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Problèmes bien posés et diffusion pour des équations non linéaires dispersives d'ordre quatre

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Résumé

On étudie l'existence en grand temps et la diffusion pour des équations modèles non linéaires dispersives d'ordre quatre. D'une part l'équation des ondes d'ordre quatre

$$\partial_t^2 u + \Delta^2 u + mu = \lambda |u|^{p-1} u,$$

et d'autre part l'équation de Schrödinger bi-harmonique

$$i\partial_t u + \Delta^2 u + \varepsilon \Delta u = \lambda |u|^{p-1} u.$$

Pour l'équation des ondes on démontre la validité de la conjecture de Levandosky et Strauss selon laquelle, dans le cas sous-critique défocalisant, l'équation diffuse en énergie arbitraire. Pour l'équation de Schrödinger bi-harmonique on démontre dans le cas défocalisant critique radial l'existence en grand temps et la diffusion pour des données arbitrairement grandes en énergie. Dans le cas L^2-critique on obtient un profil asymptotique. Enfin dans le cas de la cubique défocalisante, pour des données non nécessairement radiales, on démontre que l'équation est bien posée dès lors que $n \leq 8$, qu'elle diffuse dans l'intervalle $5 \leq n \leq 8$, et enfin qu'elle est mal posée lorsque $n \geq 9$.

Contents

Chapter 1

Introduction

The thesis here consists of a compilation of two published papers, Pausader [51] published in the *Journal of differential equations* (editor W. Strauss) and Pausader [52] published in *Dynamics of partial differential equations* (editor T.Tao), and of three preprints submitted for publication Pausader [53], Pausader [54], and Pausader and Strauss [55].

The general topic of the works we present consists in the study of local and global behavior of solutions to nonlinear dispersive partial differential equations in the case of high dispersion, and most notably on scattering results in the defocusing case. We focus on two equations: the nonlinear beam equation, or fourth-order wave equation,

$$\partial_t^2 u + \Delta^2 u + mu = \lambda |u|^{p-1} u,$$

where $m > 0$, and the biharmonic Schrödinger equation, or fourth-order Schrödinger equation,

$$i\partial_t u + \Delta^2 u + \varepsilon \Delta u = \lambda |u|^{p-1} u,$$

where $\varepsilon \in \{\pm 1, 0\}$. In both cases the unknown is a function $u : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, or $\mathbb{C}$. The high dispersive feature of these equations are best seen in frequency space (i.e. using the Fourier transform), while the nonlinearity, especially if p is not an integer, is best understood in x-space. The study of these nonlinear equations demands that one takes into account both aspects simultaneously, and harmonic analysis naturally comes into play. In order to explain our results, we first discuss the nonlinear beam equation, and then the biharmonic Schrödinger equation. In the following, we have chosen to highlight one result for each article. For each theorem we refer to the corresponding article and the corresponding chapter in the thesis.

1.1 The nonlinear beam equation

First, we study the nonlinear beam equation, or fourth-order wave equation, which is written as follows:

$$\partial_t^2 u + \Delta^2 u + mu = \lambda |u|^{p-1} u, \tag{1.1.1}$$

where $1 < p < +\infty$ if the dimension n is smaller than 4, and $1 < p \leq 2^\sharp - 1$ otherwise, where $2^\sharp = 2n/(n-4)$. As a remark, $2^\sharp$ is critical for H^2 from the viewpoint of Sobolev embeddings. We say the equation is focusing when $\lambda > 0$ and defocusing when $\lambda < 0$. Equation (1.1.1) appears in different physical settings. It is involved in the study of plate and beams, see, e.g. Love [45], but it was also used as a model for the study of interaction of water waves, see Bretherton [7], and more recently in the study of the motion of a suspension bridge, see Lazer and McKenna [35] and McKenna and Walter [46, 47]. We also refer to the book by Peletier and Troy [56] for other references. This equation was intensively studied by Levandosky [39, 40] who proved local wellposedness in the subcritical case, and studied existence and stability of travelling waves and standing waves. Later on, Levandosky and Strauss [41] derived a Morawetz estimate and proposed a conjecture about the asymptotic behavior of solutions. Other possible references on the local well-posedness and scattering study of this equation are Cui and Guo [17], Lin [42] and Miao [49]. In another direction, we point out that control for the linear plate equation has been widely investigated. Possible references are Burq [8], Fu, Zhang, and Zuazua [16], Haraux [22], Jaffard [24], Lebeau [36, 37], Lions [44], Zhang [70], and Zuazua [71].

