathbf{x})) \cdot \nabla(s_n^i(\mathbf{x}) \pm c_m^i(\mathbf{x})) &= \frac{n_i}{2} s_{2n}^i(\mathbf{x}) - \frac{m_i}{2} s_{2m}^i(\mathbf{x}) \pm \frac{l_i}{2} c_l^i(\mathbf{x}) \\ &\quad \pm \frac{n_i - m_i}{2} c_{n-m}^i(\mathbf{x}), \end{aligned}$$

we obtain that if $l_i \neq 0$, then $s_l^i(\mathbf{x}), c_l^i(\mathbf{x}) \in \mathbf{E}_{j+1}$ for any $|l| \leq 2^j, i = 1, 2, 3$. Arguing as above, we can easily prove that also in the case $l_i = 0$ we have $s_l^i(\mathbf{x}), c_l^i(\mathbf{x}) \in \mathbf{E}_{j+1}$. \square

3.3.2 Proof of Theorem 3.3.2

We shall need the concept of $(\varepsilon, \mathbf{u}_0, g_0, \mathbf{K})$ -controllability of the system. Let us fix constants $\varepsilon, \alpha > 0$, an initial point $(\mathbf{u}_0, g_0) \in \mathbf{H}^k \times G_\alpha^k$, a compact set $\mathbf{K} \subset \mathbf{H}^k \times G_\alpha^k$ and a vector space $\mathbf{X} \subset L^2(J_T, \mathbf{H}^k)$.

Définition 6. We say that system (3.3.1), (3.3.2) with $\boldsymbol{\eta} \in \mathbf{X}$ is $(\varepsilon, \mathbf{u}_0, g_0, \mathbf{K})$ -controllable at time $T > 0$ if there is a continuous mapping

$$\Psi : \mathbf{K} \rightarrow \Theta(\mathbf{u}_0, g_0, \mathbf{f}) \cap \mathbf{X}$$

such that

$$\sup_{(\hat{\mathbf{u}}, \hat{g}) \in \mathbf{K}} \|\mathcal{R}_T(\mathbf{u}_0, g_0, 0, 0, \Psi(\hat{\mathbf{u}}, \hat{g})) - (\hat{\mathbf{u}}, \hat{g})\|_{\mathbf{H}^k \times H^k} < \varepsilon,$$

where $\Theta(\mathbf{u}_0, g_0, \mathbf{f}) \cap \mathbf{X}$ is endowed with the norm of $L^2(J_T, \mathbf{H}^k)$.

The proof of Theorem 3.3.2 is deduced from the following result.

Theorem 3.3.4. If $\mathbf{E} \subset \mathbf{H}^{k+2}$, $k \geq 4$ is a finite-dimensional subspace such that \mathbf{E}_∞ is dense in \mathbf{H}^{k+1} , then for any $\varepsilon > 0$, $(\mathbf{u}_0, g_0) \in \mathbf{H}^k \times G_\alpha^k$ and $\mathbf{K} \subset \mathbf{H}^k \times G_\alpha^k$ system (3.3.1), (3.3.2) with $\boldsymbol{\eta} \in C^\infty(J_T, \mathbf{E})$ is $(\varepsilon, \mathbf{u}_0, g_0, \mathbf{K})$ -controllable at time $T > 0$.

Taking this assertion for granted, let us complete the proof of Theorem 3.3.2. Suppose ε and α are positive constants, $\mathbf{F} : \mathbf{H}^k \times G_\alpha^k \rightarrow \mathbb{R}^N$ is a continuous function and (\mathbf{u}_1, g_1) is a regular point for \mathbf{F} . Thus, there is a closed ball $\mathbf{B} \subset \mathbb{R}^N$ centred at $\mathbf{y}_1 = \mathbf{F}(\mathbf{u}_1, g_1)$ of radius $r > 0$ and a continuous function $\mathbf{G} : \mathbf{B} \rightarrow \mathbf{H}^k \times G_\alpha^k$ such that $\mathbf{G}(\mathbf{u}_1, g_1) = (\mathbf{u}_1, g_1)$ and $\mathbf{F}(\mathbf{G}(\mathbf{y})) = \mathbf{y}$ for any $\mathbf{y} \in \mathbf{B}$. Without loss of generality, we can assume that \mathbf{B} is such that

$$\sup_{\mathbf{y} \in \mathbf{B}} \|\mathbf{G}(\mathbf{y}) - (\mathbf{u}_1, g_1)\|_{\mathbf{H}^k \times H^k} \leq \frac{\varepsilon}{2}. \quad (3.3.7)$$

Let us choose a constant $0 < \varepsilon_0 < \varepsilon$ such that

$$\|F(\hat{\mathbf{y}}) - F(\tilde{\mathbf{y}})\|_{\mathbb{R}^N} < r \text{ for any } \hat{\mathbf{y}}, \tilde{\mathbf{y}} \in B, \|\hat{\mathbf{y}} - \tilde{\mathbf{y}}\|_{\mathbf{H}^k \times H^k} \leq \frac{\varepsilon_0}{2}. \quad (3.3.8)$$

Since $\mathbf{K} := \mathbf{G}(B)$ is a compact subset of $\mathbf{H}^k \times G_\alpha^k$, Theorem 3.3.4 implies that there is a continuous mapping $\Psi : \mathbf{K} \rightarrow \Theta(\mathbf{u}_0, g_0, \mathbf{f}) \cap \mathbf{X}$ such that

$$\sup_{(\hat{\mathbf{u}}, \hat{g}) \in \mathbf{K}} \|\mathcal{R}_T(\mathbf{u}_0, g_0, 0, 0, \Psi(\hat{\mathbf{u}}, \hat{g})) - (\hat{\mathbf{u}}, \hat{g})\|_{\mathbf{H}^k \times H^k} < \frac{\varepsilon_0}{2}. \quad (3.3.9)$$

Therefore, the continuous mapping

$$\Phi : B \rightarrow \mathbb{R}^N, \quad \mathbf{y} \rightarrow F(\mathcal{R}_T(\mathbf{u}_0, g_0, 0, 0, \Psi \circ G(\mathbf{y})))$$

satisfies the inequality

$$\sup_{\mathbf{y} \in B} \|\Phi(\mathbf{y}) - \mathbf{y}\|_{\mathbb{R}^N} = \sup_{\mathbf{y} \in B} \|F(\mathcal{R}_T(\mathbf{u}_0, g_0, 0, 0, \Psi \circ G(\mathbf{y}))) - F(G(\mathbf{y}))\|_{\mathbb{R}^N} < r.$$

Applying the Brouwer theorem, we see that the mapping $\mathbf{y} \rightarrow \mathbf{y}_1 + \mathbf{y} - \Phi(\mathbf{y})$ from B to B has a fixed point $\bar{\mathbf{y}} \in B$. Thus

$$F(\mathcal{R}_T(\mathbf{u}_0, g_0, 0, 0, \Psi \circ G(\bar{\mathbf{y}}))) = F(\mathbf{u}_1, g_1).$$

Using (3.3.7) and (3.3.9), we obtain

$$\begin{aligned} & \|\mathcal{R}_T(\mathbf{u}_0, g_0, 0, 0, \Psi(G(\bar{\mathbf{y}}))) - (\mathbf{u}_1, g_1)\|_{\mathbf{H}^k \times H^k} \\ & \leq \|\mathcal{R}_T(\mathbf{u}_0, g_0, 0, 0, \Psi(G(\bar{\mathbf{y}}))) - G(\bar{\mathbf{y}})\|_{\mathbf{H}^k \times H^k} + \|G(\bar{\mathbf{y}}) - (\mathbf{u}_1, g_1)\|_{\mathbf{H}^k \times H^k} < \varepsilon. \end{aligned}$$

This completes the proof.

3.4 Proof of Theorem 3.3.4

3.4.1 Reduction to controllability with E_1 -valued controls

Theorem 3.3.4 is derived from the proposition below, which is established in Subsection 3.4.2.

Proposition 3.4.1. *Suppose that $\mathbf{E} \subset \mathbf{H}^{k+2}$ is a finite-dimensional subspace. Then system (3.3.1), (3.3.2) is $(\varepsilon, \mathbf{u}_0, g_0, \mathbf{K})$ -controllable with $\boldsymbol{\eta} \in C^\infty(J_T, \mathbf{E}_1)$ if and only if it is $(\varepsilon, \mathbf{u}_0, g_0, \mathbf{K})$ -controllable with $\boldsymbol{\eta} \in C^\infty(J_T, \mathbf{E})$.*

Proof of Theorem 3.3.4. In view of Proposition 3.4.1, it suffices to prove that there is an integer $N \geq 1$, depending only on $\varepsilon, \mathbf{u}_0, g_0$ and \mathbf{K} , such that (3.3.1), (3.3.2) with $\boldsymbol{\eta} \in C^\infty(J_T, \mathbf{E}_N)$ is $(\varepsilon, \mathbf{u}_0, g_0, \mathbf{K})$ -controllable at time T . For any $\mu > 0$ and $(\hat{\mathbf{u}}, \hat{g}) \in \mathbf{K}$ let us define

$$\mathbf{u}_\mu(t; \hat{\mathbf{u}}) = T^{-1}(te^{-\mu\Delta}\hat{\mathbf{u}} + (T-t)e^{-\mu\Delta}\mathbf{u}_0), \quad (3.4.1)$$

$$g_\mu(t; \hat{g}) = \ln(T^{-1}(te^{\varphi_\mu(\hat{g})} + (T-t)e^{\varphi_\mu(g_0)})), \quad (3.4.2)$$

where $\varphi_\mu(g) \in G_\alpha^{k+1}$ is such that $\varphi_\mu(g) \rightarrow g$ as $\mu \rightarrow 0$ for all $g \in G_\alpha^k$. For example, we can take

$$\varphi_\mu(g) = \ln\left(\frac{\alpha}{\int \exp(e^{-\mu\Delta}g(\mathbf{x}))d\mathbf{x}}\right) + e^{-\mu\Delta}g.$$

Step 1. In this step, we show that there are controls $\boldsymbol{\eta}_\mu \in C^\infty(J_T, \mathbf{H}^k)$ and $\boldsymbol{\xi}_\mu \in C^\infty(J_T, \mathbf{H}^{k+1})$ satisfying

$$\mathcal{R}(e^{-\mu\Delta}\mathbf{u}_0, \varphi_\mu(g_0), \boldsymbol{\xi}_\mu, \boldsymbol{\xi}_\mu, \boldsymbol{\eta}_\mu) = (\mathbf{u}_\mu, g_\mu). \quad (3.4.3)$$

We first construct $\boldsymbol{\xi}_\mu \in C^\infty(J_T, \mathbf{H}^{k+1})$ such that

$$\partial_t g_\mu + ((\mathbf{u}_\mu + \boldsymbol{\xi}_\mu) \cdot \nabla)g_\mu + \nabla \cdot (\mathbf{u}_\mu + \boldsymbol{\xi}_\mu) = 0. \quad (3.4.4)$$

To this end, let us multiply (3.4.4) by e^{g_μ} and perform some simple transformations. We get

$$\nabla \cdot (e^{g_\mu}\boldsymbol{\xi}_\mu) = -\partial_t e^{g_\mu} - \nabla \cdot (e^{g_\mu}\mathbf{u}_\mu). \quad (3.4.5)$$

We seek a solution of this equation in the form $e^{g_\mu}\boldsymbol{\xi}_\mu = \nabla\psi_\mu$. Substituting this into (3.4.5), we get

$$\Delta\psi_\mu = -\partial_t e^{g_\mu} - \nabla \cdot (e^{g_\mu}\mathbf{u}_\mu).$$

This equation has a solution $\psi_\mu \in C^\infty(J_T, \mathbf{H}^{k+2})$ if and only if the integral of the right-hand side over \mathbb{T}^3 is zero. The definitions of \mathbf{u}_μ, g_μ imply that

$$\int (\partial_t e^{g_\mu} + \nabla \cdot (e^{g_\mu}\mathbf{u}_\mu))d\mathbf{x} = \partial_t \int e^{g_\mu}d\mathbf{x} = \partial_t \alpha = 0.$$

Thus (3.4.5) has a solution $\boldsymbol{\xi}_\mu \in C^\infty(J_T, \mathbf{H}^{k+1})$. Since $\|e^{-\mu\Delta}\mathbf{u}_0\|_{k+1}, \|\varphi_\mu(g_0)\|_{k+1}$ are bounded with respect to $\mu \in (0, 1)$, the constructions of \mathbf{u}_μ and g_μ imply that $\|\partial_t e^{g_\mu} - \nabla \cdot (e^{g_\mu}\mathbf{u}_\mu)\|_k$ is also bounded. Thus $\|\psi_\mu\|_{k+2}$ is bounded, which implies the boundedness of $\|\boldsymbol{\xi}_\mu\|_{k+1}$. If we define

$$\boldsymbol{\eta}_\mu = \partial_t \mathbf{u}_\mu + ((\mathbf{u}_\mu + \boldsymbol{\xi}_\mu) \cdot \nabla)(\mathbf{u}_\mu + \boldsymbol{\xi}_\mu) + h(g_\mu)\nabla g_\mu - \mathbf{f}, \quad (3.4.6)$$

then $\boldsymbol{\eta}_\mu \in C^\infty(J_T, \mathbf{H}^k)$ and (3.4.3) holds.

Step 2. Let us take some functions $\boldsymbol{\xi}_\mu^\delta \in C^\infty(J_T, \mathbf{H}^{k+1})$ such that $\boldsymbol{\xi}_\mu^\delta(0) = \boldsymbol{\xi}_\mu^\delta(T) = 0$ and

$$\|\boldsymbol{\xi}_\mu^\delta - \boldsymbol{\xi}_\mu\|_{L^2(J_T, \mathbf{H}^{k+1})} \rightarrow 0 \text{ as } \delta \rightarrow 0. \quad (3.4.7)$$

Using the constructions of $\boldsymbol{\xi}_\mu, \boldsymbol{\eta}_\mu$ and the fact that

$$(\mathbf{u}_\mu(T; \hat{\mathbf{u}}), g_\mu(T; \hat{g})) = (e^{-\mu\Delta}\hat{\mathbf{u}}, \varphi_\mu(\hat{g})),$$

we have

$$\mathcal{R}_T(e^{-\mu\Delta}\mathbf{u}_0, \varphi_\mu(g_0), \boldsymbol{\xi}_\mu, \boldsymbol{\xi}_\mu, \boldsymbol{\eta}_\mu) = (e^{-\mu\Delta}\hat{\mathbf{u}}, \varphi_\mu(\hat{g})). \quad (3.4.8)$$

On the other hand, $\xi_\mu^\delta(0) = \xi_\mu^\delta(T) = 0$ implies

$$\mathcal{R}_T(e^{-\mu\Delta}\mathbf{u}_0, \varphi_\mu(g_0), \xi_\mu^\delta, \xi_\mu^\delta, \boldsymbol{\eta}_\mu) = \mathcal{R}_T(e^{-\mu\Delta}\mathbf{u}_0, \varphi_\mu(g_0), 0, 0, \boldsymbol{\eta}_\mu - \partial_t \xi_\mu^\delta) \quad (3.4.9)$$

Then, by Theorem 3.2.2, (3.4.8) and (3.4.9), we obtain

$$\begin{aligned} & \|\mathcal{R}_T(e^{-\mu\Delta}\mathbf{u}_0, \varphi_\mu(g_0), \xi_\mu, \xi_\mu, \boldsymbol{\eta}_\mu) - \mathcal{R}_T(e^{-\mu\Delta}\mathbf{u}_0, \varphi_\mu(g_0), \xi_\mu^\delta, \xi_\mu^\delta, \boldsymbol{\eta}_\mu)\|_{\mathbf{H}^k \times H^k} \\ &= \|(e^{-\mu\Delta}\hat{\mathbf{u}}, \varphi_\mu(\hat{g})) - \mathcal{R}_T(e^{-\mu\Delta}\mathbf{u}_0, \varphi_\mu(g_0), 0, 0, \boldsymbol{\eta}_\mu - \partial_t \xi_\mu^\delta)\|_{\mathbf{H}^k \times H^k} \rightarrow 0 \end{aligned} \quad (3.4.10)$$

as $\delta \rightarrow 0$. Clearly

$$\|(e^{-\mu\Delta}\mathbf{u}_0, \varphi_\mu(g_0)) - (\mathbf{u}_0, g_0)\|_{\mathbf{H}^k \times H^k} + \|(e^{-\mu\Delta}\hat{\mathbf{u}}, \varphi_\mu(\hat{g})) - (\hat{\mathbf{u}}, \hat{g})\|_{\mathbf{H}^k \times H^k} \rightarrow 0 \quad (3.4.11)$$

as $\mu \rightarrow 0$. The fact that \mathbf{E}_∞ is dense in \mathbf{H}^{k+1} implies that

$$\sup_{(\hat{\mathbf{u}}, \hat{g}) \in \mathbf{K}} \|P_{\mathbf{E}_N}(\boldsymbol{\eta}_\mu - \partial_t \xi_\mu^\delta) - (\boldsymbol{\eta}_\mu - \partial_t \xi_\mu^\delta)\|_{L^2(J_T, \mathbf{H}^{k+1})} \rightarrow 0 \text{ as } N \rightarrow \infty. \quad (3.4.12)$$

Since $\|\xi_\mu\|_{k+1}$ is bounded uniformly with respect to $\mu \in (0, 1)$, equation (3.4.6) implies that $\|\boldsymbol{\eta}_\mu\|_k$ is also bounded. Taking time derivative of (3.4.5), we can show the boundedness of $\|\partial_t \xi_\mu\|_k$. Thus $\|(e^{-\mu\Delta}\mathbf{u}_0, \varphi_\mu(g_0), \mathbf{0}, \mathbf{0}, \boldsymbol{\eta}_\mu - \partial_t \xi_\mu^\delta)\|_{\mathbf{X}^k}$ is bounded uniformly with respect to $\mu \in (0, 1)$. Hence, by Theorem 3.2.2 and relations (3.4.10)-(3.4.12), a solution $\mathcal{R}(\mathbf{u}_0, g_0, 0, 0, P_{\mathbf{E}_N}(\boldsymbol{\eta}_\mu(\hat{\mathbf{u}}, \hat{g}))) \in \mathbf{Y}^k$ exists for sufficiently large $N \geq 1$ and sufficiently small $\delta, \mu > 0$. Moreover,

$$\sup_{(\hat{\mathbf{u}}, \hat{g}) \in \mathbf{K}} \|\mathcal{R}_T(\mathbf{u}_0, g_0, 0, 0, P_{\mathbf{E}_N}(\boldsymbol{\eta}_\mu(\hat{\mathbf{u}}, \hat{g}) - \partial_t \xi_\mu^\delta(\hat{\mathbf{u}}, \hat{g}))) - (\hat{\mathbf{u}}, \hat{g})\|_{\mathbf{H}^k \times H^k} < \varepsilon.$$

From (3.4.5) and the constructions of $\mathbf{u}_\mu(\hat{\mathbf{u}}), g_\mu(\hat{g})$, we have

$$\xi_\mu : (\hat{\mathbf{u}}, \hat{g}) \mapsto \xi_\mu(\hat{\mathbf{u}}, \hat{g}), \quad \partial_t \xi_\mu : (\hat{\mathbf{u}}, \hat{g}) \rightarrow \partial_t \xi_\mu(\hat{\mathbf{u}}, \hat{g})$$

are continuous from \mathbf{K} to $L^2(J_T, \mathbf{H}^{k+1})$. Then (3.4.6) implies that mapping

$$P_{\mathbf{E}_N}(\boldsymbol{\eta}_\mu - \partial_t \xi_\mu^\delta)(\cdot, \cdot) : (\hat{\mathbf{u}}, \hat{g}) \rightarrow P_{\mathbf{E}_N}(\boldsymbol{\eta}_\mu(\cdot, \hat{\mathbf{u}}, \hat{g}) - \partial_t \xi_\mu^\delta(\hat{\mathbf{u}}, \hat{g}))$$

is continuous from \mathbf{K} to $L^2(J_T, \mathbf{H}^k)$. The proof is complete. \square

3.4.2 Proof of Proposition 3.4.1

The proof of Proposition 3.4.1 is inspired by ideas from [3, 4, 39, 40]. Let us admit for the moment the following lemma.