The beam equation enjoys several conservation laws and integral estimates which are of great importance in the study of global-in-time issues. On the other hand, it does not satisfy finite speed propagation. The most important conservation law enjoyed by the equation is the conservation of energy, namely

$$E(u,u_t) = \int_{\mathbb{R}^n} \left(\frac{|u_t|^2 + |\Delta u|^2 + m|u|^2}{2} - \frac{\lambda |u|^{p+1}}{p+1} \right) dx,$$

which, in case $\lambda < 0$, gives a global bound on the H^2 norm. Other conservation laws are the conservation of momentum, and conservation of charge for complex-valued solutions. Another very important a priori estimate is the Morawetz estimate, see (1.1.2) below. In order to exploit these laws, we work at the general level of strong H^2 solutions, that is solutions $u \in C(H^2) \cap C^1(L^2) \cap C^2(H^{-2})$ that satisfy (1.1.1) in H^{-2}. The formal structure of this equation is quite close to the structure of the Klein-Gordon equation, and this allows one to deduce various analogues of second-order theorems. For example, any initial data with negative energy will lead to a solution that blows up in finite time. As a rule of thumb, any result on the Klein-Gordon equation that does not rely on the dispersive properties of the equation (most notably the finite speed propagation), or on the conformal invariance properties, remains true for (1.1.1). However, the dispersive behavior of (1.1.1) is much different from the one of the wave equation. This equation enjoys a "generic" dispersion with decay of the homogeneous linear solution in L^∞-norm like $t^{-\frac{n}{2}}$ as predicted by standard stationary phase in the high frequency case ($|\xi| \geq 1$), and a higher "fourth-order" dispersion mode in the low frequency case ($|\xi| \leq 1$). We refer to the next section for a more precise description of the "fourth-order" dispersion. For small time, the contribution of the low frequencies can be ignored, and using the Schrödinger structure of the highest order part of (1.1.1) given by the decomposition

$$\partial_t^2 + \Delta^2 = (\partial_t + i\Delta)(\partial_t - i\Delta),$$

one can obtain good local-in-time space-time bounds on the solution and prove local well posedness for this equation. Our main focus, however lies in the analysis of the global behavior of solutions for equation (1.1.1).

While the local properties above are based solely on basic conservation laws and a rough dispersive study, to address the global questions one needs a better understanding both of the formal structure to derive more subtle a priori estimates and of the dispersive structure of the equation.

On what concerns the question of global behavior of solutions, Levandosky [40] proved scattering for initial data with small $H^2 \times L^2$-norm when $p \geq 1+8/n$ if $n \leq 4$, and $1 + 8/n \leq p < 1 + 8/(n-4)$ otherwise. Besides, Levandosky and Strauss showed the following a priori Morawetz estimates

$$\int_{\mathbb{R}} \int_{\mathbb{R}^n} \frac{|u(t,x)|^{p+1}}{|x|} dxdt \lesssim \sup_t \|u_t(t)\|_{L^2} \|\nabla u(t)\|_{L^2} \tag{1.1.2}$$

for u a solution of (1.1.1), when $\lambda < 0$ and $n \geq 5$. This was an important hint for suspecting scattering in case $n \geq 5$ and $1 + 8/n < p < 2^\sharp - 1$. This led Levandosky and Strauss to propose the following conjecture.