Lemma 3.4.2. *For any $(\mathbf{u}_0, g_0) \in \mathbf{H}^{k+2} \times H^{k+2}$, for any $\varepsilon > 0$ and for any continuous mapping $\Psi_1 : \mathbf{K} \rightarrow \Theta(\mathbf{u}_0, g_0, \mathbf{f}) \cap C^\infty(J_T, \mathbf{E}_1)$ there is a constant $\nu > 0$ and a continuous mapping $\Psi : \mathbf{K} \rightarrow \Theta(\mathbf{u}_0, g_0, \mathbf{f}) \cap C^\infty(J_T, \mathbf{E})$ such that*

$$\sup_{(\hat{\mathbf{u}}, \hat{g}) \in \mathbf{K}} \|\mathcal{R}_T(\tilde{\mathbf{u}}_0, \tilde{g}_0, 0, 0, \Psi(\hat{\mathbf{u}}, \hat{g})) - \mathcal{R}_T(\tilde{\mathbf{u}}_0, \tilde{g}_0, 0, 0, \Psi_1(\hat{\mathbf{u}}, \hat{g}))\|_{\mathbf{H}^k \times H^k} < \varepsilon \quad (3.4.13)$$

for any $(\tilde{\mathbf{u}}_0, \tilde{g}_0) \in \mathbf{H}^{k+2} \times H^{k+2}$ with $\|\mathbf{u}_0 - \tilde{\mathbf{u}}_0\|_k + \|g_0 - \tilde{g}_0\|_k < \nu$.

Let $(\mathbf{u}_0, g_0) \in \mathbf{H}^k \times H^k$ and $\Psi_1 : \mathbf{K} \rightarrow \Theta(\mathbf{u}_0, g_0, \mathbf{f}) \cap C^\infty(J_T, \mathbf{E}_1)$ be such that

$$\sup_{(\hat{\mathbf{u}}, \hat{g}) \in \mathbf{K}} \|\mathcal{R}_T(\mathbf{u}_0, g_0, 0, 0, \Psi_1(\hat{\mathbf{u}}, \hat{g})) - (\hat{\mathbf{u}}, \hat{g})\|_{\mathbf{H}^k \times H^k} < \frac{\varepsilon}{2}. \quad (3.4.14)$$

Take any sequence $(\mathbf{u}_0^n, g_0^n) \in \mathbf{H}^{k+2} \times H^{k+2}$ such that

$$\|(\mathbf{u}_0, g_0) - (\mathbf{u}_0^n, g_0^n)\|_{\mathbf{H}^k \times H^k} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

As \mathbf{K} is compact, Theorem 3.2.2 implies that $\Psi_1(\mathbf{K}) \subset \Theta(\mathbf{u}_0^n, g_0^n, \mathbf{f})$ for sufficiently large n . By Lemma 3.4.2, there is a continuous mapping

$$\Psi : \mathbf{K} \rightarrow \Theta(\mathbf{u}_0^n, g_0^n, \mathbf{f}) \cap C^\infty(J_T, \mathbf{E})$$

such that

$$\sup_{(\hat{\mathbf{u}}, \hat{g}) \in \mathbf{K}} \|\mathcal{R}_T(\mathbf{u}_0^n, g_0^n, 0, 0, \Psi(\hat{\mathbf{u}}, \hat{g})) - \mathcal{R}_T(\mathbf{u}_0^n, g_0^n, 0, 0, \Psi_1(\hat{\mathbf{u}}, \hat{g}))\|_{\mathbf{H}^k \times H^k} < \frac{\varepsilon}{2}.$$

Choosing n sufficiently large and using the fact that \mathcal{R} is uniformly continuous on the compact set $\Psi(\mathbf{K}) \cup \Psi_1(\mathbf{K})$, we get

$$\begin{aligned} & \sup_{(\hat{\mathbf{u}}, \hat{g}) \in \mathbf{K}} \|\mathcal{R}_T(\mathbf{u}_0, g_0, 0, 0, \Psi(\hat{\mathbf{u}}, \hat{g})) - \mathcal{R}_T(\mathbf{u}_0, g_0, 0, 0, \Psi_1(\hat{\mathbf{u}}, \hat{g}))\|_{\mathbf{H}^k \times H^k} \\ & \leq \sup_{(\hat{\mathbf{u}}, \hat{g}) \in \mathbf{K}} \|\mathcal{R}_T(\mathbf{u}_0, g_0, 0, 0, \Psi(\hat{\mathbf{u}}, \hat{g})) - \mathcal{R}_T(\mathbf{u}_0^n, g_0^n, 0, 0, \Psi(\hat{\mathbf{u}}, \hat{g}))\|_{\mathbf{H}^k \times H^k} \\ & \quad + \sup_{(\hat{\mathbf{u}}, \hat{g}) \in \mathbf{K}} \|\mathcal{R}_T(\mathbf{u}_0^n, g_0^n, 0, 0, \Psi_1(\hat{\mathbf{u}}, \hat{g})) - \mathcal{R}_T(\mathbf{u}_0, g_0, 0, 0, \Psi_1(\hat{\mathbf{u}}, \hat{g}))\|_{\mathbf{H}^k \times H^k} \\ & \quad + \sup_{(\hat{\mathbf{u}}, \hat{g}) \in \mathbf{K}} \|\mathcal{R}_T(\mathbf{u}_0^n, g_0^n, 0, 0, \Psi(\hat{\mathbf{u}}, \hat{g})) - \mathcal{R}_T(\mathbf{u}_0^n, g_0^n, 0, 0, \Psi_1(\hat{\mathbf{u}}, \hat{g}))\|_{\mathbf{H}^k \times H^k} \\ & < \frac{\varepsilon}{2}. \end{aligned} \quad (3.4.15)$$

Combining (3.4.14) and (3.4.15), we complete the proof of Proposition 3.4.1.

Proof of Lemma 3.4.2. Step 1. We shall need the following lemma, which can be proved by literal repetition of the arguments of the proof of [40, Lemma 3.5].

Lemma 3.4.3. *For any continuous mapping $\Psi_1 : \mathbf{K} \rightarrow \Theta(\mathbf{u}_0, g_0, \mathbf{f}) \cap L^2(J_T, \mathbf{E}_1)$ there is a set $\mathbf{A} = \{\eta_1^l, l = 1, \dots, m\} \subset \mathbf{E}_1$ an integer $s \geq 1$ and a mapping $\Psi_s : \mathbf{K} \rightarrow \Theta(\mathbf{u}_0, g_0, \mathbf{f}) \cap L^2(J_T, \mathbf{E}_1)$ such that*

$$\Psi_s(\hat{\mathbf{u}}, \hat{g}) = \sum_{l=1}^m \sum_{r=0}^{s-1} c_{l,r}(\hat{\mathbf{u}}, \hat{g}) I_{r,s}(t) \eta_1^l,$$

where $c_{l,r}$ are non-negative functions such that $\sum_{l=1}^m c_{l,r} = 1$, $I_{r,s}$ is the indicator function of the interval $[t_r, t_{r+1})$ with $t_r = rT/s$ and

$$\sup_{(\hat{\mathbf{u}}, \hat{g}) \in \mathbf{K}} \|\mathcal{R}_T(\mathbf{u}_0, g_0, 0, 0, \Psi_1(\hat{\mathbf{u}}, \hat{g})) - \mathcal{R}_T(\mathbf{u}_0, g_0, 0, 0, \Psi_s(\hat{\mathbf{u}}, \hat{g}))\|_{\mathbf{H}^k \times H^k} < \varepsilon.$$

Let Ψ_s be the function constructed in Lemma 3.4.3 :

$$\Psi_s(\hat{\mathbf{u}}, \hat{g}) = \sum_{l=1}^m \varphi_l(t, \hat{\mathbf{u}}, \hat{g}) \boldsymbol{\eta}_1^l.$$

We claim that there are vectors $\boldsymbol{\zeta}^{l,1}, \dots, \boldsymbol{\zeta}^{l,2n}, \boldsymbol{\eta}^l \in \mathbf{E}$ and positive constants $\lambda_{l,1}, \dots, \lambda_{l,2n}$ whose sum is equal to 1 such that

$$\begin{aligned} \boldsymbol{\zeta}^i &= -\boldsymbol{\zeta}^{i+n} \text{ for } i = 1, \dots, n, \\ (\mathbf{u} \cdot \nabla) \mathbf{u} - \boldsymbol{\eta}_1^l &= \sum_{j=1}^{2n} \lambda_{l,j} ((\mathbf{u} + \boldsymbol{\zeta}^{l,j}) \cdot \nabla) (\mathbf{u} + \boldsymbol{\zeta}^{l,j}) - \boldsymbol{\eta}^l \text{ for any } \mathbf{u} \in \mathbf{H}^1. \end{aligned} \quad (3.4.16)$$

Indeed, by the definition of $\mathcal{F}(\mathbf{E})$, for any $\boldsymbol{\eta}_1^l \in \mathcal{F}(\mathbf{E})$ there are $\boldsymbol{\xi}^{l,1}, \dots, \boldsymbol{\xi}^{l,n}, \boldsymbol{\eta}^l \in \mathbf{E}$ such that

$$\boldsymbol{\eta}_1^l = \boldsymbol{\eta}^l - \sum_{i=1}^n (\boldsymbol{\xi}^{l,i} \cdot \nabla \boldsymbol{\xi}^{l,i}).$$

Let us set

$$\lambda^{l,i} = \lambda^{l,i+n} = \frac{1}{2n}, \quad \boldsymbol{\zeta}^{l,i} = -\boldsymbol{\zeta}^{l,i+n} = \sqrt{n} \boldsymbol{\xi}^i, \quad i = 1, \dots, n.$$

Then (3.4.16) holds for any $\mathbf{u} \in \mathbf{H}^1$.

Let $(\mathbf{u}_1, g_1) = \mathcal{R}(\mathbf{u}_0, g_0, 0, 0, \Psi_s(\hat{\mathbf{u}}, \hat{g}))$. It follows from (3.4.16) that (\mathbf{u}_1, g_1) satisfies the problem

$$\begin{aligned} \dot{\mathbf{u}}_1 + \sum_{j=1}^{2n} \sum_{l=1}^m \lambda_{l,j} \varphi_l(t, \hat{\mathbf{u}}, \hat{g}) ((\mathbf{u}_1 + \boldsymbol{\zeta}^{l,j}) \cdot \nabla) (\mathbf{u}_1 + \boldsymbol{\zeta}^{l,j}) + h(g_1) \nabla g_1 \\ = \mathbf{f}(t) + \sum_{l=1}^m \varphi_l(t, \hat{\mathbf{u}}, \hat{g}) \boldsymbol{\eta}^l, \\ (\partial_t + \mathbf{u}_1 \cdot \nabla) g_1 + \nabla \cdot \mathbf{u}_1 = 0. \end{aligned} \quad (3.4.17)$$

Taking $q = m \cdot n$, $\{\boldsymbol{\zeta}^i\}_{i=1}^q := \{\boldsymbol{\zeta}^{l,j}\}_{l=1}^m, j=1}^n, \boldsymbol{\zeta}^{i+q} := -\boldsymbol{\zeta}^i, i = 1, \dots, q$, we rewrite (3.4.17) in the form

$$\dot{\mathbf{u}}_1 + \sum_{i=1}^{2q} \psi_i(t, \hat{\mathbf{u}}, \hat{g}) ((\mathbf{u}_1 + \boldsymbol{\zeta}^i) \cdot \nabla) (\mathbf{u}_1 + \boldsymbol{\zeta}^i) + h(g_1) \nabla g_1 = \mathbf{f}(t) + \boldsymbol{\eta}(t, \hat{\mathbf{u}}, \hat{g}), \quad (3.4.18)$$

where

$$\begin{aligned} \boldsymbol{\eta}(t, \hat{\mathbf{u}}, \hat{g}) &= \sum_{l=1}^m \varphi_l(t, \hat{\mathbf{u}}, \hat{g}) \boldsymbol{\eta}^l, \\ \psi_i(t, \hat{\mathbf{u}}, \hat{g}) &= \sum_{r=0}^{s-1} d_{i,r}(\hat{\mathbf{u}}, \hat{g}) I_{r,s}(t), \end{aligned} \quad (3.4.19)$$

and $d_{i,r} \in C(\mathbf{K})$ are some non-negative functions such that

$$\sum_{i=1}^q d_{i,r} = \sum_{i=q+1}^{2q} d_{i,r} = \frac{1}{2}.$$

Step 2. Let us show that it suffices to consider the case $s = 1$. Indeed, let us assume that for any constant $\varepsilon_0 > 0$ and for any interval $I_r := [t_{r-1}, t_r]$ there exists a continuous mapping $\Psi_{\varepsilon_0}^r : \mathbf{K} \rightarrow \Theta^r(\mathbf{u}_1(t_{r-1}), g_1(t_{r-1})) \cap C^\infty(J_T, \mathbf{E})$ such that

$$\sup_{(\hat{\mathbf{u}}, \hat{g}) \in \mathbf{K}} \|\mathcal{R}_{t_r - t_{r-1}}(\mathbf{u}_1(t_{r-1}), g_1(t_{r-1}), 0, 0, \Psi_{\varepsilon_0}^r(\hat{\mathbf{u}}, \hat{g})) - (\mathbf{u}_1(t_r), g_1(t_r))\|_{\mathbf{H}^k \times H^k} < \varepsilon_0.$$

Here $\Theta^r(\mathbf{u}_1(t_{r-1}), g_1(t_{r-1}))$ is the set of functions $\boldsymbol{\eta} \in L^2(I_r, \mathbf{H}^k)$ for which problem (3.3.1)-(3.3.2) has a solution in $C(I_r, \mathbf{H}^k) \times C(I_r, H^k)$ satisfying the initial condition

$$\mathbf{u}(t_{r-1}) = \mathbf{u}_1(t_{r-1}), \quad g(t_{r-1}) = g_1(t_{r-1}).$$

In view of Theorem 3.2.2, there is $\delta_s > 0$ such that for any $(\tilde{\mathbf{u}}_0, \tilde{g}_0) \in \mathbf{H}^{k+2} \times H^{k+2}$ with $\|(\tilde{\mathbf{u}}_0, \tilde{g}_0) - (\mathbf{u}_1(t_{s-1}), g_1(t_{s-1}))\|_{\mathbf{H}^k \times H^k} < \delta_s$ we have the inequality

$$\sup_{(\hat{\mathbf{u}}, \hat{g}) \in \mathbf{K}} \|\mathcal{R}_{T - t_{s-1}}(\tilde{\mathbf{u}}_0, \tilde{g}_0, 0, 0, \Psi_\varepsilon^s(\hat{\mathbf{u}}, \hat{g})) - (\mathbf{u}_1(T), g_1(T))\|_{\mathbf{H}^k \times H^k} < \varepsilon.$$

Similarly, we can find $\delta_r > 0$, $r = s - 1, \dots, 1$ such that

$$\sup_{(\hat{\mathbf{u}}, \hat{g}) \in \mathbf{K}} \|\mathcal{R}_{t_{r+1} - t_r}(\tilde{\mathbf{u}}_0, \tilde{g}_0, 0, 0, \Psi_{\delta_{r+1}}^r(\hat{\mathbf{u}}, \hat{g})) - (\mathbf{u}_1(t_{r+1}), g_1(t_{r+1}))\|_{\mathbf{H}^k \times H^k} < \delta_{r+1}$$

for any $(\tilde{\mathbf{u}}_0, \tilde{g}_0) \in \mathbf{H}^{k+2} \times H^{k+2}$ satisfying

$$\|(\tilde{\mathbf{u}}_0, \tilde{g}_0) - (\mathbf{u}_1(t_r), g_1(t_r))\|_{\mathbf{H}^k \times H^k} < \delta_r.$$

Let us denote by $\hat{\Psi} : \mathbf{K} \rightarrow L^2(J_T, \mathbf{E})$ the continuous operator defined by the relations

$$\hat{\Psi}(\hat{\mathbf{u}}, \hat{g})(t) = \Psi_{\delta_{r+1}}^r(\hat{\mathbf{u}}, \hat{g})(t) \text{ for } t \in I_r,$$

where $\delta_{s+1} = \varepsilon$. Then

$$\sup_{(\hat{\mathbf{u}}, \hat{g}) \in \mathbf{K}} \|\mathcal{R}_T(\tilde{\mathbf{u}}_0, \tilde{g}_0, 0, 0, \hat{\Psi}(\hat{\mathbf{u}}, \hat{g})) - (\mathbf{u}_1(T), g_1(T))\|_{\mathbf{H}^k \times H^k} < \varepsilon.$$

To complete the proof, it suffices to approximate $\hat{\Psi}$ in $L^2(J_T, \mathbf{H}^k)$ by a continuous mapping $\Psi : \mathbf{K} \rightarrow \Theta(\mathbf{u}_0, g_0) \cap C^\infty(J_T, \mathbf{E})$.

Step 3. We now assume that $s = 1$. Then (3.4.18) takes the form

$$\dot{\mathbf{u}}_1 + \sum_{i=1}^{2q} d_i(\hat{\mathbf{u}}, \hat{g})((\mathbf{u}_1 + \boldsymbol{\zeta}^i) \cdot \nabla)(\mathbf{u}_1 + \boldsymbol{\zeta}^i) + h(g_1) \nabla g_1 = \mathbf{f}(t) + \boldsymbol{\eta}(\hat{\mathbf{u}}, \hat{g}), \quad (3.4.20)$$

where $d_i \in C(\mathbf{K})$ and $\boldsymbol{\eta} \in C(\mathbf{K}, \mathbf{E})$. For any $n \in \mathbb{N}$, let $\zeta_n(t, \hat{\mathbf{u}}, \hat{g}) = \zeta(\frac{nt}{T}, \hat{\mathbf{u}}, \hat{g})$, where $\zeta(t, \hat{\mathbf{u}}, \hat{g})$ is a 1-periodic function such that

$$\zeta(s, \hat{\mathbf{u}}, \hat{g}) = \zeta^j \text{ for } 0 \leq s - (d_1(\hat{\mathbf{u}}, \hat{g}) + \dots + d_{j-1}(\hat{\mathbf{u}}, \hat{g})) < d_j(\hat{\mathbf{u}}, \hat{g}), \quad j = 1, \dots, q.$$

Note that $\zeta(t, \hat{\mathbf{u}}, \hat{g}) = -\zeta(t - \frac{1}{2}, \hat{\mathbf{u}}, \hat{g})$ for $t \in (\frac{1}{2}, 1)$. Eq. (3.4.20) is equivalent to

$$\begin{aligned} \dot{\mathbf{u}}_1 + ((\mathbf{u}_1 + \zeta_n(t, \hat{\mathbf{u}}, \hat{g})) \cdot \nabla)(\mathbf{u}_1 + \zeta_n(t, \hat{\mathbf{u}}, \hat{g})) + h(g_1)\nabla g_1 \\ = \mathbf{f} + \boldsymbol{\eta}(t, \hat{\mathbf{u}}, \hat{g}) + \mathbf{f}_n(t, \hat{\mathbf{u}}, \hat{g}), \end{aligned}$$

where

$$\begin{aligned} \mathbf{f}_n(t, \hat{\mathbf{u}}, \hat{g}) &= ((\mathbf{u}_1 + \zeta_n(t, \hat{\mathbf{u}}, \hat{g})) \cdot \nabla)(\mathbf{u}_1 + \zeta_n(t, \hat{\mathbf{u}}, \hat{g})) \\ &\quad - \sum_{i=1}^{2q} d_i(\hat{\mathbf{u}}, \hat{g})((\mathbf{u}_1 + \zeta^i) \cdot \nabla)(\mathbf{u}_1 + \zeta^i). \end{aligned}$$

Let us define

$$\mathcal{K}\mathbf{f}_n(t) = \int_0^t \mathbf{f}_n(s) ds.$$

Then $\mathbf{v}_n = \mathbf{u}_1 - \mathcal{K}\mathbf{f}_n$ is a solution of the problem

$$\begin{aligned} \dot{\mathbf{v}}_n + ((\mathbf{v}_n + \zeta_n(t, \hat{\mathbf{u}}, \hat{g}) + \mathcal{K}\mathbf{f}_n(t, \hat{\mathbf{u}}, \hat{g})) \cdot \nabla)(\mathbf{v}_n + \zeta_n(t, \hat{\mathbf{u}}, \hat{g}) + \mathcal{K}\mathbf{f}_n(t, \hat{\mathbf{u}}, \hat{g})) \\ + h(g_1)\nabla g_1 = \mathbf{f}(t) + \boldsymbol{\eta}(t, \hat{\mathbf{u}}, \hat{g}), \\ (\partial_t + (\mathbf{v}_n + \mathcal{K}\mathbf{f}_n(t, \hat{\mathbf{u}}, \hat{g})) \cdot \nabla)g_1 + \nabla \cdot (\mathbf{v}_n + \mathcal{K}\mathbf{f}_n(t, \hat{\mathbf{u}}, \hat{g})) = 0, \\ \mathbf{v}_n = \mathbf{u}_0. \end{aligned}$$

It is straightforward to see that

$$\sup_{(\hat{\mathbf{u}}, \hat{g}) \in \mathbf{K}} \|\mathcal{K}\mathbf{f}_n(t, \hat{\mathbf{u}}, \hat{g})\|_{C(J_T, \mathbf{H}^{k+1})} \rightarrow 0,$$

(e.g. see [28, Chapter 3] or [36, Section 6]). Thus

$$\|\mathbf{v}_n - \mathbf{u}_1\|_{C(J_T, \mathbf{H}^{k+1})} \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (3.4.21)$$

On the other hand, Theorem 3.2.2 implies that

$$\|(\mathbf{v}_n, g_1) - (\tilde{\mathbf{u}}_n, \tilde{g}_n)\|_{\mathbf{Y}^k} \rightarrow 0 \text{ as } n \rightarrow \infty, \quad (3.4.22)$$

where $(\tilde{\mathbf{u}}_n, \tilde{g}_n)$ satisfies the problem

$$\begin{aligned} \partial_t \tilde{\mathbf{u}}_n + ((\tilde{\mathbf{u}}_n + \zeta_n(t, \hat{\mathbf{u}}, \hat{g})) \cdot \nabla)(\tilde{\mathbf{u}}_n + \zeta_n(t, \hat{\mathbf{u}}, \hat{g})) + h(\tilde{g}_n)\nabla \tilde{g}_n = \mathbf{f}(t) + \boldsymbol{\eta}(t, \hat{\mathbf{u}}, \hat{g}), \\ (\partial_t + \tilde{\mathbf{u}}_n \cdot \nabla)\tilde{g}_n + \nabla \cdot \tilde{\mathbf{u}}_n = 0, \\ \tilde{\mathbf{u}}_n(0) = \tilde{\mathbf{u}}_0, \quad \tilde{g}_n(0) = \tilde{g}_0. \end{aligned}$$

We want to apply Theorem 3.2.3 to the above system. To this end, let $\chi_n : J_T \rightarrow \mathbf{H}^k$ be a uniformly equicontinuous sequence and let $t_0 \in J_T$. Then

$$\begin{aligned} \int_0^{t_0} \zeta_n(t) \cdot \chi_n(t) dt &= \int_0^{t_0} \zeta\left(\frac{nt}{T}\right) \cdot \chi_n(t) dt = \int_0^{\frac{nt_0}{T}} \zeta(t) \cdot \chi_n\left(\frac{tT}{n}\right) \frac{T}{n} dt \\ &= \sum_{i=0}^{\lceil \frac{nt_0}{T} \rceil - 1} \int_i^{i+1} \zeta(t) \cdot \chi_n\left(\frac{tT}{n}\right) \frac{T}{n} dt + \int_{\lceil \frac{nt_0}{T} \rceil}^{\frac{nt_0}{T}} \zeta(t) \cdot \chi_n\left(\frac{tT}{n}\right) \frac{T}{n} dt. \end{aligned} \quad (3.4.23)$$

Using the construction of $\zeta(t)$, we get

$$\begin{aligned} \int_i^{i+1} \zeta(t) \cdot \chi_n\left(\frac{tT}{n}\right) dt &= \int_i^{i+\frac{1}{2}} \zeta(t) \cdot \chi_n\left(\frac{tT}{n}\right) dt + \int_{i+\frac{1}{2}}^{i+1} -\zeta\left(t - \frac{1}{2}\right) \cdot \chi_n\left(\frac{tT}{n}\right) dt \\ &= \int_i^{i+\frac{1}{2}} \zeta(t) \cdot \left(\chi_n\left(\frac{tT}{n}\right) - \chi_n\left(\frac{tT}{n} + \frac{T}{2n}\right)\right) dt. \end{aligned}$$

As χ_n is uniformly equicontinuous and ζ is bounded, we have

$$\sup_{t \in [0, n]} \|\zeta(t) \cdot \left(\chi_n\left(\frac{tT}{n}\right) - \chi_n\left(\frac{tT}{n} + \frac{T}{2n}\right)\right)\|_k \rightarrow 0, \quad n \rightarrow \infty.$$

The boundedness of $\zeta \cdot \chi_n$ implies that the second term of the right-hand side of (3.4.23) goes to zero. Thus

$$\int_0^{t_0} \zeta_n(t) \cdot \chi_n(t) dt \rightarrow 0 \text{ in } \mathbf{H}^k.$$

Using Theorem 3.2.3 and limits (3.4.21), (3.4.22), we get

$$\sup_{(\hat{u}, \hat{g}) \in \mathbf{K}} \|\mathcal{R}_T(\mathbf{u}_0, g_0, \zeta_n, \zeta_n, \boldsymbol{\eta}(\hat{u}, \hat{g})) - (\mathbf{u}_1(T, \hat{u}, \hat{g}), g_1(T, \hat{u}, \hat{g}))\|_{\mathbf{H}^k \times H^k} < \varepsilon$$

for sufficiently large n . Let us take some functions $\zeta_n^m \in C^\infty(J_T, E)$ such that $\zeta_n^m(0) = \zeta_n^m(T) = 0$ and

$$\|\zeta_n^m - \zeta_n\|_{L^2(J_T, \mathbf{H}^{k+1})} \rightarrow 0 \text{ as } m \rightarrow \infty. \quad (3.4.24)$$

Then Theorem 3.2.2 implies

$$\sup_{(\hat{u}, \hat{g}) \in \mathbf{K}} \|\mathcal{R}_T(\mathbf{u}_0, g_0, \zeta_n, \zeta_n, \boldsymbol{\eta}(\hat{u}, \hat{g})) - \mathcal{R}_T(\mathbf{u}_0, g_0, \zeta_n^m, \zeta_n^m, \boldsymbol{\eta}(\hat{u}, \hat{g}))\|_{\mathbf{H}^k \times H^k} < \varepsilon.$$

For $m \gg 1$, the operator

$$\Psi : \mathbf{K} \rightarrow L^2(J_T, \mathbf{E}), \quad (\hat{u}, \hat{g}) \rightarrow \boldsymbol{\eta}(\hat{u}, \hat{g}) - \partial_t \zeta_n^m$$

satisfies

$$\sup_{(\hat{u}, \hat{g}) \in \mathbf{K}} \|\mathcal{R}_T(\mathbf{u}_0, g_0, 0, 0, \boldsymbol{\Psi}_1(\hat{u}, \hat{g})) - \mathcal{R}_T(\mathbf{u}_0, g_0, 0, 0, \boldsymbol{\Psi}(\hat{u}, \hat{g}))\|_{\mathbf{H}^k \times H^k} < \varepsilon,$$

which completes the proof. \square

CHAPITRE 4

Stabilisation de l'équation d'Euler 2D incompressible dans un cylindre infini

Stabilization of the 2D incompressible Euler system in an infinite strip

Abstract. The paper is devoted to the study of a stabilization problem for the 2D incompressible Euler system in an infinite strip with boundary controls. We show that for any stationary solution $(c, 0)$ of the Euler system there is a control which is supported in a given bounded part of the boundary of the strip and stabilizes the system to $(c, 0)$.

4.1 Introduction

We consider the incompressible two-dimensional Euler system

$$\dot{u} + \langle u, \nabla \rangle u + \nabla p = 0, \quad \operatorname{div} u = 0, \quad (4.1.1)$$

where $u = (u_1, u_2)$ and p are unknown velocity field and pressure of the fluid, and

$$\langle u, \nabla \rangle v = \sum_{i=1}^3 u_i(t, x) \frac{\partial}{\partial x_i} v.$$

The space variable $x = (x_1, x_2)$ belongs to the strip D defined by

$$D := \{(x_1, x_2) : x_1 \in \mathbb{R}, x_2 \in (-1, 1)\}. \quad (4.1.2)$$

Let us take two open intervals $(a, b), (a + d, b + d) \subset \mathbb{R}$ and denote

$$\Gamma_0 = (a, b) \times \{1\} \cup (a + d, b + d) \times \{-1\}. \quad (4.1.3)$$

The aim of this paper is the study of stabilization of (4.1.1) with boundary controls supported by Γ_0 . System (4.1.1) is completed with the boundary and initial conditions

$$u \cdot n = 0 \text{ on } \Gamma \setminus \Gamma_0, \quad (4.1.4)$$

$$u(x, 0) = u_0(x), \quad (4.1.5)$$

where $\Gamma := \partial D$ and n is the outward unit normal vector on Γ . In particular, (4.1.4) is equivalent to $u_2 = 0$ on $\Gamma \setminus \Gamma_0$.

For any integer $s \geq 0$ we denote by $H^s(D)$ the space of vector functions $u = (u_1, u_2)$ whose components belong to the Sobolev space of order s and by $\|\cdot\|_{s,D}$ the corresponding norm. If there is no confusion, we drop the index D . In the case $s = 0$, we write $\|\cdot\| := \|\cdot\|_0$. For any integer $s > 0$ we define $\mathcal{H}^s(D)$ as the space of distributions u in D with $\nabla u \in H^{s-1}(D)$. We equip $\mathcal{H}^s(D)$ with the semi-norm

$$\|u\|_{\mathcal{H}^s(D)} := \|\nabla u\|_{s-1}.$$

We denote by $\dot{H}^s(D)$ the quotient space $\mathcal{H}^s(D)/\mathbb{R}$. The following theorem is our main result.

Main result. *Let $s \geq 4$ be an integer. Then for any constant $c \in \mathbb{R}$ and initial function $u_0 \in H^s(D)$ that decays fast at infinity and satisfies the relations*

$$\operatorname{div} u_0 = 0, \quad u_0 \cdot n = 0 \text{ on } \Gamma \setminus \Gamma_0$$

there exists a solution $(u, p) \in C(\mathbb{R}_+, C(\overline{D}) \cap \dot{H}^s(D)) \times C(\mathbb{R}_+, \dot{H}^s(D))$ of (4.1.1), (4.1.4) and (4.1.5) such that

$$\lim_{t \rightarrow \infty} (\|u(\cdot, t) - (c, 0)\|_{L^\infty} + \|\nabla u\|_{s-1} + \|\nabla p\|_{s-1}) = 0.$$

For the exact statement see Theorem 4.3.1. In this formulation the control is not given explicitly, but we can assume that control acts on the system as a boundary condition on Γ_0 .

Before turning to the ideas of the proof, let us describe in a few words some previous results on the controllability of Euler and Navier–Stokes systems. Coron [12] introduced the return method to study a stabilization problem for ODE’s, then using this method he proved in [13] the exact boundary controllability of 2D incompressible Euler system in a bounded domain. Glass [23] generalized this result for 3D Euler system. Chapouly [10] using return method proved the global null controllability of the Navier–Stokes system in rectangle. Recently, Glass and Rosier [26] proved the controllability of the motion of a rigid body, which is surrounded by an incompressible fluid. The asymptotic stabilization of 2D Euler equation by stationary feedback boundary controls is studied by Coron [15] and Glass [25].

Controllability of Euler and Navier–Stokes systems with distributed controls is studied in [3, 22, 36, 39]; see also the book [16] for further references.

Notice that the above papers concern the problem of controllability of the fluid in a bounded domain. In this paper, we develop Coron’s return method to get the controllability of velocity of 2D Euler system in an unbounded strip. This method consists in reducing the controllability of nonlinear system to the linear one. To this end, one constructs a particular solution (\bar{u}, \bar{p}) of Euler system and a sequence of balls $\{B_i\}$ covering \overline{D} , such that

(P) Any ball B_i driven by the flow of \bar{u} leaves \overline{D} through Γ_0 at some time.

Then the linearized system around \bar{u} is controllable. In our case, since the domain D is unbounded, the number of balls B_i is infinite, thus we cannot construct a bounded function \bar{u} , whose flow moves all balls outside D in a finite time. However, we can find a particular solution \bar{u} such that property (P) holds in infinite time. This proves the stabilization of linearized system in infinite time.

To show that controllability of linearized system implies that of the nonlinear system, we need to prove that (P) also holds for any \tilde{u} sufficiently close to \bar{u} . This is obvious in the case of bounded domain. In our case, to prove this, we need some additional properties for \bar{u} . In particular, we need to construct a solution \bar{u} , which decays at infinity faster than $1/x_1^2$. As our particular solution \bar{u} is a combination of the Green functions of the Laplacian with Neumann boundary condition, we need to prove that Green functions decay at infinity. This property is a consequence of

elliptic regularity and some explicit formulas for solutions of the Laplace equation in a strip.

The paper is organized as follows. In Section 4.2, we give preliminaries on Poisson and Euler equations in an unbounded strip. The main results of the paper are presented in Section 4.3. In Section 4.4, we construct the particular solution \bar{u} . In the Appendix, we prove an auxiliary result used in Section 4.2.

Acknowledgments. The author would like to express deep gratitude to Armen Shirikyan for drawing his attention to this problem and for many fruitful suggestions and also to Nikolay Tzvetkov for useful remarks on the Euler system.

Notation.

Let $J_T := [0, T)$. The space of continuous functions $u : J_T \rightarrow X$ is denoted by $C(J_T, X)$. For any integer $s \geq 0$ or $s = \infty$, we denote

$$C_b^s(D) = \{u \in C^s(D) : \|u\|_{L^\infty(D)} < \infty\}.$$

We set $\dot{H}^\infty(D) := \cap_{s=0}^\infty \dot{H}^s(D)$. Define

$$\mathcal{S}(D) := \{u \in L^2(D) : |x_1|^\alpha \partial^\beta u(x_1, x_2) \in L^2(D) \text{ for any } \alpha \in \mathbb{R}_+, \beta \in \mathbb{Z}_+^2\}.$$

For a vector field $u = (u_1, u_2)$ we set

$$\text{curl } u = \partial_1 u_2 - \partial_2 u_1.$$

The interior of a set K is denoted by $\text{int}(K)$. Let $B(x_0, r)$ be the closed ball in \mathbb{R}^2 of radius r centred at x_0 . We denote by C a universal constant whose value may change from line to line.

4.2 Preliminaries

In this section, we present some auxiliary results on Poisson and Euler equations in an unbounded strip. The methods used in their proofs are well known and in many cases we confine ourselves to a brief description of the main ideas.

4.2.1 Poisson equations in an unbounded strip

First, let us describe the spaces $\dot{H}^s(D)$.

Proposition 4.2.1. *For any integer $s \geq 1$ we have*

- (i) *The space $\dot{H}^s(D)$ is complete.*
- (ii) $\mathcal{H}^s(D) = \{u \in H_{loc}^s(D) : \nabla u \in H^{s-1}\}$.
- (iii) *If $s \geq 3$, then for any $u \in \mathcal{H}^s(D)$ there is a constant C depending on u such that*

$$|u(x_1, x_2)| \leq C|x_1| + C$$

holds for all $x \in D$.

Démonstration. Let $\{u_n\} \subset \dot{H}^s(D)$ be a Cauchy sequence. Then there is $v \in H^{s-1}(D)$ such that $\nabla u_n \rightarrow v$ in $H^{s-1}(D)$ as $n \rightarrow \infty$, and for any $\varphi \in C_0^\infty(D)$ such that $\operatorname{div} \varphi = 0$, we have

$$0 = \lim_{n \rightarrow \infty} (\nabla u_n, \varphi)_{L^2} = (v, \varphi)_{L^2}.$$

Hence, $v = \nabla z$, where $z \in \dot{H}^s(D)$. This proves that $\dot{H}^s(D)$ is complete. Now let us prove assertion (ii). Clearly the space in the right-hand side is contained in $\dot{H}^s(D)$. Let us take a function $u \in \dot{H}^s(D)$, a compact set $K \subset D$ and let us show that $u \in H^s(K)$. Take two functions $\chi, \chi_1 \in C_0^\infty(D)$ and a compact set $K_1 \subset D$ with $\operatorname{int}(K_1) \supset K$ such that $\chi = 1$ in K_1 and $\chi_1 = 1$ in $\tilde{K}_1 := \operatorname{supp} \chi$. Then there exists $r \in \mathbb{N}$ such that $\chi_1 u \in H^{-r}(D)$. This implies that $u \in H^{-r}(\tilde{K}_1)$, hence

$$\Delta(\chi u) = 2\nabla\chi\nabla u + \chi\Delta u + u\Delta\chi \in H^{\min(-r; s-2)}(\tilde{K}_1).$$

The elliptic regularity implies $\chi u \in H^{\min(-r+2; s)}(D)$, thus $u \in H^{\min(-r+2; s)}(K_1)$. Repeating this argument for a compact set $K_2 \subset K_1$ with $\operatorname{int}(K_2) \supset K$ we can show that $u \in H^{\min(-r+4; s)}(K_2)$. Iterating this, we get $u \in H^s(K)$. This completes the proof of assertion (ii).