Conjecture 1 (LS conjecture)**.** *Scattering holds true in the energy space for the defocusing beam equation* (1.1.1) *when* $n \geq 5$ *and* $1 + 8/n < p < 1 + 8/(n-4)$.

Scattering for a nonlinear dispersive equation means that every nonlinear solution approaches a linear solution as time goes to infinity, and that the pairing nonlinear solution/asymptotic behavior (i.e associated linear solution) is bijective. This gives a neat description of the behavior of the nonlinear solutions, since in particular, they all decay to 0 in many norms. This also allows one to define the scattering operator which maps a behavior at time $-\infty$ (solution before self-interaction) to the behavior at $+\infty$ (solution after the process of self-interaction). As a remark, Levandosky and Strauss also conjectured that scattering holds true when $n \leq 4$ and $1 + \frac{8}{n} < p$.

The exponent $1+8/n$ can be seen as the L^2-critical exponent and $1+8/(n-4)$ is the $\dot{H}^2$-critical exponent. For small data, see Levandosky [40], scattering holds true in the whole range $1 + 8/n \leq p \leq 1 + 8/(n-4)$. In the focusing case, see Pausader [51], it can be proved that scattering does not hold true even for very small energies when $p < 1 + 8/n$. Needless to say, because of the existence of ground states, scattering does not hold true in general in the focusing case for arbitrary energies. In Pausader [51] we prove that scattering holds in the defocusing case $\lambda < 0$, in the natural range $1 + 8/n < p < 2^\sharp - 1$. In particular, we prove that Conjecture 1 is true. The main result of Pausader [51] states as follows:

Theorem 1 ([51], Chapter 2)**.** *Let* $n \geq 5$, $1 + 8/n < p < 1 + 8/(n-4)$, *and* $\lambda < 0$. *For any* $(u_0, u_1) \in H^2 \times L^2$, *there exists a unique globally defined strong solution* u *of* (1.1.1) *such that* $(u(0), \partial_t u(0)) = (u_0, u_1)$. *Besides, there exists a unique solution* $v \in C(H^2) \cap C^1(L^2)$ *of the linear equation* (1.1.1) *with* $\lambda = 0$ *such that*

$$\|u(t) - v(t)\|_{H^2} + \|\partial_t u(t) - \partial_t v(t)\|_{L^2} \to 0 \qquad (1.1.3)$$

as $t \to +\infty$. *In particular, Conjecture 1 holds true.*

As already mentioned, scattering cannot hold for all solutions in the focusing case. Below is a short explanation of how the proof in Pausader [51] goes. The first step is to obtain good global bounds on space-time integrals, i.e. Strichartz-type estimates for solutions of the linear equation. To achieve this, we need to take into account the fourth-order dispersion at the 0-frequency mode. The second step is to overcome the lack of local stability of the solution which was derived in the second-order dispersive case either from the conservation of the mass for the Schrödinger equation, or from the finite speed propagation for the wave equation. This is achieved in two steps. First, following a strategy initiated by Tao [63], we prove that global solutions remain almost bounded in frequency. Then, we use this to prove that nonlinear solutions satisfy an almost finite speed propagation principle which forces them to essentially live in large cones. The final step is to use the a priori estimate given by Levandosky and Strauss [41] to obtain decay for the nonlinear solution, in the spirit of Lin and Strauss [43], and then use again the linear decay to upgrade this and prove scattering.

Once scattering has been proved for the beam equation, one can define the scattering operator $\mathcal{S}$ as follows: let $(u_0^-, u_1^-) \in H^2 \times L^2$, and let v^- be the linear solution of (1.1.1) with initial data $(v^-(0), \partial_t v^-(0)) = (u_0^-, u_1^-)$. Then there exists a unique nonlinear solution u of (1.1.1) such that (1.1.3) holds for $v = v^-$ as $t \to -\infty$. Besides, by the above theorem, there exists v, a linear solution of (1.1.1) with $\lambda = 0$ such that (1.1.3) holds true. The scattering operator is defined by