It is easy to see that (ii) implies (iii). Indeed, from (ii) we get

$$u(x_1, x_2) = \int_0^{x_1} \partial_1 u(y, x_2) dy + u(0, x_2).$$

The Sobolev inequality yields (iii). □

Now we summarize some facts about Poisson equation. Let us take a non-negative function $\gamma \in C_0^\infty(\mathbb{R})$ such that $\operatorname{supp} \gamma = [a, b]$ and $\gamma \neq 0$ in (a, b) and define

$$\tilde{D} := \{(x_1, x_2) : x_1 \in \mathbb{R}, x_2 \in (-1 - \gamma(x_1 - d), 1 + \gamma(x_1))\} \tag{4.2.1}$$

(see figure 1).

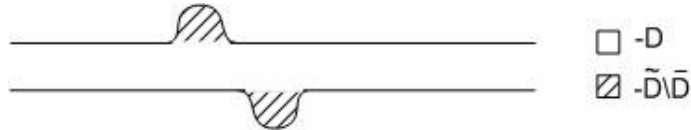


FIGURE 4.1 – Domain \tilde{D}

Let us take $D' = D$ or $D' = \tilde{D}$ and consider the Dirichlet problem for the Poisson equation :

$$\Delta u = f \quad \text{in } D', \tag{4.2.2}$$

$$u = 0 \quad \text{on } \Gamma', \tag{4.2.3}$$

where $\Gamma' = \partial D'$ and $f \in L^2(D')$. We say that $u \in H_0^1(D')$ is a solution of (4.2.2), (4.2.3) if

$$\int_{D'} \nabla u \nabla \theta dx = - \int_{D'} f \theta dx$$

for any $\theta \in H_0^1(D')$. We have the following result for the well-posedness of this problem.

Proposition 4.2.2. *For any integer $s \geq 0$ and for any $f \in H^s(D')$ problem (4.2.2), (4.2.3) has a unique solution $u \in H^{s+2}(D')$. Moreover,*

$$\|u\|_{s+2} \leq C \|f\|_s, \tag{4.2.4}$$

where C depends only on s .

Démonstration. The existence of the solution $u \in H_0^1(D')$ is a consequence of the Riesz representation theorem. Clearly, we have

$$\|\nabla u\|^2 \leq C \|f\| \|u\|. \tag{4.2.5}$$

The Poincaré inequality applied to $u(x_1, \cdot)$ gives

$$\|u\| \leq C \|\partial_2 u\|.$$

Combining this with (4.2.5), we obtain

$$\|u\|_1 \leq C \|f\|. \tag{4.2.6}$$

To show the regularity of the solution and estimate (4.2.4), we need the following lemma.

Lemma 4.2.3. *For any integer $s \geq 1$ we have*

$$H^s(D') = \{z \in L^2(D') : \text{curl } z \in H^{s-1}(D'), \text{div } z \in H^{s-1}(D'), z \cdot n \in H^{s-1/2}(\Gamma')\},$$

where n is the outward unit normal vector on Γ' . Moreover, any function $z \in H^s(D')$ satisfies the inequality

$$\|z\|_s \leq C (\|z\| + \|\text{curl } z\|_{s-1} + \|\text{div } z\|_{s-1} + \|z \cdot n\|_{s-1/2}),$$

where C depends only on s .

The proof of this lemma is given in the Appendix. Let us denote $z = \nabla^\perp u := (\partial_2 u, -\partial_1 u)$. Then $\text{curl } z = -\Delta u = -f$, $\text{div } z = 0$. Notice that (4.2.3) implies that $z \cdot n = 0$. It follows from Lemma 4.2.3 and inequality (4.2.6) that $z \in H^{s+1}(D')$ and $\|z\|_{s+1} \leq C \|f\|_s$. Thus, we obtain $u \in H^{s+2}(D')$ and (4.2.4). \square

Let us take $g \in H^1(D')$ and consider the Neumann problem for the Poisson equation :

$$\Delta u = \operatorname{div} g \quad \text{in } D', \quad (4.2.7)$$

$$\frac{\partial u}{\partial n} = g \cdot n \quad \text{on } \Gamma'. \quad (4.2.8)$$

We say that $u \in \dot{H}^1(D')$ is a solution of (4.2.7), (4.2.8) if for any $\theta \in H^1(D')$ we have

$$\int_{D'} \nabla u \nabla \theta \, dx = \int_{D'} g \nabla \theta \, dx.$$

Proposition 4.2.4. *For any integer $s \geq 1$ and $g \in H^s(D')$ problem (4.2.7), (4.2.8) has a unique solution $u \in \dot{H}^{s+1}(D')$. Moreover,*

$$\|u\|_{\dot{H}^{s+1}} \leq C \|g\|_s. \quad (4.2.9)$$

Démonstration. The Riesz representation theorem implies the existence of the solution $u \in \dot{H}^1(D')$. Lemma 4.2.3 applied to $z := \nabla u$ gives (4.2.9). \square

Now we consider the problem

$$\Delta G_a = \partial_1 \delta_a \quad \text{in } \tilde{D}, \quad (4.2.10)$$

$$\frac{\partial G_a}{\partial n} = 0 \quad \text{on } \partial \tilde{D}, \quad (4.2.11)$$

where δ_a is the Dirac delta function concentrated at $a = (a_1, a_2) \in \tilde{D}$.

Proposition 4.2.5. *Problem (4.2.10), (4.2.11) has a solution $G_a \in C^\infty(\overline{\tilde{D}} \setminus \{a\})$. Moreover, the following assertions hold :*

- (i) *For any open neighbourhood Q of a and for any integer $s \geq 1$, the solution G_a is uniquely determined by the additional condition that it belongs to $\dot{H}^s(\tilde{D} \setminus \overline{Q})$.*
- (ii) *For any $x \in \tilde{D} \setminus \{a\}$*

$$\nabla G_a(x) = -\frac{1}{2\pi} \left(\frac{|x-a|^2 - 2(x_1 - a_1)^2}{|x-a|^4}, \frac{-2(x_1 - a_1)(x_2 - a_2)}{|x-a|^4} \right) + \psi_a(x), \quad (4.2.12)$$

where $\psi_a \in H^\infty(\tilde{D})$.

- (iii) *Let $a \in \tilde{D} \setminus \overline{D}$, then $G_a \in \dot{H}^\infty(D)$ and for any integers $1 \leq i, j \leq 2$ we have*

$$\partial_i \partial_j G_a(x_1, x_2) \in \mathcal{S}(D). \quad (4.2.13)$$

- (iv) *For any fixed $x \in \tilde{D}$ the function $G_a(x)$ is analytic in $a \in \tilde{D} \setminus \{x\}$.*

Démonstration. The existence of a solution $G_a \in C^\infty(\overline{\tilde{D}} \setminus \{a\})$ will be established when proving assertion (ii). To prove the uniqueness of the solution, we assume that there are two solutions $G_{1,a}$ and $G_{2,a}$. For $\tilde{G} = G_{1,a} - G_{2,a}$ we have

$$\begin{aligned} \Delta \tilde{G} &= 0 \quad \text{in } \tilde{D}, \\ \frac{\partial \tilde{G}}{\partial n} &= 0 \quad \text{on } \partial \tilde{D}, \end{aligned}$$

Let $\chi \in C_0^\infty(\tilde{D})$ with $\chi = 1$ in Q . Then

$$\Delta(\chi \tilde{G}) = h,$$

where $h \in C_0^\infty(\tilde{D})$. The elliptic regularity for a bounded domain implies that $\chi \tilde{G} \in H^\infty(\tilde{D})$. Since $\tilde{G} \in \dot{H}^s(\tilde{D} \setminus \overline{Q})$, we get $\tilde{G} \in \dot{H}^s(\tilde{D})$. It follows from Proposition 4.2.4 that $\tilde{G} = 0$.

To prove (ii), we seek the solution in the form

$$G_a = \partial_1(F_a \chi) + u_a, \tag{4.2.14}$$

where $F_a(x) = -\frac{1}{2\pi} \ln|x-a|$ is the fundamental solution of the Laplace operator in \mathbb{R}^2 , $\chi \in C_0^\infty(\tilde{D})$, χ is 1 in a neighborhood of a . Then u_a must be the solution of the problem

$$\begin{aligned} \Delta u_a &= -\partial_1(2\nabla F_a \cdot \nabla \chi + F_a \Delta \chi) := \partial_1 f \quad \text{in } \tilde{D}, \\ \frac{\partial u_a}{\partial n} &= 0 \quad \text{on } \partial \tilde{D}. \end{aligned}$$

Since $f \in C_0^\infty(\tilde{D})$, applying Proposition 4.2.4 for $g = (f, 0)$, we conclude that this problem has a solution $u_a \in H^\infty(\tilde{D})$. Property (4.2.12) follows from the construction of G_a .

Now let us show (4.2.13). We have that G_a satisfies the following problem in D :

$$\Delta G_a = 0 \quad \text{in } D, \tag{4.2.15}$$

$$\frac{\partial G_a}{\partial n} = \varphi \quad \text{on } \Gamma, \tag{4.2.16}$$

where $\varphi \in C^\infty(\Gamma)$ and $\text{supp } \varphi \subset \overline{\Gamma}_0$. To show that the second derivatives of the solution belong to $\mathcal{S}(D)$, let us apply the Fourier transform in x_1 to (4.2.15), (4.2.16). We obtain

$$\begin{aligned} \frac{d^2}{dx_2^2} \hat{G}_a - \xi^2 \hat{G}_a &= 0 \quad \text{in } D, \\ \frac{d\hat{G}_a}{dx_2}(\xi, -1) &= \hat{\varphi}_1(\xi), \\ \frac{d\hat{G}_a}{dx_2}(\xi, 1) &= \hat{\varphi}_2(\xi), \end{aligned}$$

where \hat{G}_a , $\hat{\varphi}_1$ and $\hat{\varphi}_2$ are Fourier transforms of G_a , $\varphi(\cdot, -1)$ and $\varphi(\cdot, 1)$, respectively. The solution of this ODE is given by

$$\hat{G}_a(\xi, x_2) = \frac{\hat{\varphi}_2 - \hat{\varphi}_1}{2\xi \sinh(\xi)} \cosh(\xi x_2) + \frac{\hat{\varphi}_2 + \hat{\varphi}_1}{2\xi \cosh(\xi)} \sinh(\xi x_2).$$

Since φ_1 and φ_2 are compactly supported, we have

$$\mathcal{F}(\partial_i \partial_j G_a) \in \mathcal{S}(D), \quad 1 \leq i, j \leq 2,$$

whence it follows that $\partial_i \partial_j G_a \in \mathcal{S}(D)$. This completes the proof of (iii).

Let Ω be any domain such that $\bar{\Omega} \subset \tilde{D}$ and $\Omega \cap (\tilde{D} \setminus \bar{D}) \neq \emptyset$. Then for any fixed $x \in \Omega$ the function $G_a(x)$ is analytic in $a \in \Omega \setminus \{x\}$. Indeed, let χ in (4.2.14) be 1 in Ω . Then the analyticity of $G_a(x)$ is consequence of the facts that F_a is analytic in a and u_a is a linear operator in F_a . Since G_a is the unique solution of (4.2.10), (4.2.11), we have the analyticity of $G_a(x)$ in $\tilde{D} \setminus \{x\}$. \square

4.2.2 Euler equations in an unbounded strip

We consider the incompressible Euler system :

$$\dot{u} + \langle u, \nabla \rangle u + \nabla p = 0, \quad \operatorname{div} u = 0 \quad \text{in } D, \quad (4.2.17)$$

$$u \cdot n = 0 \quad \text{on } \Gamma, \quad (4.2.18)$$

$$u(x, 0) = u_0(x), \quad (4.2.19)$$

It is well known that if D is a bounded domain or if $D = \mathbb{R}^2$, then problem (4.2.17)-(4.2.19) is well posed in various function spaces (e.g., see [29, 31, 46]).

In this subsection, we study the well-posedness of Euler system in D defined by (4.1.2).

Définition 7. *For any integer $s \geq 3$ we say that (u, p) is a solution of Euler system if $(u, p) \in C(J_T, H^s(D)) \times C(J_T, \dot{H}^{s+1}(D))$ and (4.2.17) is satisfied in the sense of distributions.*

Let us show that the Euler system is equivalent to the problem

$$\dot{w} + \langle u, \nabla \rangle w = 0, \quad w(x, 0) = \operatorname{curl} u_0(x), \quad (4.2.20)$$

$$\operatorname{curl} u = w, \quad \operatorname{div} u = 0, \quad u \cdot n|_{\Gamma} = 0. \quad (4.2.21)$$

Clearly, if (u, p) is a solution of the Euler system, then (4.2.20), (4.2.21) hold. Now let us show that to any solution

$$(u, w) \in C(J_T, H^s(D)) \cap C^1(J_T, H^{s-1}(D)) \times C(J_T, H^{s-1}(D))$$

of (4.2.20), (4.2.21) there corresponds a unique solution $(u, p) \in C(J_T, H^s(D)) \times C(J_T, \dot{H}^{s+1}(D))$ of (4.2.17)-(4.2.19). From (4.2.20) and (4.2.21) it follows that

$$\operatorname{curl}(\dot{u} + \langle u, \nabla \rangle u) = 0.$$

Hence, there exists $p \in C(J_T, \dot{H}^s(D))$ such that $-\nabla p = \dot{u} + \langle u, \nabla \rangle u$. It is easy to see that

$$\begin{aligned} -\operatorname{div} \nabla p &= \operatorname{div}(\langle u, \nabla \rangle u) = \sum_{i,j=1}^2 \partial_i u_j \partial_j u_i \in H^{s-1}, \quad \operatorname{curl} \nabla p = 0, \\ -\frac{\partial p}{\partial n} &= (\langle u, \nabla \rangle u) \cdot n = \langle u, \nabla \rangle (u \cdot \tilde{n}) - \sum_{i,j=1}^2 u_j u_i \partial_j \tilde{n}_i \\ &= - \sum_{i,j=1}^2 u_j u_i \partial_j \tilde{n}_i \in H^{s-1/2}, \end{aligned}$$

where \tilde{n} is a regular extension of n . Thus, it follows from Lemma 4.2.3 that $\nabla p \in C(J_T, H^s(D))$, whence we conclude that $p \in C(J_T, \dot{H}^{s+1}(D))$.

We have the following result on the local well-posedness of Euler system. The ideas used in the proof of existence of a solution play an important role in the study of stabilization problem (see Section 4.3). Therefore we present a rather complete proof, even though we do not really need this result.

Theorem 4.2.6. *Let $s \geq 4$. For any $u_0 \in H^s(D)$ satisfying the conditions*

$$\begin{aligned} \operatorname{div} u_0 &= 0, \\ u_0 \cdot n &= 0 \text{ on } \Gamma, \end{aligned}$$

there is $T_ = T_*(\|u_0\|_s)$ such that system (4.2.17)-(4.2.19) has a unique solution $(u, p) \in C(J_{T_*}, H^s(D)) \times C(J_{T_*}, \dot{H}^{s+1}(D))$.*

Démonstration. Uniqueness. To prove the uniqueness, we argue as in the case of bounded domain. We assume that there are two solutions u_1 and u_2 . Then for $v = u_1 - u_2$, we have

$$\begin{aligned} \dot{v} + \langle u_1, \nabla \rangle v + \langle v, \nabla \rangle u_2 + \nabla p &= 0, \\ \operatorname{div} v &= 0, \quad v \cdot n|_{\Gamma} = 0, \quad v(x, 0) = 0. \end{aligned} \tag{4.2.22}$$

Multiplying (4.2.22) by v and integrating over D , we get

$$\partial_t \|v(\cdot, t)\|^2 \leq - \int_D \langle u_1, \nabla \rangle v \cdot v dx + C \|v(\cdot, t)\|^2 - \int_D \nabla p \cdot v dx, \tag{4.2.23}$$

where $C > 0$ is a constant depending only on u_2 . Since $u_1 \cdot n = 0$, the first term on the right-hand side of (4.2.23) is zero. Let us show that the last term is also zero. Let us denote

$$\Omega_{(R)} := \{x \in D : |x_1| < R\},$$

and let $\chi \in C^\infty(\bar{D})$ be such that

$$\chi(x) = \begin{cases} 0, & \text{if } x \notin \bar{\Omega}_{(2)}, \\ 1, & \text{if } x \in \Omega_{(1)}. \end{cases}$$

Clearly, we have

$$\lim_{R \rightarrow \infty} \int_D \chi\left(\frac{x}{R}\right) \nabla p(x) \cdot v(x) dx = \int_D \nabla p(x) \cdot v(x) dx.$$

On the other hand, integrating by parts, we obtain

$$\int_D \chi\left(\frac{x}{R}\right) \nabla p(x) \cdot v(x) dx = - \int_{\Omega_{(2R)} \setminus \Omega_{(R)}} \nabla \chi\left(\frac{x}{R}\right) \frac{p(x)}{R} \cdot v(x) dx.$$

Since $p \in \dot{H}^{s+1}$, from assertion (iii) of Proposition 4.2.1 we have

$$\sup_{x \in \Omega_{(2R)}} \left| \frac{p(x)}{R} \right| < C,$$

where C does not depend on R . Thus, dominated convergence theorem yields

$$\int_D \nabla p(x) \cdot v(x) dx = 0.$$

Applying the Gronwall inequality to (4.2.23), we obtain $v = 0$.

Existence. To prove the existence of the solution, we shall need the following result.

Lemma 4.2.7. *Let $\tilde{u} \in C(\mathbb{R}_+, H^s)$, $\tilde{u} \cdot n|_{\Gamma \times \mathbb{R}_+} = 0$, $f \in C(\mathbb{R}_+, H^s)$ and $w_0 \in H^s$, $s \geq 3$. Then the problem*

$$\partial_t w + \langle \tilde{u}, \nabla \rangle w = f, \quad (4.2.24)$$

$$w(x, 0) = w_0, \quad (4.2.25)$$

has a unique solution $w \in C(\mathbb{R}_+, H^s)$, which satisfies the inequality

$$\|w(\cdot, t)\|_s \leq \|w_0\|_s + \int_0^t (\|f(\cdot, \tau)\|_s + C\|w(\cdot, \tau)\|_s \|\nabla \tilde{u}(\cdot, \tau)\|_{s-1}) d\tau. \quad (4.2.26)$$

Démonstration. Let us denote by $\phi^g : \tilde{D} \times \mathbb{R}_+ \rightarrow \tilde{D}$ the flow associated to g , i.e., the solution of the problem

$$\begin{aligned} \frac{\partial \phi^g}{\partial t} &= g(\phi^g, t), \\ \phi^g(x, 0) &= x. \end{aligned}$$

Since (4.2.24), (4.2.25) is an inhomogeneous transport equation, its solution is given by

$$w(\phi^{\tilde{u}}(x, t), t) = w_0(x) + \int_0^t f(\phi^{\tilde{u}}(x, \tau), \tau) d\tau.$$

Let us derive formally inequality (4.2.26). Taking the $\partial^\alpha := \frac{\partial^\alpha}{\partial x^\alpha}$, $|\alpha| \leq s$ derivative of (4.2.24) and multiplying the resulting equation by $\partial^\alpha w$, we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\partial^\alpha w\|^2 &= \int_D \partial^\alpha f \partial^\alpha w dx - \int_D \partial^\alpha (\tilde{u} \cdot \nabla) w \cdot \partial^\alpha w dx \\ &\leq \left| \int_D (\tilde{u} \cdot \nabla) \partial^\alpha w \cdot \partial^\alpha w dx \right| + \|f\|_s \|w\|_s + C \|\nabla \tilde{u}\|_{s-1} \|w\|_s^2. \end{aligned}$$

Integrating by parts, one verifies that the first integral in the right-hand side vanishes. Integrating in time, we obtain (4.2.26). \square

Lemma 4.2.8. *Let $w \in H^s, s \geq 0$. Then the problem*

$$\operatorname{curl} z = w, \quad (4.2.27)$$

$$\operatorname{div} z = 0, \quad (4.2.28)$$

$$z \cdot n|_{\Gamma} = 0 \quad (4.2.29)$$

has a unique solution $z \in H^{s+1}$. Moreover, there is $C > 0$ depending only on s such that

$$\|z\|_{s+1} \leq C\|w\|_s. \quad (4.2.30)$$

Démonstration. Let us consider the following Dirichlet problem for the Poisson equation :

$$\Delta v = w \quad \text{in } D,$$

$$v = 0 \quad \text{on } \Gamma.$$

By Proposition 4.2.2, $v \in H^{s+2}$ and $\|v\|_{s+2} \leq C\|w\|_s$. Then for $z = -\nabla^\perp v$ properties (4.2.27)-(4.2.30) are satisfied. \square

We now return to the proof of the theorem. The proof is based on some ideas from [6] and [9].

Step 1. Let

$$E : H^k(D) \rightarrow H^k(\mathbb{R}^2), \quad 0 \leq k \leq s+1$$

be an extension operator. Let $\rho \in \mathcal{S}(\mathbb{R}^2)$ be the function such that

$$\hat{\rho}(\xi) = \begin{cases} \exp(-\frac{|\xi|^2}{1-|\xi|^2}) & |\xi| < 1, \\ 0 & |\xi| \geq 1. \end{cases}$$

Define $J_m : H^s(D) \rightarrow H^{s+1}(D)$ by

$$J_m(v) := (m^2 \rho(mx) * E(v))|_D. \quad (4.2.31)$$

For $u_0 \in H^s(D)$ we define $u_0^m := J_m(u_0)$. Then

$$u_0^m \rightarrow u_0 \text{ in } H^s(D), \quad \|u_0^m\|_s \leq C\|u_0\|_s, \quad \|u_0^m\|_{s+1} \leq mC\|u_0\|_s, \quad (4.2.32)$$

$$\|u_0^m - u_0^k\|_s = o(1) \text{ and } \|u_0^m - u_0^k\|_1 = o\left(\frac{1}{m^{s-1}}\right) \text{ as } m \rightarrow \infty, \quad (4.2.33)$$

where (4.2.33) holds uniformly in $k > m$. Using Lemmas 4.2.7 and 4.2.8, we define the sequences $u^m \in C(\mathbb{R}_+, H^{s+1})$ and $w^m \in C(\mathbb{R}_+, H^s)$ by

$$\begin{aligned} u^0 &= u_0, \\ \partial_t w^{m+1} + \langle u^m, \nabla \rangle w^{m+1} &= 0, \quad w^{m+1}(0) = \operatorname{curl} u_0^{m+1}, \\ \operatorname{curl} u^{m+1} &= w^{m+1}, \quad \operatorname{div} u^{m+1} = 0, \quad u^{m+1} \cdot n|_{\Gamma} = 0. \end{aligned}$$

Our strategy is to show that sequence u^m is convergent and the limit is the solution of Euler system. From (4.2.26) we derive

$$\|w^m(\cdot, t)\|_i \leq \|\operatorname{curl} u_0^m\|_i + C_1 \int_0^t \|w^m(\cdot, \tau)\|_i \|u^{m-1}(\cdot, \tau)\|_i d\tau \quad (4.2.34)$$

for $i = s - 1, s$.

Step 2. In this step, we show that there exists a time $T_* = T_*(\|u_0\|_s)$ such that for any $t \in J_{T_*}$

$$\|w^m(\cdot, t)\|_{s-1} \leq C\|u_0^m\|_s, \quad \|w^m(\cdot, t)\|_s \leq C\|u_0^m\|_{s+1} \leq mC\|u_0\|_s. \quad (4.2.35)$$

By induction, let us prove for $i = s - 1, s$ the inequality

$$\|w^m(\cdot, t)\|_i \leq y_m(t), \quad (4.2.36)$$

where C does not depend on m and $y_m(t)$ is the solution of

$$\dot{y}_m = C_1 y_m^2, \quad y_m(0) = \|\operatorname{curl} u_0^m\|_i. \quad (4.2.37)$$

Clearly (4.2.36) holds for $m = 0$ for a sufficiently large C . Assume that it holds also for $m - 1$ and let us prove it for m . From the construction of $\hat{\rho}$ we have $\|u_0^{m-1}\|_i \leq \|u_0^m\|_i$, hence $y_{m-1} \leq y_m$. Thus, from (4.2.34), (4.2.37) and induction hypothesis, we have

$$\begin{aligned} \|w^m(\cdot, t)\|_i - y_m &\leq C_1 \int_0^t (\|w^m(\cdot, \tau)\|_i \|u^{m-1}(\cdot, \tau)\|_i - y_m^2) d\tau \\ &\leq C_1 \int_0^t y_m (\|w^m(\cdot, \tau)\|_i - y_m) d\tau. \end{aligned}$$

Inequality (4.2.36) follows from the Gronwall inequality. It is easy to see that (4.2.36) yields (4.2.35).

Step 3. Now let us show that w^m converges in $C(J_{T_*}, H^{s-1})$. In view of Lemma 4.2.8, sequence u^m converges in $C(J_{T_*}, H^s)$ and the limit u is the solution of Euler problem.

Notice that for $m < k$ we have

$$\partial_t (w^m - w^k) + \langle u^{k-1}, \nabla \rangle (w^m - w^k) = \langle u^{k-1} - u^{m-1}, \nabla \rangle w^m. \quad (4.2.38)$$

Denote $K^{m,k}(t) := \|w^m(\cdot, t) - w^k(\cdot, t)\|_{s-1}$. Lemma 4.2.7 implies

$$\begin{aligned} K^{m,k}(t) &\leq \|u_0^m - u_0^k\|_s + C \int_0^t (K^{m,k}(\tau) \|u^{k-1}(\cdot, \tau)\|_{s-1} \\ &\quad + \|u^{m-1}(\cdot, \tau) - u^{k-1}(\cdot, \tau)\|_{s-1} \|w^m(\cdot, \tau)\|_s) d\tau. \end{aligned} \quad (4.2.39)$$

On the other hand,

$$\|w^m\|_s \leq Cm, \quad \|u^{m-1} - u^{k-1}\|_{s-1} \leq \|u^{m-1} - u^{k-1}\|_1^{\frac{1}{s-1}} \|u^{m-1} - u^{k-1}\|_s^{\frac{s-2}{s-1}}. \quad (4.2.40)$$

Assume for a moment that

$$U^{m,k} := \|w^{m-1} - w^{k-1}\| \leq o\left(\frac{1}{m^{s-1}}\right). \quad (4.2.41)$$

Substituting (4.2.40) into (4.2.39) and using (4.2.33) and (4.2.41), we obtain

$$K^{m,k}(t) \leq o(1) + C \int_0^t (K^{m,k}(\tau) \|u^{k-1}(\cdot, \tau)\|_{s-1}) d\tau.$$

Using the Gronwall inequality, we obtain the convergence of w^m in $C(J_{T_*}, H^{s-1}(D))$.

Step 4. To complete the proof of the theorem, it remains to show (4.2.41). Taking the scalar product of (4.2.38) with $w^m - w^k$ in L^2 , we get

$$U^{m,k}(t) \leq C \|u_0^m - u_0^k\|_1 + C \int_0^t U^{m-1,k-1}(t_1) dt_1.$$

Iterating this inequality, one deduces

$$\begin{aligned} U^{m+p,k+p}(t) &\leq C \|u_0^{m+p} - u_0^{k+p}\|_1 + C \int_0^t U^{m+p-1,k+p-1}(t_1) dt_1 \\ &\leq C \|u_0^{m+p} - u_0^{k+p}\|_1 + C \int_0^t (C \|u_0^{m+p-1} - u_0^{k+p-1}\|_1 \\ &\quad + C \int_0^{t_1} U^{m+p-2,k+p-2}(t_2)) dt_2 dt_1 \\ &\leq C \|u_0^{m+p} - u_0^{k+p}\|_1 + C \int_0^t (C \|u_0^{m+p-1} - u_0^{k+p-1}\|_1 \\ &\quad + \dots + C \int_0^{t_{p-1}} (C \|u_0^{m+1} - u_0^{k+1}\|_1 + C \int_0^{t_p} U^{m,k}(t_p))) dt_p \dots dt_2 dt_1. \end{aligned}$$

Hence, for any $t \in J_{T_*}$ we obtain

$$U^{m+p,k+p} \leq \sum_{j=1}^p \frac{C^{p-j+1} T_*^{p-j}}{(p-j)!} \|u_0^{m+j} - u_0^{k+j}\|_1 + \frac{C^{p+1} T_*^p}{p!} \max_{t \in [0, T_*]} U^{m,k}. \quad (4.2.42)$$

Since

$$\sum_{j=1}^{\infty} \frac{C^{j+1} T_*^j}{j!} < \infty,$$

inequalities (4.2.33) and (4.2.42) imply (4.2.41). □

Remark 4.2.9. We have the following assertions :

- Adapting the Beale–Kato–Majda criterion (see [7]) for an unbounded strip, one can prove that the solution of (4.2.17)-(4.2.19) is global in time. However, we shall not need this result.

- Let us take any non-zero function $g \in H^{s-1/2}(\Gamma)$. If the homogeneous boundary condition (4.2.18) is replaced by $u \cdot n|_{\Gamma} = g$, then, the result of Theorem 4.2.6 holds if we add the boundary condition

$$\operatorname{curl} u = \phi \quad \text{where} \quad u \cdot n < 0.$$

See [47] for the case of a bounded domain.

4.3 Main result

Let D and Γ_0 be defined by (4.1.2) and (4.1.3). Consider the Euler system :

$$\dot{u} + \langle u, \nabla \rangle u + \nabla p = 0 \quad \text{in} \quad D \times (0, \infty), \quad (4.3.1)$$

$$\operatorname{div} u = 0, \quad (4.3.2)$$

$$u \cdot n = 0 \quad \text{on} \quad \Gamma \setminus \Gamma_0 \times \mathbb{R}_+, \quad (4.3.3)$$

$$u(x, 0) = u_0(x). \quad (4.3.4)$$

For any integer s we denote

$$\mathcal{X}^s(D) = C(\mathbb{R}_+, C_b(\overline{D}) \cap \dot{H}^s(D)),$$

and $\langle x_1 \rangle := (1 + x_1^2)^{1/2}$. The following theorem is our main result.

Theorem 4.3.1. *For any constants $\alpha, \beta > 0$, $c \in \mathbb{R}$ and integer $s \geq 4$, for any initial data $u_0 \in H^s(D)$ such that*

$$\operatorname{div} u_0 = 0, \quad (4.3.5)$$

$$u_0 \cdot n = 0 \text{ on } \Gamma \setminus \Gamma_0, \quad (4.3.6)$$

$$\|\exp(\alpha \langle x_1 \rangle^{2+\beta}) \operatorname{curl} u_0(x_1, x_2)\|_{s-1} < \infty \quad (4.3.7)$$

there is a solution $(u, p) \in \mathcal{X}^s(D) \times C(\mathbb{R}_+, \dot{H}^s(D))$ of (4.3.1)-(4.3.4) with

$$\lim_{t \rightarrow \infty} (\|u(\cdot, t) - (c, 0)\|_{L^\infty(D)} + \|\nabla u(\cdot, t)\|_{s-1} + \|\nabla p(\cdot, t)\|_{s-1}) = 0. \quad (4.3.8)$$

As explained in Introduction, in this formulation the control is not given explicitly, but we can assume that control acts on the system as a boundary condition on Γ_0 . So we show that there exists control η such that there is a solution of our system with $u \cdot n|_{\Gamma_0} = \eta$ verifying (4.3.8). As we mentioned in Remark 4.2.9, we are not able to show that this solution is unique.

Using a standard scaling argument for Euler system, we can reduce this theorem to a small neighborhood of the origin.

Theorem 4.3.2. *There exists $\varepsilon > 0$ such that for any $u_0 \in H^s(D)$ and $c \in \mathbb{R}$ verifying (4.3.5)-(4.3.7) and*

$$\|u_0\|_s < \varepsilon, \quad |c| < \varepsilon$$

there is a solution $(u, p) \in \mathcal{X}^s(D) \times C(\mathbb{R}_+, \dot{H}^s(D))$ of (4.3.1)-(4.3.4) satisfying (4.3.8).

Proof of Theorem 4.3.1. Let $\varepsilon > 0$ be the constant in Theorem 4.3.2. Take any $u_0 \in H^s(D)$ and $c \in \mathbb{R}$ verifying (4.3.5)-(4.3.7). Let $M > 0$ be such that

$$\left\| \frac{u_0}{M} \right\|_s < \varepsilon, \quad \left| \frac{c}{M} \right| < \varepsilon.$$

By Theorem 4.3.2, there exists a solution (u_M, p_M) of (4.3.1)-(4.3.3) with initial condition $u_M(0) = \frac{u_0}{M}$, such that

$$\lim_{t \rightarrow \infty} (\|u_M(\cdot, t) - (\frac{c}{M}, 0)\|_{L^\infty(D)} + \|\nabla u_M(\cdot, t)\|_{s-1} + \|\nabla p_M(\cdot, t)\|_{s-1}) = 0.$$

Then $(u, p) = (Mu_M(x, Mt), M^2p_M(x, Mt))$ is a solution of our system with $u(0) = u_0$ and it satisfies (4.3.8). \square

Proof of Theorem 4.3.2. The proof of this theorem is based on generalization of the Coron return method to the case of an unbounded strip. It consists in construction of a particular solution (\bar{u}, \bar{p}) of (4.3.1)-(4.3.3) such that the solution of linearized system around (\bar{u}, \bar{p}) verifies property (4.3.8). Then, in the small neighborhood of \bar{u} , we construct a solution u of Euler system satisfying (4.3.8).

Step 1. In this step, we construct a particular solution (\bar{u}, \bar{p}) of (4.3.1)-(4.3.3) such that any point of strip D , driven by the flow of \bar{u} , leaves \bar{D} at some time. Let $\hat{D} \subset \mathbb{R}^2$ be the strip

$$\hat{D} := \{(x_1, x_2) : x_1 \in \mathbb{R}, x_2 \in (-2, 2)\}.$$

Let us admit the proposition below, which is proved in Section 4.4.1.

Proposition 4.3.3. *There are scalar functions $\theta^i \in C^1(\hat{D} \times \mathbb{R}_+)$ with $\nabla \theta^i \in \mathcal{X}^s(\hat{D})$, open balls B^i , a sequence $\tau_i \subset \mathbb{R}_+$, constants L, λ and an integer $N \in \mathbb{N}$ such that the following properties are true.*

1. Covering. For any integer $k \geq 0$, we have

$$[k, k+1] \times [-1, 1] \subset \bigcup_{j=1}^N B^{2kN+j}, \quad (4.3.9)$$

$$[-k-1, -k] \times [-1, 1] \subset \bigcup_{j=1}^N B^{(2k+1)N+j}. \quad (4.3.10)$$

In particular, the union of balls B^i covers \bar{D} and any square $[k, k+1] \times [-1, 1]$ is covered by N balls.

2. Support.

$$\text{supp } \theta^i \subset \hat{D} \times (0, \tau_i). \quad (4.3.11)$$

3. Vector field. The time dependent vector field $\nabla\theta^i$ is divergence-free in D and tangent to $\Gamma \setminus \Gamma_0$ and $\partial\hat{D}$:

$$\Delta\theta^i = 0 \quad \text{in } D \times [0, \tau_i], \quad (4.3.12)$$

$$\frac{\partial\theta^i}{\partial n} = 0 \quad \text{on } (\Gamma \setminus \Gamma_0) \cup \partial\hat{D} \times [0, \tau_i]. \quad (4.3.13)$$

4. Time decay. For any $i \geq 1$ we have

$$\|\nabla\theta^i(\cdot, t)\|_{\mathcal{X}^s(\hat{D})} \leq \frac{1}{i} \text{ for any } t \in [0, \tau_i], \quad (4.3.14)$$

$$\tau_i \leq Li. \quad (4.3.15)$$

5. Flow. For any $i \geq 1$ and $c \in \mathbb{R}$ with $|c| < \lambda$ the flow associated with $\nabla\theta^i + (c, 0)$ is such that

$$\phi^{\nabla\theta^i + (c, 0)}(B^i, \tau_i) \subset \hat{D} \setminus \bar{D}. \quad (4.3.16)$$

Moreover, there are two closed balls $\tilde{B}_1, \tilde{B}_2 \subset \hat{D} \setminus \bar{D}$ such that

$$\cup_{i=1}^{\infty} \phi^{\nabla\theta^i + (c, 0)}(B^i, \tau_i) \subset \tilde{B}_1 \cup \tilde{B}_2. \quad (4.3.17)$$

Let us set $t_0 = 0$,

$$t_i = 2 \sum_{j=1}^i \tau_j, \quad t_{i+1/2} = \frac{t_i + t_{i+1}}{2}, \quad i \geq 1. \quad (4.3.18)$$

We define $\bar{\theta}$ in the following way :

$$\bar{\theta}(x, t) = \theta^i(x, t - t_{i-1}) \text{ for } t \in [t_{i-1}, t_{i-1/2}], \quad (4.3.19)$$

$$\bar{\theta}(x, t) = -\theta^i(x, t_i - t) \text{ for } t \in [t_{i-1/2}, t_i]. \quad (4.3.20)$$

Notice that from the construction of t_i we have $t_i - t_{i-1/2} = \tau_i$. Thus (4.3.11) shows that $\bar{\theta} \in C^1(\hat{D} \times \mathbb{R}_+)$ and $\nabla\bar{\theta} \in \mathcal{X}^s(\hat{D})$. We define

$$\bar{u} := \nabla\bar{\theta} + (c, 0),$$

$$\bar{p} := -\partial_t\bar{\theta} - \frac{|\nabla\bar{\theta}|^2}{2} - c\partial_1\theta.$$

Then (\bar{u}, \bar{p}) is a solution of (4.3.1)-(4.3.3). Indeed, by construction, (\bar{u}, \bar{p}) satisfies (4.3.1). Properties (4.3.12) and (4.3.13) imply (4.3.2) and (4.3.3), respectively. Moreover, it follows from (4.3.14), (4.3.16) that for any $i \in \mathbb{N}$, we have

$$\begin{aligned} \phi^{\bar{u}}(B^i, t_{i-1/2}) &\not\subset \bar{D}, \\ \lim_{t \rightarrow \infty} (\|\bar{u}(\cdot, t) - (c, 0)\|_{L^\infty(D)} + \|\nabla\bar{u}(\cdot, t)\|_{s-1}) &= 0. \end{aligned} \quad (4.3.21)$$

We deduce from (4.3.19) and (4.3.20) that

$$\phi^{\bar{u}}(x, t_i) = x \quad (4.3.22)$$

for any $i \geq 1$ and $x \in \hat{D}$. We shall need the following result, which is proved in Section 4.4.1.

Proposition 4.3.4. *There is a constant $\nu > 0$ such that the functions θ^i in Proposition 4.3.3 can be chosen in a way that, for any $u \in \mathcal{X}^s(\hat{D})$ satisfying the inequality*

$$\int_0^\infty \|u(t) - \bar{u}(t)\|_{s, \hat{D}} dt \leq \nu,$$

we have $\phi^u(B^i, t_{i-1/2}) \subset \hat{D} \setminus \bar{D}$ for any $i \geq 1$.