$$\mathcal{S}(u_0^-, u_1^-) = (v(0), \partial_t v(0)). \tag{1.1.4}$$

It is shown in Pausader [51] that for fixed $m, \lambda < 0$, $\mathcal{S}$ is a homeomorhism of $H^2 \times L^2$ when $n \geq 5$ and $1 + 8/n < p < 1 + 8/(n-4)$. For general background on scattering, we refer to Strauss [61]. A natural question is then to study in depth the scattering operator. This is the purpose of Pausader and Strauss [55] which was written in collaboration with Walter Strauss. In this short paper we prove with a very direct argument that the scattering operator is as smooth as the nonlinearity allows. We also investigate its dependency with respect to the parameters, and prove that different choices of λ, p lead to different scattering operators. Finally, we discuss situations in which scattering cannot hold. The main result of Pausader and Strauss [55] is the following.

Theorem 2 ([55], Chapter 3)**.** *Let $n \geq 5$, $\lambda < 0$ and $1+8/n < p < 1+8/(n-4)$. Then $\mathcal{S}$ as defined in* (1.1.4) *is $C^{p-1,1}$ if p is an even integer, and analytic if p is an odd integer.*

Concerning nonexistence, we prove in Pausader and Strauss [55] that if the nonlinearity is not sufficiently flat around 0, namely, if $1 < p < 1 + 2/n$ and $n \geq 2$, then there exist finite energy linear solutions which cannot be approximated by nonlinear solutions. In particular, the scattering operator cannot be defined in the whole energy space. We refer to Baez and Zhou [1], Glassey [18], Kumlin [34] and Morawetz and Strauss [50] for previous results in this direction established for the wave and Klein-Gordon equations, and to Carles and Gallagher [10] for related results independently obtained.

1.2 The nonlinear Biharmonic Schrödinger equation

Fourth-order Schrödinger equations have been introduced by Karpman [25, 26] and Karpman and Shagalov [27] to take into account the role of small fourth-order dispersion terms in the propagation of intense laser beams in a bulk medium with Kerr nonlinearity. Sharp dispersive estimates for the biharmonic Schrödinger operator we consider, namely for the linear group associated to $i\partial_t + \Delta^2 + \varepsilon\Delta$, have recently been obtained in Ben-Artzi, Koch, and Saut [4], while specific nonlinear fourth-order Schrödinger equations have been recently discussed in Fibich, Ilan, and Papanicolaou [14], Guo and Wang [19], Hao, Hsiao, and Wang [20, 21], and Segata [60]. Fibich, Ilan and Papanicolaou [14] describe various properties of the equations in the subcritical regime, with part of their analysis relying on very interesting numerical developments. Guo and Wang [19] prove global well-posedness and scattering in H^s for small data. Hao, Hsiao and Wang [20, 21] discuss the Cauchy problem in a high-regularity setting. Segata [60] proves scattering in case the space dimension is one. Related equations also appeared in Fibich, Ilan, and Schochet [15], Huo and Jia [23], and Segata [58, 59].

Here again, we only consider the model case of the fourth order Schrödinger equation, namely

$$i\partial_t u + \Delta^2 u + \varepsilon \Delta u = \lambda |u|^{p-1} u, \tag{1.2.1}$$

where $\varepsilon \in \{\pm 1, 0\}$, $\lambda < 0$, $1 \leq p < +\infty$ if the dimension n is smaller than 4, and $1 \leq p \leq 2^\sharp - 1$ otherwise. Equation (1.2.1) looks formally close to the Schrödinger equation, and shares the important property of having two conserved quantities at different regularity level. It has conserved mass

$$M(u(t)) = \int_{\mathbb{R}^n} |u(t,x)|^2 dx = M(u(0)), \tag{1.2.2}$$

as well as conserved energy

$$E(u(t)) = \int_{\mathbb{R}^n} \left(\frac{|\Delta u(t,x)|^2 - \varepsilon |\nabla u(t,x)|^2}{2} - \frac{\lambda |u(t,x)|^{p+1}}{p+1} \right) dx = E(u(0)). \tag{1.2.3}$$

In order to exploit at best these two conservation laws, we work with strong H^2 solutions, that are functions $u \in C(H^2) \cap C^1(H^{-2})$ that satisfy (1.2.1) in H^{-2}. The conservations laws (1.2.2) and (1.2.3) allow, as in Fibich, Ilan and Papanicolaou [14], to prove global well-posedness in case $n \leq 3$ and $p < 1 + 8/n$.