From now on, we assume that functions θ^i verify this proposition.

Step 2. In this step, we construct an application F_{u_0} such that its fixed point is a solution of our stabilization problem. First, for any constant $\nu > 0$ let us introduce the set

$$\begin{aligned} \mathcal{Y}_\nu(u_0) := \{u \in \mathcal{X}^s(D) : \operatorname{div} u = 0, \int_0^\infty \|u(t) - \bar{u}(t)\|_{s, D} dt \leq \nu, \\ u(x, t) \cdot n(x) = (u_0(x)\mu(t) + \bar{u}(x, t)) \cdot n(x) \text{ on } \Gamma \times \mathbb{R}_+\}, \end{aligned}$$

where $\mu \in C_0^\infty([0, \infty))$ is a non-negative function such that

$$\mu(0) = 1, \quad \int_0^\infty \mu(t) dt < 1.$$

Let $D_1 := \mathbb{R} \times (-\frac{3}{2}, \frac{3}{2})$ and $\pi : H^s(D) \rightarrow H^s(\hat{D})$ be any linear bounded extension operator such that $\operatorname{supp} \pi u \subset D_1$ for any $u \in H^s(D)$. Let $\kappa^i \in C_0^\infty(\hat{D})$ be a partition of unity subordinate to B^i , i.e.,

$$\begin{aligned} \operatorname{supp} \kappa^i \subset B^i, \\ \sum_{i=1}^\infty \kappa^i = 1 \text{ in } \bar{D}. \end{aligned}$$

Take any $u \in \mathcal{Y}_\nu(u_0)$ and let $w^l \in C(\mathbb{R}_+, H^{s-1}(\hat{D}))$ be the solution of the linear problem

$$\dot{w}^l + \langle \tilde{u}, \nabla \rangle w^l = 0 \text{ in } \hat{D} \times \mathbb{R}_+, \quad (4.3.23)$$

$$w^l(0) = \kappa^l \operatorname{curl}(\pi u_0), \quad (4.3.24)$$

where

$$\tilde{u} = \bar{u} + \pi(u - \bar{u}). \quad (4.3.25)$$

Take ν such that Proposition 4.3.4 holds. Since $\operatorname{supp} w^l(0) \subset B_l$, we obtain

$$w^l(x, t_{l-1/2}) = 0 \text{ for any } x \in \bar{D}. \quad (4.3.26)$$

For any $t \in \mathbb{R}_+$ we define the function

$$w(\cdot, t) = \sum_{l=i+1}^\infty w^l(\cdot, t), \quad \text{when } t \in [t_{i-1/2}, t_{i+1/2}], \quad (4.3.27)$$

where $t_{-1/2} := 0$ and $i \geq 0$. Let us show that for any $t \in [t_{i-1/2}, t_{i+1/2}]$ the sum in the right-hand side of (4.3.27) exists and belongs to $C(\mathbb{R}_+, H^{s-1}(D))$. Applying Lemma 4.2.7 to (4.3.23), (4.3.24), we obtain

$$\|w^l(t)\|_{s-1, \hat{D}} \leq C(\|\kappa^l \operatorname{curl}(\pi u_0)\|_{s-1, \hat{D}} + \int_0^t \|\nabla \tilde{u}(\tau)\|_{s-1, \hat{D}} \|w^l(\tau)\|_{s-1, \hat{D}} d\tau).$$

It follows from the Gronwall inequality and relation (4.3.25) that

$$\begin{aligned} \|w^l(t)\|_{s-1, \hat{D}} &\leq C\|\kappa^l \operatorname{curl}(\pi u_0)\|_{s-1, \hat{D}} \exp\left(C \int_0^t \|\nabla \tilde{u}(\tau)\|_{s-1, \hat{D}} d\tau\right) \\ &\leq C\|\kappa^l \operatorname{curl}(\pi u_0)\|_{s-1, \hat{D}} \exp\left(C \int_0^t (\|\nabla \bar{u}(\tau)\|_{s-1, \hat{D}} + \|\bar{u}(\tau) - \tilde{u}(\tau)\|_{s, \hat{D}}) d\tau\right). \end{aligned}$$

Using the fact that $\bar{u} \in \mathcal{X}^s(\hat{D})$, we get

$$\|w^l(t)\|_{s-1, \hat{D}} \leq C\|\kappa^l \operatorname{curl}(\pi u_0)\|_{s-1, \hat{D}} \exp(C(t_{i+1/2} + \nu))$$

for any $t \in [t_{i-1/2}, t_{i+1/2}]$. Thus

$$\sum_{l=i}^{\infty} \|w^l(t)\|_{s-1, \hat{D}} \leq C \exp(Ct_{i+1/2}) \sum_{l=i}^{\infty} \|\kappa^l \operatorname{curl}(\pi u_0)\|_{s-1, \hat{D}}. \quad (4.3.28)$$

Using (4.3.7) and assertion 1 of Proposition 4.3.3, we derive that the right-hand side of (4.3.28) is finite. Hence, $w \in C([t_{i-1/2}, t_{i+1/2}], H^{s-1}(\hat{D}))$ for any $i \geq 0$. Moreover, assertion (4.3.26) yields that w is continuous at $t_{i-1/2}$, thus $w \in C(\mathbb{R}_+, H^{s-1}(D))$ (we emphasize that, in general, this is not true for \hat{D}). Furthermore, we have

$$\begin{aligned} \dot{w} + \langle \tilde{u}, \nabla \rangle w &= 0 \quad \text{in} \quad \hat{D} \times [t_{i-1/2}, t_{i+1/2}], \\ w(0) &= \sum_{l=1}^{\infty} \kappa^l \operatorname{curl} \pi u_0 \quad \text{in} \quad \hat{D}. \end{aligned}$$

In Step 3, we prove that for this w there exists a $v \in \mathcal{Y}_\nu(u_0)$ such that

$$\operatorname{curl} v = w. \quad (4.3.29)$$

For any $u \in \mathcal{Y}_\nu(u_0)$, let $F_{u_0}(u) := v$. In Step 4, we show that the mapping $F_{u_0} : \mathcal{Y}_\nu(u_0) \rightarrow \mathcal{Y}_\nu(u_0)$ has a fixed point. We shall prove that this fixed point is a solution of our stabilization problem.

Step 3. In this step, we prove the existence of the solution $v \in \mathcal{Y}_\nu(u_0)$ of (4.3.29). By Lemma 4.2.8, there is a function $z \in C(\mathbb{R}_+, H^s(D))$ such that

$$\begin{aligned} \operatorname{curl} z &= w, \\ \operatorname{div} z &= 0, \\ z \cdot n &= 0, \\ \|z(\cdot, t)\|_{s, D} &\leq C\|w(\cdot, t)\|_{s-1, D}. \end{aligned} \quad (4.3.30)$$

Let us take the solution of the following problem

$$\begin{aligned}\Delta\varphi &= 0 \text{ in } D, \\ \frac{\partial\varphi}{\partial n} &= (u_0\mu) \cdot n \text{ on } \Gamma.\end{aligned}$$

From Proposition 4.2.4 we have $\varphi \in C(\mathbb{R}_+, \dot{H}^{s+1}(D))$ and

$$\|\varphi(\cdot, t)\|_{\dot{H}^{s+1}(D)} \leq C\|u_0\mu(t)\|_{s,D}.$$

Denote $v = z + \nabla\varphi + \bar{u}$. Let us show that $v \in \mathcal{Y}_\nu(u_0)$ and (4.3.29) is verified. Clearly

$$\begin{aligned}\operatorname{curl} v &= \operatorname{curl} z = w, \\ \operatorname{div} v &= \operatorname{div} z + \Delta\varphi = 0, \\ v \cdot n &= (u_0(x)\mu + \bar{u}) \cdot n \text{ on } \Gamma \times \mathbb{R}_+.\end{aligned}$$

Hence, to show $v \in \mathcal{Y}_\nu(u_0)$, it suffices to prove for sufficiently small u_0 that

$$\int_0^\infty \|v(t) - \bar{u}(t)\|_{s,D} dt \leq \nu. \quad (4.3.31)$$

It follows from the construction of v that

$$\|v(\cdot, t) - \bar{u}(t)\|_{s,D} \leq \|\varphi(\cdot, t)\|_{\dot{H}^{s+1}(D)} + \|z(\cdot, t)\|_{s,D}.$$

Proposition 4.2.4 and (4.3.30) imply

$$\int_0^\infty \|v(t) - \bar{u}(t)\|_{s,D} dt \leq \|u_0\|_{s,D} \int_0^\infty \mu(t) dt + C \int_0^\infty \|w(\cdot, t)\|_{s-1,D} dt.$$

From (4.3.27) we have

$$\int_0^\infty \|w(\cdot, t)\|_{s-1,D} dt = \sum_{i=0}^\infty \int_{t_{i-1/2}}^{t_{i+1/2}} \left\| \sum_{l=i+1}^\infty w^l(\cdot, t) \right\|_{s-1,D} dt.$$

Applying Lemma 4.2.7 to $\sum_{l=i+1}^\infty w^l$, we obtain

$$\left\| \sum_{l=i+1}^\infty w^l(x, t) \right\|_{s-1,D} \leq C \exp\left(C \int_0^t \|\nabla \tilde{u}(\cdot, \tau)\|_{s-1,D} d\tau\right) \left\| \sum_{l=i+1}^\infty \kappa^l \operatorname{curl} u_0 \right\|_{s-1,D}.$$

Thus

$$\begin{aligned}\int_0^\infty \|w(\cdot, t)\|_{s-1,D} dt &\leq C \sum_{i=0}^\infty \int_{t_{i-1/2}}^{t_{i+1/2}} \left\| \sum_{l=i+1}^\infty \kappa^l \operatorname{curl} u_0 \right\|_{s-1,D} \times \\ &\quad \times \exp\left(C \int_0^t \|\nabla \tilde{u}(\cdot, \tau)\|_{s-1,D} d\tau\right) dt \\ &\leq C_1 \sum_{i=0}^\infty \int_{t_{i-1/2}}^{t_{i+1/2}} \exp(Ct_{i+1/2}) \|\operatorname{curl} u_0\|_{s-1,D \setminus \cup_{l=1}^i B_l} dt.\end{aligned}$$

Combining (4.3.7), (4.3.15), (4.3.18) and assertion 1 of Proposition 4.3.3, we get

$$(t_{i+1/2} - t_{i-1/2}) \exp(Ct_{i+1/2}) \|\operatorname{curl} u_0\|_{s-1, D \setminus \cup_{l=1}^i B_l} \leq C_2 \frac{1}{i^2}$$

for any $i > 0$, where C_2 does not depend on i . Let K be a constant such that

$$C_1 C_2 \sum_{i=K}^{\infty} \frac{1}{i^2} < \frac{\nu}{2}.$$

Taking u_0 sufficiently small such that

$$\begin{aligned} \|u_0\|_{s,D} + \sum_{i=1}^K \int_{t_{i-1/2}}^{t_{i+1/2}} \sum_{l=i+1}^{\infty} \|\kappa^l \operatorname{curl} u_0\|_{s-1, D} \exp\left(\int_0^t \|\nabla \tilde{u}(\cdot, \tau)\|_{s-1, D} d\tau\right) dt \\ \leq \frac{\nu}{2}, \end{aligned}$$

we get (4.3.31).

Step 4. In this step, we show that the mapping $F_{u_0} : \mathcal{Y}_\nu(u_0) \rightarrow \mathcal{Y}_\nu(u_0)$ admits a fixed point, which is the solution of our stabilization problem. Let us take a sequence $u_0^m := J_m(u_0)$, where J_m is the operator defined by (4.2.31). We have that $u_0^m \in H^{s+1}(D)$ verifies (4.2.32), (4.2.33). Take $u^0(x, t) = \mu(t)u_0(x) + \bar{u}(x, t)$. For sufficiently small u_0 we have $u^0 \in \mathcal{Y}_\nu(u_0)$. Let $u^1 = F_{u_0^1}(u^0)$ and let w_1 be defined as in (4.3.27) with $u = u^0$ and $u_0(x) = u_0^1(x)$. In this way we introduce the sequences $u^m \in \mathcal{X}^s$ and $w_m \in C(\mathbb{R}_+, H^s(D))$ by the relations

$$\begin{cases} u^{m+1} = F_{u_0^{m+1}}(u^m), \\ w_{m+1} \text{ defined as in (4.3.27) with } u = u^m \text{ and } u_0 = u_0^{m+1}. \end{cases}$$

Let us show the convergence of w_m in $C([0, t_{1/2}], H^{s-1}(\hat{D}))$. This will be proved by using the same arguments as in the proof of Theorem 4.2.6. It is easy to see

$$\partial_t (w_m - w_k) + \langle \tilde{u}^{k-1}, \nabla \rangle (w_m - w_k) = \langle \tilde{u}^{k-1} - \tilde{u}^{m-1}, \nabla \rangle w_m.$$

Setting $K^{m,k}(t) := \|w_m(\cdot, t) - w_k(\cdot, t)\|_{s-1, \hat{D}}$ and using Lemmas 4.2.7 and 4.2.8, we obtain

$$\begin{aligned} K^{m,k}(t) &\leq \|u_0^m - u_0^k\|_s + C \int_0^t (K^{m,k}(\tau) \|\nabla \tilde{u}^{k-1}(\cdot, \tau)\|_{s-1} \\ &\quad + \|\tilde{u}^{m-1}(\cdot, \tau) - \tilde{u}^{k-1}(\cdot, \tau)\|_{s-1} \|w_m(\cdot, \tau)\|_s) d\tau. \end{aligned} \quad (4.3.32)$$

Let us show that for any $m \in \mathbb{N}$

$$\sup_{t \in [0, t_{1/2}]} \|w_m(\cdot, t)\|_{s-1, \hat{D}} < C \|u_0^m\|_{s, \hat{D}}, \quad (4.3.33)$$

where C depends only on $\|\bar{u}(t)\|_{L^1((0,t_{1/2}),\dot{H}^s(\hat{D}))}$ and does not depend on m . From the construction of w_m , we have

$$\begin{aligned} \dot{w}_m + \langle \tilde{u}^{m-1}, \nabla \rangle w_m &= 0 & \text{in } \hat{D} \times \mathbb{R}_+, \\ w_m(0) &= \sum_{l=1}^{\infty} \kappa^l \operatorname{curl} \pi u_0^m & \text{in } \hat{D}. \end{aligned}$$

Applying Lemma 4.2.7, we get

$$\begin{aligned} \|w_m(t)\|_{s-1,\hat{D}} &\leq C \left(\|u_0^m\|_{s,\hat{D}} + \int_0^t \|w_m\|_{s-1,\hat{D}} \|\nabla \tilde{u}^{m-1}\|_{s-1,\hat{D}} dt \right) \\ &\leq C \left(\|u_0^m\|_{s,\hat{D}} + \int_0^t \|w_m\|_{s-1,\hat{D}} \left(\|\nabla \bar{u}\|_{s-1,\hat{D}} + \|\bar{u} - \tilde{u}^{m-1}\|_{s,\hat{D}} \right) dt \right). \end{aligned}$$

Using the Gronwall inequality and the fact that $\tilde{u}_{m-1} \in \mathcal{Y}_\nu(u_0^m)$, we derive

$$\|w_m(t)\|_{s-1,\hat{D}} \leq C(\|u_0\|_s \exp(\int_0^{t_{1/2}} (\|\nabla \bar{u}\|_{s-1,\hat{D}} + \|\bar{u} - \tilde{u}_{m-1}\|_{s,\hat{D}}) dt)) \leq C_1,$$

where C_1 does not depend on m . Thus, we obtain (4.3.33). The construction of u^m implies boundedness of $\sup_{t \in [0, t_{1/2}]} \|u^m\|_{s,\hat{D}}$ uniformly in m . In the same way we can show that

$$\sup_{t \in [0, t_{1/2}]} \|w_m(\cdot, t)\|_{s,\hat{D}} \leq C \|u_0^m\|_{s+1,\hat{D}}.$$

Combining this with (4.2.32) and (4.2.33), we get

$$\begin{aligned} \|\tilde{u}^{m-1}(\cdot, \tau) - \tilde{u}^{k-1}(\cdot, \tau)\|_{s-1} \|w_m(\cdot, \tau)\|_s &\leq \|\tilde{u}^{m-1}(\cdot, \tau) - \tilde{u}^{k-1}(\cdot, \tau)\|^{1/s} \times \\ &\times \|\tilde{u}^{m-1}(\cdot, \tau) - \tilde{u}^{k-1}(\cdot, \tau)\|_s^{1-1/s} \|w_m(\cdot, \tau)\|_s \leq a_{m,k} \end{aligned} \quad (4.3.34)$$

for any $t \in J_{t_{1/2}}$, where $\sup_{k \geq m} a_{m,k} \rightarrow 0$ as $m \rightarrow \infty$ and $a_{m,k}$ is decreasing sequence in m for any fixed $k > m$ (this properties we can obtain arguing in the same way as in Theorem 4.2.6). Using this with (4.3.32) and (4.3.33), for any $t \in J_{t_{1/2}}$ we get

$$K^{m,k}(t) \leq C \int_0^t (K^{m-1,k-1}(t_1) + K^{m,k}(t_1)) dt_1 + a_{m,k}.$$

By the Gronwall inequality, for any $t \in [0, t_{1/2}]$ we have

$$\begin{aligned}
K^{m+p,k+p}(t) &\leq C \int_0^t K^{m+p-1,k+p-1}(\sigma_1) e^{Ct_1} d\sigma_1 + Ca_{m+p,k+p} \\
&\leq C^2 \int_0^t \int_0^{\sigma_1} K^{m+p-2,k+p-2}(\sigma_2) e^{C\sigma_1} e^{C\sigma_2} d\sigma_2 d\sigma_1 \\
&\quad + Ce^{Ct_{1/2}} a_{m+p-1,k+p-1} + Ca_{m+p,k+p} \\
&\leq C^3 \int_0^t \int_0^{\sigma_1} \int_0^{\sigma_2} K^{m+p-3,k+p-3}(\sigma_3) e^{C\sigma_1} e^{C\sigma_2} e^{C\sigma_3} d\sigma_3 d\sigma_2 d\sigma_1 \\
&\quad + C \frac{e^{2Ct_{1/2}}}{2} a_{m+p-2,k+p-2} + Ce^{Ct_{1/2}} a_{m+p-1,k+p-1} + Ca_{m+p,k+p} \\
&\leq C^p \int_0^t \int_0^{\sigma_1} \cdots \int_0^{\sigma_{p-1}} K^{m,k}(\sigma_p) e^{C\sigma_1 + C\sigma_2 + \cdots + C\sigma_p} d\sigma_p \cdots d\sigma_2 d\sigma_1 \\
&\quad + \sum_{j=0}^{p-1} C \frac{(e^{Ct_{1/2}})^j}{j!} a_{m+p-j,k+p-j}.
\end{aligned}$$

Thus, we derive

$$K^{m+p,k+p} \leq \frac{Ce^{pC}}{p!} \max_{t \in [0, T]} K^{m,k} + Ca_{m,k}.$$

Hence, w_m is a convergent sequence in $C([0, t_{1/2}], H^{s-1}(\hat{D}))$. In the same way we can get the convergence of w_m in $C([t_{i-1/2}, t_{i+1/2}], H^{s-1}(\hat{D}))$. Finally, the fact $w_m \in C(\mathbb{R}_+, H^{s-1}(D))$ implies that w_m converges to some w^* in $C(\mathbb{R}_+, H^{s-1}(D))$. The convergence of w_m implies the convergence of u^m to some u^* in $\mathcal{X}^s(D)$. We have

$$\operatorname{curl} u^* = w^*, \quad (4.3.35)$$

$$\operatorname{div} u^* = 0, \quad (4.3.36)$$

$$u^*(x, t) \cdot n(x) = (u_0(x)\mu(t) + \bar{u}(x, t)n(x)) \text{ on } \Gamma \times \mathbb{R}_+. \quad (4.3.37)$$

Let us show that

$$w^*(\cdot, t) = \sum_{l=i+1}^{\infty} w^{*l}(\cdot, t) \quad \text{for } t \in [t_{i-1/2}, t_{i+1/2}], \quad (4.3.38)$$

where w^{*l} is the solution of

$$\partial_t w^{*l} + \langle \tilde{u}^*, \nabla \rangle w^{*l} = 0 \text{ in } \hat{D} \times \mathbb{R}_+, \quad (4.3.39)$$

$$w^{*l}(0) = \kappa^l \operatorname{curl}(\pi u_0). \quad (4.3.40)$$

To this end, recall that

$$w_m(\cdot, t) = \sum_{l=i+1}^{\infty} w_m^l(\cdot, t), \quad \text{when } t \in [t_{i-1/2}, t_{i+1/2}],$$

where w_m^l is the solution of

$$\begin{aligned} \dot{w}_m^l + \langle \tilde{u}_{m-1}, \nabla \rangle w_m^l &= 0 \text{ in } \hat{D} \times \mathbb{R}_+, \\ w_m^l(0) &= \kappa^l \operatorname{curl}(\pi u_0^{m+1}). \end{aligned}$$

We have that $w_m^l \rightarrow w^{*l}$ in $C(\mathbb{R}_+, H^{s-1}(\hat{D}))$ uniformly with respect to l as $m \rightarrow \infty$ (this can be proved in the same way as in the proof of the convergence of w_m). Thus we have (4.3.38). Clearly (4.3.35)-(4.3.40) imply that u^* is a solution of the Euler system (4.3.1)-(4.3.3).