As for the dispersion properties, we remark that equation (1.2.1) exhibits a fundamental fourth-order dispersion (pure fourth-order dispersion if $\varepsilon = 0$, combined fourth-order and second order dispersion if $\varepsilon < 0$, and competing fourth order and second order dispersion when $\varepsilon > 0$). The fourth-order dispersion scaling property leads to the heuristic that solutions of the free homogeneous equation (equation (1.2.1) with $\varepsilon = 0$ and $\lambda = 0$) have their L^∞ norm which decays like $t^{-\frac{n}{4}}$. However, the situation is not so transparent. In fact, all frequency parts of the function have their L^∞-norm that decays much faster, like $t^{-\frac{n}{2}}$, but at a rate which depends on the frequency, so that uniformly, the rate of decay $t^{-\frac{n}{4}}$ is the best possible, but it is not optimal when the solution is localized in frequency. This subtelty leads to various differences between the dispersion behaviors of the Schrödinger equation and of (1.2.1).

The local subcritical theory for (1.2.1) is an easy consequence of the dispersive estimates in Ben-Artzi, Koch and Saut [4], of the Strichartz-type estimates derived from Keel and Tao [28] and of compactness arguments derived for the Schrödinger equation. In contrast the local theory for the critical equation (1.2.1) in case $p = 2^\sharp - 1$ is slightly more subtle. Indeed, to obtain H^2-level space-time bounds for equation (1.2.1), one should derive the equation twice, and if one wants to prove some uniform continuity of the solution flow at the H^2-level, one should derive a third time. However, for high dimensions, the nonlinearity is not even $C^{1,1}$. To overcome this, in Pausader [52], it was shown that equation (1.2.1) has the surprising property of regularizing the data in L^p spaces, $p \neq 2$. Note that the linear flow of (1.2.1) is an isometry on any H^s space, and therefore cannot give any regularization on these spaces. This regularizing property is in sharp contrast with the second order Schrödinger equation, since it would violate Galilean invariance. In order to obtain uniform continuity of the solution flow, this regularizing effect is not even sufficient, and one needs to work with exotic norms to first bound a weaker scale invariant space time norm, and then upgrade this to a bound on the difference of the solutions.

The problem of local well posedness settled, one can now address the question of the global behavior. We investigate that problem in the defocusing case. In this case, global existence in the subcritical case $1 < p < 2^\sharp - 1$ is provided by the conservation of energy. Besides, one can derive an analogue of the Morawetz estimate, namely, when $\varepsilon \leq 0$, there holds that

$$\int_{\mathbb{R}} \int_{\mathbb{R}^n} \frac{|u(t,x)|^{p+1}}{|x|} dt dx \lesssim \sup_t \|u(t)\|^2_{\dot{H}^{\frac{1}{2}}} \tag{1.2.4}$$

for all solutions u of (1.2.1) when $n \geq 5$. One can use (1.2.4) together with the exact conservation of the L^2 mass (1.2.2), to prove that in dimensions $n \geq 5$, for $1 + 8/n < p < 2^\sharp - 1$, scattering holds for (1.2.1) when $\varepsilon \leq 0$. This is the analogous result of the scattering statement of the beam equation. The proof, however is much simpler, due to the L^2 conservation law.