As in (4.3.28), using (4.3.35)-(4.3.40) for any $t \in [t_{i-1/2}, t_{i+1/2}]$ and (4.3.7), we can show that

$$\begin{aligned} \sum_{l=i}^{\infty} \|w^{*l}(t)\|_{s-1, \hat{D}} &\leq C \sum_{l=i}^{\infty} \exp(Ci^2) \|\kappa^l \operatorname{curl}(\pi u_0)\|_{s-1, \hat{D}} \\ &\leq C \sum_{l=i}^{\infty} \exp(Ci^2) \exp(-Ci^{2+\beta}). \end{aligned}$$

Thus

$$\lim_{t \rightarrow \infty} \|u^*(t) - \bar{u}(t)\|_{s, D} = 0. \quad (4.3.41)$$

Combining this with (4.3.21), we see that the first two terms on the left-hand side of (4.3.8) go to zero as $t \rightarrow \infty$. Recall that

$$\begin{aligned} \Delta p^* &= -\operatorname{div}(\langle u^*, \nabla \rangle u^*) \\ \frac{\partial p^*}{\partial n} &= -(\langle u^*, \nabla \rangle u^*) \cdot n. \end{aligned}$$

Thus, Proposition 4.2.4 implies $\lim_{t \rightarrow \infty} \|\nabla p^*(t)\|_{s-1} = 0$. This completes the proof of Theorem 4.3.1. □

4.4 Construction of the particular solution

4.4.1 Proof of Proposition 4.3.3

We have the following simplified version of Proposition 4.3.3.

Lemma 4.4.1. *For any $x_0 \in \bar{D}$ there exist a function $\theta \in C^\infty([0, 1], \dot{H}^{s+1}(\hat{D}))$ and a constant $\lambda > 0$ such that*

$$\Delta \theta = 0 \quad \text{in } D \times [0, 1], \quad (4.4.1)$$

$$\frac{\partial \theta}{\partial n} = 0 \quad \text{on } (\Gamma \setminus \Gamma_0) \times [0, 1], \quad (4.4.2)$$

$$\operatorname{supp} \theta \subset \hat{D} \times (0, 1), \quad (4.4.3)$$

$$\phi^{\nabla \theta + (c, 0)}(x_0, 1) \notin \bar{D} \text{ for any } |c| < \lambda. \quad (4.4.4)$$

This lemma is proved at the end of this subsection.

Proof of Proposition 4.3.3. It follows from Lemma 4.4.1 that there are functions $\tilde{\theta}^i \in C^\infty([0, 1], \dot{H}^{s+1}(\hat{D}))$ and open balls $B^i = B(x_i, r_i) \subset \mathbb{R}^2$, $i = 1, \dots, N$ covering the rectangle $[0, 1] \times [-1, 1]$ such that properties (4.3.11)-(4.3.13) and (4.3.16) are verified for $\tau_i = 1$. For $i = 1, \dots, N$ let us take

$$\tau_i := i \sup_{t \in [0, 1]} \|\nabla \tilde{\theta}^i(\cdot, t)\|_{s, \hat{D}}, \quad (4.4.5)$$

$$\theta^i(x, t) := \frac{\tilde{\theta}^i(x, \frac{t}{\tau_i})}{\tau_i}. \quad (4.4.6)$$

Then B^i, τ_i and θ^i verify (4.3.9)-(4.3.16) for $i = 1, \dots, N$. Moreover, there are closed balls $\tilde{B}_1, \tilde{B}_2 \subset \hat{D} \setminus \bar{D}$ such that

$$\cup_{i=1}^N \phi^{\nabla \theta^i + (c, 0)}(B^i, \tau_i) \subset \tilde{B}_1 \cup \tilde{B}_2.$$

We denote $B^{2kN+j} := B(x_j, r_j) + (k, 0)$ and $B^{(2k+1)N+j} := B(x_j, r_j) - (k+1, 0)$, $j = 1, \dots, N$. Then properties (4.3.9) and (4.3.10) are satisfied. Let $h \in C^\infty([0, 1])$ be such that

$$\begin{aligned} h(t) &= 0 & \text{for any } t \in [0, 1/4], \\ h(t) &= 1 & \text{for any } t \in [3/4, 1], \\ |h(t)| &\leq 1 & \text{for any } t \in [0, 1]. \end{aligned}$$

For any $x = (x_1, x_2) \in \hat{D}$ and $c \in \mathbb{R}$ define

$$\tilde{\theta}^{2kN+j}(x, t) = \begin{cases} (-k - c)x_1 h'(t) & \text{for } t \in [0, 1], \\ \tilde{\theta}^j(x, t - 1) & \text{for } t \in [1, 2]. \end{cases} \quad (4.4.7)$$

It follows from the constructions of $\tilde{\theta}^j$, $j = 1, \dots, N$ that (4.3.11)-(4.3.13) are verified for $\tau_i = 2$. It is easy to see that for any $t \in [0, 1]$ we have

$$\phi^{\nabla \tilde{\theta}^{2kN+j} + (c, 0)}(x, t) = (-k - c, 0)h(t) + (c, 0)t + x. \quad (4.4.8)$$

Thus $\phi^{\nabla \tilde{\theta}^{2kN+j} + (c, 0)}(B^{2kN+j}, 2) = \phi^{\nabla \tilde{\theta}^j + (c, 0)}(B^j, 1) \not\subset \bar{D}$, which implies (4.3.16) and (4.3.17). Notice that $\nabla \tilde{\theta}^i \in \mathcal{X}^s(\hat{D})$. In order to have also (4.3.14) and (4.3.15), we define τ_i by (4.4.5) and

$$\theta^i(x, t) := \frac{2\tilde{\theta}^i(x, \frac{2t}{\tau_i})}{\tau_i}. \quad (4.4.9)$$

This completes the proof. \square

Proof of Lemma 4.4.1. The proof is based on the ideas of [14, Lemma A.1].

Step 1. We denote by \mathcal{A} the vector space of functions $\xi \in \dot{H}^{s+1}(\hat{D})$ with the following properties

$$\begin{aligned} \Delta \xi &= 0 \text{ in } D, \\ \frac{\partial \xi}{\partial n} &= 0 \text{ on } \Gamma \setminus \Gamma_0, \\ \text{supp } \xi &\subset \hat{D}. \end{aligned} \tag{4.4.10}$$

First, let us show that for any $x_0 \in D \cup \Gamma_0$ we have

$$\mathbb{R}^2 = \{\nabla \xi(x_0) : \xi \in \mathcal{A}\}. \tag{4.4.11}$$

Suppose that (4.4.11) does not hold. Then, there is a vector $V \in \mathbb{R}^2$, $V \neq 0$ such that

$$V \cdot \nabla \xi(x_0) = 0$$

for all $\xi \in \mathcal{A}$. Let \tilde{D} be the domain defined in (4.2.1) and let $\tilde{D} \subset D_1$. Take any $a \in \tilde{D} \setminus \bar{D}$, and let G_a be the solution of (4.2.10), (4.2.11). Let $B_1, B_2 \subset \tilde{D} \setminus \bar{D}$ be two open neighborhoods of a such that $\bar{B}_1 \subset B_2$ and let $\rho \in C^\infty(\tilde{D})$ be such that

$$\rho(x) = \begin{cases} 1, & \text{if } x \notin \bar{B}_2, \\ 0, & x \in B_1. \end{cases}$$

Clearly $\pi(\rho G_a) \in \mathcal{A}$, thus $V \cdot \nabla \pi(\rho G_a)(x_0) = 0$. Since $x_0 \notin \bar{B}_2$, we have

$$V \cdot \nabla G_a(x_0) = 0 \tag{4.4.12}$$

for all $a \in \tilde{D} \setminus \bar{D}$. On the other hand G_a is analytic in $a \in \tilde{D} \setminus \{x_0\}$ (see Proposition 4.2.5, (iii)). Thus, we have (4.4.12) for all $a \in \tilde{D} \setminus \{x_0\}$. Using (4.2.12), one can find a sequence $a_n \rightarrow x_0$ such that $V \cdot \nabla G_{a_n}(x_0) \rightarrow \infty$ as $n \rightarrow \infty$, which is a contradiction to $V \neq 0$.

Step 2. Take any $x_0 \in D \cup \Gamma_0$, $x^1 \in \hat{D} \setminus \bar{D}$ and let $F : [0, 1] \rightarrow \hat{D}$ be a continuous function such that

$$\begin{aligned} F(t) &= x_0 & \text{for any } t \in [0, 1/4], \\ F(t) &= x^1 & \text{for any } t \in [3/4, 1], \\ F(t) &\notin \Gamma \setminus \Gamma_0 & \text{for any } t \in [0, 1]. \end{aligned}$$

Then for any $\varepsilon > 0$ we can find $\xi_i \in \mathcal{A}$, $h_i \in C^\infty([0, 1])$, $i = 1, \dots, k$ with $\text{supp } h_i \subset [1/4, 3/4]$ such that for $\theta(x, t) := \sum_{i=1}^k \xi_i(x) h_i(t)$ we have

$$|F(t) - \phi^{\nabla \theta}(x_0, t)| < \varepsilon \tag{4.4.13}$$

for any $t \in [0, 1]$. It is easy to see that there is a constant $\lambda > 0$ such that for any $|c| < \lambda$

$$|\phi^{\nabla \theta}(x_0, t) - \phi^{\nabla \theta + (c, 0)}(x_0, t)| < \varepsilon. \tag{4.4.14}$$

Since $\xi_i \in \mathcal{A}$ and $\text{supp } h_i \subset [1/4, 3/4]$, we have (4.4.1)-(4.4.3). The construction of F , inequalities (4.4.13) and (4.4.14) imply $\phi^{\nabla\theta+(c,0)}(x_0, 1) \notin \overline{D}$ for sufficiently small $\varepsilon > 0$.

Step 3. It remains to study the case $x_0 \in \Gamma \setminus \Gamma_0$. Let $y_0 \in \Gamma_0$ and $k \in \mathbb{R}$ be such that $x_0 = y_0 + (k, 0)$. Then, the function

$$\theta(x, t) = \begin{cases} (-c - k)x_1 h'(t) & \text{for } t \in [0, 1/2], \\ 2\theta_{y_0}(x, 2(t - 1/2)) & \text{for } t \in [1/2, 1] \end{cases}$$

satisfies (4.4.1)-(4.4.4), where $h \in C^\infty([0, 1/2])$ is any function with $h(0) = 0$, $h(1/2) = 1$ and θ_{y_0} is the function constructed in Step 2 for $y_0 \in \Gamma_0$. \square

4.4.2 Proof of Proposition 4.3.4

For any $m \in \mathbb{R}_+$, let us denote

$$D_-^m := (-\infty, -m] \times [-2, 2] \text{ and } D_+^m := [m, +\infty) \times [-2, 2]. \quad (4.4.15)$$

We shall need the following lemma.

Lemma 4.4.2. *The functions θ^i constructed in the proof of Proposition 4.3.3 are such that there exist $\varphi^i \in C(\mathbb{R}_+)$ with*

$$\sup_{x \in \overline{D}} |\phi^{\nabla\theta^i+(c,0)}(x, t) - x| \leq \left[\frac{i}{2N} \right] + M \text{ for any } t \in [0, \tau_i], \quad (4.4.16)$$

$$|\nabla\theta^i(x, t) - \nabla\theta^i(y, t)| \leq \frac{\varphi^i(t)}{(m+1)^2} |x - y| \text{ for any } x, y \in D_+^m \text{ or } x, y \in D_-^m, \quad (4.4.17)$$

where $\int_0^{\tau_i} \varphi^i(t) dt \leq M$, N is the integer introduced in Proposition 4.3.3 and $M \in \mathbb{R}$ does not depend on i .

Démonstration. It is easy to see that (4.4.7) and (4.4.9) imply

$$\phi^{\nabla\theta^i+(c,0)}(x, t) = \begin{cases} (-k - c, 0)h(\frac{2t}{\tau_i}) + (c, 0)\frac{2t}{\tau_i} + x & \text{for } t \in [0, \tau_i/2], \\ \phi^{\nabla\tilde{\theta}^i+(c,0)}(x, \frac{2t}{\tau_i} - 1) & \text{for } t \in [\tau_i/2, \tau_i]. \end{cases}$$

where $k = \lfloor \frac{i}{2N} \rfloor$ and $j = i - 2Nk$. This yields (4.4.16) for a sufficiently large M . To prove (4.4.17), notice that in the proof of Lemma 4.4.1, the functions θ can be chosen such that

$$\|x_1^2 \partial^\beta \theta\|_{s, \hat{D}} < C,$$

where $|\beta| = 2$. Indeed, since Proposition 4.2.5 implies that the second order derivatives of G_a belong to $\mathcal{S}(D)$, one can replace (4.4.11) by

$$\mathbb{R}^2 = \{\nabla\xi(x_0) : \xi \in \mathcal{A} \text{ and } \|x_1^2 \partial^\beta \xi\|_{s, \hat{D}} < \infty, \quad |\beta| = 2\}.$$

Hence, we can find a constant M_1 such that

$$\sup_{i=1,\dots,N,|\beta|=2} \int_0^1 \|x_1^2 \partial^\beta \tilde{\theta}^i(t, \cdot)\|_{L^\infty(\hat{D})} dt < M_1.$$

Combining this with (4.4.7) and (4.4.9), we get (4.4.17). \square

Now we return to the proof of Proposition 4.3.4. It suffices to show that for any $\varepsilon > 0$ there is $\nu > 0$ such that the inequality

$$\sup_{x \in B_i} |\phi^u(x, t) - \phi^{\bar{u}}(x, t)| \leq \varepsilon \quad (4.4.18)$$

holds for any $i \geq 1$ and $t \in [0, t_{i-1/2}]$. Let us denote

$$\begin{aligned} X(t) &= \phi^u(x, t), \\ Y(t) &= \phi^{\bar{u}}(x, t), \end{aligned}$$

where $x \in B^i$. We shall prove (4.4.18) in the case when i is even. The proof when i is odd is similar. Let $k := \lfloor \frac{i}{2N} \rfloor$, then

$$B^i \subset [k-2, k+3] \times [-2, 2]. \quad (4.4.19)$$

Step 1. First let us show that to establish (4.4.18) it suffices to prove that

$$|X(t) - Y(t)| < 1 \text{ for all } t \in \mathbb{R}_+. \quad (4.4.20)$$

It is easy to see that

$$\begin{aligned} \partial_t (X(t) - Y(t)) &= u(X(t), t) - \bar{u}(Y(t), t) \\ &= (u(X(t), t) - \bar{u}(X(t), t)) + (\bar{u}(X(t), t) - \bar{u}(Y(t), t)) =: I_1(t) + I_2(t). \end{aligned} \quad (4.4.21)$$

We have that

$$\int_0^\infty |I_1(t)| dt \leq \nu. \quad (4.4.22)$$

From (4.3.19), (4.3.20), (4.4.16) and (4.4.19) it follows that

$$Y(t) \in [k-2 - \lfloor \frac{j}{2N} \rfloor - M, k+3 + \lfloor \frac{j}{2N} \rfloor + M] \times [-2, 2]$$

for any $t \in [0, t_{j-1/2}]$. Hence, (4.4.20) implies

$$X(t) \in [k-3 - \lfloor \frac{j}{2N} \rfloor - M, k+4 + \lfloor \frac{j}{2N} \rfloor + M] \times [-2, 2].$$

We derive from (4.3.19), (4.3.20) and (4.4.17) that

$$\int_0^{t_{i-1/2}} |I_2(t)| dt \leq \int_0^{t_{i-1/2}} \Psi(t) |X(t) - Y(t)| dt, \quad (4.4.23)$$

where

$$\Psi(t) = \begin{cases} \frac{\varphi^j(t-t_{j-1})}{(k-2-\lfloor \frac{j}{2N} \rfloor - M)^2}, & t \in [t_{j-1}, t_{j-1/2}], \\ \frac{\varphi^j(t_j-t)}{(k-2-\lfloor \frac{j}{2N} \rfloor - M)^2}, & t \in [t_{j-1/2}, t_j] \end{cases}$$

for $j < 2N(k-3-M)$ (here we use (4.4.17) for $m = k-3 - \lfloor \frac{j}{2N} \rfloor - M$) and

$$\Psi(t) = \begin{cases} \varphi^j(t), & t \in [t_{j-1}, t_{j-1/2}], \\ \varphi^j(t_j-t), & t \in [t_{j-1/2}, t_j] \end{cases}$$

for $j \geq 2N(k-3-M)$ (in this case we use (4.4.17) for $m = 0$). Thus we have

$$\begin{aligned} \int_0^{t_{i-1/2}} \Psi(t) dt &= \int_0^{t_{2N(k-3-M)-1}} \Psi(t) dt + \int_{t_{2N(k-3-M)-1}}^{t_{i-1/2}} \Psi(t) dt \\ &\leq \sum_{j=1}^{2N(k-3-M)-1} \frac{2M}{(k-2-\lfloor \frac{j}{2N} \rfloor - M)^2} + (2N(M+4) + 1)2M. \end{aligned} \quad (4.4.24)$$

Integrating (4.4.21), using (4.4.22)-(4.4.24) and the Gronwall inequality, we obtain

$$\begin{aligned} &|X(t_{i-1/2}) - Y(t_{i-1/2})| \\ &\leq \nu \exp \left(\sum_{j=1}^{2N(k-3-M)-1} \frac{2M}{(k-2-\lfloor \frac{j}{2N} \rfloor - M)^2} + (2N(M+4) + 1)2M \right) \\ &\leq \nu \exp \left(\sum_{j=1}^{\infty} \frac{8MN^2}{j^2} + (2N(M+4) + 1)2M \right). \end{aligned} \quad (4.4.25)$$

Choosing ν such that the right-hand side of (4.4.25) is smaller than ε , we prove (4.4.18) for all i .

Step 2. To complete the proof, it remains to show (4.4.20). To this end, let us assume that (4.4.20) does not hold for some $t > 0$. Denote by \tilde{t}_0 the first time such that $|X(\tilde{t}_0) - Y(\tilde{t}_0)| = 1$. Hence, we have (4.4.20) for all $t < \tilde{t}_0$. Step 1 implies

$$|X(\tilde{t}_0) - Y(\tilde{t}_0)| \leq \nu \exp \left(\sum_{j=1}^{\infty} \frac{8MN^2}{j^2} + (2N(M+4) + 1)2M \right). \quad (4.4.26)$$

Since the right-hand side of (4.4.26) does not depend on \tilde{t}_0 , choosing ν sufficiently small, we get (4.4.20).

4.5 Appendix : proof of Lemma 4.2.3

Let us consider the space

$$H_0(D') = \{z \in L^2(D') : \operatorname{curl} z \in L^2(D'), \operatorname{div} z \in L^2(D'), z \cdot n|_{\Gamma'} = 0\}$$

endowed with the norm

$$\|z\|_{H_0} = \|z\| + \|\operatorname{curl} z\| + \|\operatorname{div} z\|.$$

Here D' is a strip or is the domain \tilde{D} defined in (4.2.1). Recall the following result (see [18, Chapter 7, Theorem 6.1]).

Theorem 4.5.1. *The following equality holds*

$$\{z \in H^1(D') : z \cdot n|_{\Gamma'} = 0\} = H_0.$$

In the case of bounded domains it is shown in [44, Appendix 1, Proposition 1.4] that

$$H^s(\Omega) = \{z \in L^2(\Omega) : \operatorname{curl} z \in H^{s-1}(\Omega), \operatorname{div} z \in H^{s-1}(\Omega), z \cdot n \in H^{s-1/2}(\partial\Omega)\}. \quad (4.5.1)$$

Let us generalize this result to the case of domain D' . We shall need the following lemma.

Lemma 4.5.2. *Let $g \in H^{1/2}(\Gamma')$. Then the problem*

$$\Delta u - u = 0 \quad \text{in } D', \quad (4.5.2)$$

$$\frac{\partial u}{\partial n} = g \quad \text{on } \Gamma' \quad (4.5.3)$$

has a unique solution $u \in H^2(D')$, which satisfies

$$\|u\|_2 \leq C\|g\|_{1/2}. \quad (4.5.4)$$

Démonstration. Problem (4.5.2), (4.5.3) is equivalent to

$$\int_{D'} \nabla u \nabla \theta \, dx + \int_{D'} u \theta \, dx = \int_{\Gamma'} g \theta \, d\sigma \quad \text{for any } \theta \in H^1(D').$$

Since $g \in H^{-1/2}(\Gamma')$, the Riesz representation theorem implies the existence of a unique solution $u \in H^1(D')$.

Case 1. Assume $D' = D$, and let us prove that $u \in H^2(D)$. It is easy to see that $v := \partial_1 u$ is the solution of the problem

$$\Delta v - v = 0 \quad \text{in } D,$$

$$\frac{\partial v}{\partial n} = \partial_1 g \quad \text{on } \Gamma.$$

Thus $\partial_1 u \in H^1(D)$ and

$$\|\partial_1 u\|_1 \leq C\|g\|_{1/2}.$$

Combining this with the fact that $\Delta u \in H^1(D)$, we obtain $u \in H^2(D)$ and (4.5.4).

Case 2. Now consider the case $D' = \tilde{D}$. Let

$$\Omega_1 := \{x \in \tilde{D} : |x_1| < N\} \text{ and } \Omega_2 := \{x \in \tilde{D} : |x_1| < N + 1\},$$

where N is so large that $\tilde{D} \setminus D \subset \Omega_1$. Let us take some function $\chi \in C^\infty(\overline{\tilde{D}})$ such that

$$\chi(x) = \begin{cases} 0, & \text{if } x \notin \overline{\Omega_2}, \\ 1, & \text{if } x \in \Omega_1. \end{cases}$$

Then $w := \chi u$ is the solution of

$$\Delta w - w = 2\nabla\chi\nabla u + \Delta\chi u =: f \quad \text{in } \Omega_2, \quad (4.5.5)$$

$$\frac{\partial w}{\partial n} =: \tilde{g} \quad \text{on } \partial\Omega_2. \quad (4.5.6)$$

It is easy to see that $f \in L^2(\Omega_2)$ and $\tilde{g} \in H^{1/2}(\partial\Omega_2)$. This implies that $w \in H^2(\Omega_2)$ (e.g., see [1]). Thus $u \in H^2(\Omega_1)$. On the other hand, from the fact $\Gamma_0 \subset \Omega_1$ we derive $\frac{\partial u}{\partial n}|_\Gamma \in H^{1/2}(\Gamma)$. Hence, using the result for $D' = D$, we see that $u \in H^2(D)$. This completes the proof of Lemma 4.5.2. \square

Now let us prove (4.5.1) for $\Omega = D'$. Clearly the space in the left-hand side is contained in the right-hand side of (4.5.1). By induction, let us show the other inclusion. Assume $s = 1$. Let us take some function z from the right-hand side of (4.5.1) and consider the problem :

$$\begin{aligned} \Delta p - p &= 0 \quad \text{in } D, \\ \frac{\partial p}{\partial n} &= z \cdot n \quad \text{on } \Gamma. \end{aligned}$$

By Lemma 4.5.2, we have $p \in H^2(D')$ and $\|p\|_2 \leq C\|z \cdot n\|_{1/2}$. Let us take $w = z - \nabla p$. Clearly $w \in H_0$, thus Theorem 4.5.1 implies $w \in H^1(D')$. Hence, $z \in H^1(D')$ and

$$\|z\|_1 \leq \|w\|_1 + \|p\|_2 \leq C(\|z\| + \|\text{curl } z\| + \|\text{div } z\| + \|z \cdot n\|_{1/2}). \quad (4.5.7)$$

Now assume that (4.5.1) holds for $s - 1$ and let us prove it for s . Let \tilde{n} be a regular extension of n in D' such that $|\tilde{n}(x)| = 1$. Let us show that such an extension exists. To simplify the proof, let us assume that $d = 0$ in the definition of \tilde{D} (see (4.2.1)). We define

$$\begin{aligned} \tilde{n}_1(x_1, x_2) &= -\frac{\gamma'(x_1)}{\sqrt{1 + \gamma'(x_1)^2}} + h(x_1, x_2), \\ \tilde{n}_2(x_1, x_2) &= \frac{x_2}{(1 + \gamma(x_1))\sqrt{1 + \gamma'(x_1)^2}}, \end{aligned}$$

where $h \in C_b^\infty(\overline{\tilde{D}})$, $h|_{\partial\tilde{D}} = 0$ and $h(x_1, 0) = 1 + \frac{\gamma'(x_1)}{\sqrt{1 + \gamma'(x_1)^2}}$. Then we have $(\tilde{n}_1, \tilde{n}_2)|_{\partial\tilde{D}} = n$ and $|(\tilde{n}_1, \tilde{n}_2)| > \delta$ for sufficiently small $\delta > 0$. Hence, $\tilde{n}(x) = \frac{(\tilde{n}_1, \tilde{n}_2)}{|(\tilde{n}_1, \tilde{n}_2)|}$

is an extension of n . Let us take $v := \nabla^\perp(z \cdot \tilde{n})$. Then $v \in L^2$, $\operatorname{div} v = 0$. Since $v \cdot \tilde{n}$ is the tangential derivative of $z \cdot \tilde{n}$ along Γ' , we have $v \cdot \tilde{n} \in H^{s-3/2}(\Gamma')$. On the other hand

$$-\operatorname{curl} v = \Delta(z \cdot \tilde{n}) = (\Delta z_1)\tilde{n}_1 + (\Delta z_2)\tilde{n}_2 + \tilde{v},$$

where $\tilde{v} \in H^{s-2}$. It follows from the facts $\Delta z_1 = \partial_1 \operatorname{div} z + \partial_2 \operatorname{curl} z$ and $\Delta z_2 = \partial_2 \operatorname{div} z - \partial_1 \operatorname{curl} z$ that $\operatorname{curl} v \in H^{s-2}$. Thus the induction hypothesis yields $\nabla^\perp(z \cdot \tilde{n}) \in H^{s-1}$. Hence,

$$\begin{aligned} (\partial_2 z_1)\tilde{n}_1 + (\partial_2 z_2)\tilde{n}_2 &\in H^{s-1}, \\ (\partial_1 z_1)\tilde{n}_1 + (\partial_1 z_2)\tilde{n}_2 &\in H^{s-1}. \end{aligned}$$

Combining this with $\operatorname{div} z \in H^{s-1}$ and $\operatorname{curl} z \in H^{s-1}$, we obtain $\tilde{n} \cdot \nabla^\perp z_i \in H^{s-1}$ and $\tilde{n} \cdot \nabla z_i \in H^{s-1}$ for $i = 1, 2$. Thus $\nabla z_i \in H^{s-1}$, which completes the proof.

Références

- [1] S. Agmon, A. Douglis, and L. Nirenberg. Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. II. *Comm. Pure Appl. Math.*, 17 :35–92, 1964. (Cité en page 97.)
- [2] A. Agrachev and A. Sarychev. Reduction of a smooth system that is linear with respect to the control,. *Mat. Sb. (N.S.)*, 130(172) :18–34, 1986. (Cité en page 7.)
- [3] A. Agrachev and A. Sarychev. Navier–Stokes equations controllability by means of low modes forcing. *J. Math. Fluid Mech.*, 7 :108–152, 2005. (Cité en pages 6, 25, 29, 36, 47, 48, 60 et 69.)
- [4] A. Agrachev and A. Sarychev. Controllability of 2D Euler and Navier–Stokes equations by degenerate forcing. *Comm. Math. Phys.*, 265(3) :673–697, 2006. (Cité en pages 6, 25, 29, 36, 47, 48 et 60.)
- [5] J. M. Ball, J. E. Marsden, and M. Slemrod. Controllability for distributed bilinear systems. *SIAM J. Control Optim.*, 20(4) :575–597, 1982. (Cité en page 11.)
- [6] C. Bardos and U. Frisch. Finite-time regularity for bounded and unbounded ideal incompressible fluids using Hölder estimates. *Lecture Notes in Math.*, 565 :1–13, 1976. (Cité en page 78.)
- [7] J. T. Beale, T. Kato, and A. Majda. Remarks on the breakdown of smooth solutions for the 3-D Euler equations. *Comm. Math. Phys.*, 94(3) :61–66, 1984. (Cité en pages 3, 24, 27 et 80.)
- [8] H. Beirão da Veiga. Perturbation theorems for linear hyperbolic mixed problems and applications to the compressible Euler equations. *Comm. Pure Appl. Math.*, 46(2) :221–259, 1993. (Cité en page 52.)
- [9] J. L. Bona and R. Smith. The initial-value problem for the Korteweg–de Vries equation. *Philos. Trans. Roy. Soc. London Ser. A*, 278(1287) :555–601, 1975. (Cité en page 78.)
- [10] M. Chapouly. On the global null controllability of a Navier-Stokes system with Navier slip boundary conditions. *Journal of Differential Equations*, 247(7) :2094–2123, 2009. (Cité en page 69.)
- [11] P. Constantin and C. Foias. Navier–Stokes Equations. *University of Chicago Press, Chicago*, 1988. (Cité en page 26.)
- [12] J.-M. Coron. Global asymptotic stabilization for controllable systems without drift. *Math. Control Signals Systems*, 5 :295–312, 1992. (Cité en pages 6 et 69.)
- [13] J.-M. Coron. On the controllability of 2-D incompressible perfect fluids. *J. Math. Pures Appl.*, 75(2) :155–188, 1996. (Cité en pages 6, 17, 25, 47 et 69.)

- [14] J.-M. Coron. On the controllability of the 2-D incompressible Navier–Stokes equations with the Navier slip boundary conditions. *J. ESAIM Control Optim. Calc. Var.*, 1 :35–75, 1996. (Cité en pages 19 et 91.)
- [15] J.-M. Coron. On the null asymptotic stabilization of 2-D incompressible Euler equation in a simply connected domain. *SIAM J. Control Optim.*, 37(6) :1874–1896, 1999. (Cité en pages 6 et 69.)
- [16] J.-M. Coron. *Control and nonlinearity*, volume 136. Mathematical Surveys and Monographs, 2007. (Cité en page 69.)
- [17] J.-M. Coron and A. V. Fursikov. Global exact controllability of the 2D Navier–Stokes equations on a manifold without boundary. *Russian J. Math. Phys.*, 4(4) :429–448, 1996. (Cité en pages 6 et 47.)
- [18] G. Duvaut and J.-L. Lions. *Les inéquations en mécanique et en physique*. Dunod, Paris, 1972. (Cité en page 96.)
- [19] D. G. Ebin and J. E. Marsden. Groups of diffeomorphisms and the motion of an incompressible fluid. *Ann. of Math.*, 92, :102–163, 1970. (Cité en page 3.)
- [20] D. E. Edmunds and H. Triebel. *Function Spaces, Entropy Numbers, Differential Operators*. Cambridge University Press, Cambridge, UK, 1996. (Cité en page 43.)
- [21] E. Fernández-Cara, S. Guerrero, O. Yu. Imanuvilov, and J. P. Puel. Local exact controllability of the Navier–Stokes system. *J. Math. Pures Appl.*, 83(12) :1501–1542, 2004. (Cité en pages 6, 25 et 47.)
- [22] A. V. Fursikov and O. Yu. Imanuvilov. Exact controllability of the Navier–Stokes and Boussinesq equations. *Russian Math. Surveys*, 54(3) :93–146, 1999. (Cité en pages 6, 25, 47 et 69.)
- [23] O. Glass. Exact boundary controllability of 3-D Euler equation. *ESAIM Control Optim. Calc. Var.*, 5 :1–44, 2000. (Cité en pages 6, 25, 47 et 69.)
- [24] O. Glass. On the controllability of the 1-D isentropic Euler equation. *J. Eur. Math. Soc. (JEMS)*, 9(3) :427–486, 2007. (Cité en pages 13 et 47.)
- [25] O. Glass. Asymptotic stabilizability by stationary feedback of the two-dimensional Euler equation : the multiconnected case. *SIAM J. Control Optim.*, 44(3) :1105–1147, 2005. (Cité en pages 6 et 69.)
- [26] O. Glass and L. Rosier. On the control of the motion of a boat. *Preprint*, 2011. (Cité en page 69.)
- [27] O. Yu. Imanuvilov. Remarks on exact controllability for the Navier–Stokes equations. *ESAIM Control Optim. Calc. Var.*, 6 :39–72, 2001. (Cité en pages 6 et 47.)
- [28] V. Jurdjevic. *Geometric control theory*. 52 :xviii+492, 1997. (Cité en page 64.)
- [29] T. Kato. On classical solutions of the two-dimensional non-stationary Euler equation. *Arch. Rational Mech. Anal.*, 25 :188–200, 1967. (Cité en pages 3 et 75.)

- [30] T. Kato. The Cauchy problem for quasi-linear symmetric hyperbolic systems. *Arch. Rational Mech. Anal.*, 58(3) :181–205, 1975. (Cité en pages 5, 46 et 49.)
- [31] T. Kato and G. Ponce. Commutator estimates and the Euler and Navier–Stokes equations. *Comm. Pure Appl. Math.*, 41(7) :891–907, 1988. (Cité en page 75.)
- [32] L. Lichtenstein. Über einige Existenzprobleme der Hydrodynamik homogener, unzusammendrückbarer, reibungsloser Flüssigkeiten und die Helmholtzchen Wirbelsätze *Mathematische Zeitschrift*, 23 :89–154, 1925 ; 26 :196–323 1927 ; 28 :387–415, 1928 and 32 :608–640, 1930. (Cité en page 3.)
- [33] T. Li and B. Rao. Exact boundary controllability for quasi-linear hyperbolic systems. *SIAM J. Control Optim.*, 41(6) :1748–1755, 2003. (Cité en pages 13 et 47.)
- [34] G. Lorentz. Approximation of Functions. *Chelsea Publishing Co., New York*, 1986. (Cité en pages 10 et 42.)
- [35] V. Nersisyan and H. Nersisyan. Global exact controllability in infinite time of multidimensional Schrödinger equation. *Accepté pour publication dans J. Math. Pures Appl.*, 2011. (Cité en page 10.)
- [36] H. Nersisyan. Controllability of 3D incompressible Euler equations by a finite-dimensional external force. *ESAIM Control Optim. Calc. Var.*, 16(3) :677–694, 2010. (Cité en pages 47, 48, 64 et 69.)
- [37] H. Nersisyan. Controllability of the 3D compressible Euler system. *Communications in Partial Differential Equations*, 36(9) : 1544–1564, 2011. (Cité en page 12.)
- [38] S. S. Rodrigues. Navier–Stokes equation on the rectangle : controllability by means of low mode forcing. *J. Dyn. Control Syst.*, 12(4) :517–562, 2006. (Cité en pages 6, 25 et 47.)
- [39] A. Shirikyan. Approximate controllability of three-dimensional Navier–Stokes equations. *Comm. Math. Phys.*, 266(1) :123–151, 2006. (Cité en pages 3, 6, 9, 25, 29, 30, 36, 47, 48, 60 et 69.)
- [40] A. Shirikyan. Exact controllability in projections for three-dimensional Navier–Stokes equations. *Annales de l’IHP, Analyse Non Linéaire*, 24 :521–537, 2007. (Cité en pages 6, 25, 36, 41, 47, 48, 60 et 61.)
- [41] A. Shirikyan. Euler equations are not exactly controllable by a finite-dimensional external force. *Physica D*, 237 :1317–1323, 2008. (Cité en pages 7, 10, 25, 42, 43 et 47.)
- [42] M. E. Taylor. Partial Differential Equations, III. *Springer-Verlag, New York*, 1996. (Cité en pages 5, 7, 24, 26, 27, 46, 49 et 53.)
- [43] R. Temam. Local existence of C^∞ solution of the Euler equation of incompressible perfect fluids. *Lecture Notes in Mathematics*, 565 :184–194, 1976. (Cité en pages 3, 24 et 27.)
- [44] R. Temam. *Navier–Stokes equations. Theory and numerical analysis*, volume 2. North-Holland Publishing Co., 1977. (Cité en page 96.)

- [45] G. Turinici. On the controllability of bilinear quantum systems. *Lecture Notes in Chem.*, 74, 2000. (Cité en page 11.)
- [46] W. Wolibner. Un théorème sur l'existence du mouvement plan d'un fluide parfait, homogène, incompressible, pendant un temps infiniment long. *Math. Z.*, 146 :561–564, 1962. (Cité en pages 3 et 75.)
- [47] V. Yudovich. The flow of a perfect, incompressible liquid through a given region. *Dokl. Akad. Nauk SSSR*, 37(1) :698–726, 1933. (Cité en page 81.)