However, in the $\dot{H}^2$-critical case, where $p = 2^\sharp - 1$, even global well posedness does not follow from the local theory and the conservation laws. A way to get global well posedness, together with scattering, is to prove that a suitable scale-invariant space-time norm is finite. In Pausader [52] we prove that there is such a norm and such a bound in the radially symmetrical case, globally when $\varepsilon \leq 0$, and locally when $\varepsilon = 1$. That the two cases $\varepsilon \leq 0$ and $\varepsilon > 0$ give rise to distinct behaviors was alreay noticed in Karpman [25]. This also corresponds to different decay rates of the fundamental solution, see Ben-Artzi, Koch and Saut [2]. More precisely, we prove the following in Pausader [52]:

Theorem 3 ([52], Chapter 4). *Let $n \geq 5$, $\lambda < 0$ and $p = 1 + 8/(n-4)$. For any radially symmetrical element $u_0 \in H^2$, there exists a unique global strong solution u of* (1.2.1) *such that $u(0) = u_0$. Besides we have that*

$$\|u\|_{L^{\frac{2(n+4)}{n-4}}(\mathbb{R}^{1+n})} \lesssim_{E(u_0)} 1$$

when $\varepsilon \leq 0$. Moreover, the solution scatters.

Here is a brief explanation of the proof. To work in the critical setting requires to manipulate only scale invariant quantities, therefore, all estimates which do not scale like the $\dot{H}^2$-norm, as the mass (1.2.2) and the Morawetz estimate (1.2.4) should be useless. However, estimates that scale like a subcritical norm $\dot{H}^s$ with $s < 2$, can be replaced by localized versions which are often weaker, but nevertheless allow some additional control on solutions. This leads to localized mass conservation laws and localized Morawetz estimates. Following a strategy initiated by Bourgain [5], and developed by Tao [62], we cut the maximal time of existence of a solution into "elementary events", use harmonic analysis technique to isolate a concentration of the solution in space-time on each such "event", and then use the localized conservation laws to get combinatoric laws on the organization of these "elementary events", and eventually bound their number.

Now, we discuss our result in Pausader [53] about the homogeneous L^2-critical case. The homogeneous L^2-critical fourth-order Schrödinger equation is written as

$$i\partial_t u + \Delta^2 u = \lambda |u|^{\frac{8}{n}} u, \tag{1.2.5}$$

where $\lambda \in \mathbb{R}$. The equation involves the borderline exponent $p = 1 + 8/n$ in the scattering range, and is thought of to be more difficult, partly because the remaining a priori bounds, energy conservation (1.2.3) and Morawetz estimates (1.2.4) now correspond to supercritical norms $\dot{H}^s$, $s > 0$. Note that even in the model case of Schrödinger equation, the scattering problem for the corresponding L^2-problem is not yet completely settled. In Pausader [53] we study the lack of compactness for equation (1.2.5). We give a structure theorem for bounded sequences of solutions. This structure theorem reduces the problem of establishing scattering to proving the nonexistence of certain solutions with very specific behavior. Previous work concerning the lack of compactness of the Strichartz estimates and its consequences can be found in Bahouri and Gerard [2] and Kenig and Merle [30] for the wave equation, Keraani [32] and Kenig and Merle [29] for the energy-critical Schrödinger equation, and Begout and Vargas [3], Bourgain [6], Carles and Keraani [11], Keraani [31], Killip Tao and Visan [33], Merle and Vega [48] and Tao, Visan and Zhang [64, 65] for the L^2-critical Schrödinger equation. The advantage to carry over this study at the L^2-level is that it can then be used for any higher regularity level $\dot{H}^s$, $s \geq 0$, with only minor additional work. Our main result in the defocusing case states as follows:

Theorem 4 ([53], Chapter 5)**.** *There exists $E = E(n) > 0$ such that a bubble tree decomposition for* (1.2.5) *holds true for all sequences of solutions of mass less than E. Moreover, if $E < +\infty$, then there exists a solution u of* (1.2.1) *such that $M(u) = E$ and*

$$\|u\|_{L^{\frac{2(n+4)}{n}}(I\times\mathbb{R}^n)} = \infty,$$

where I is the maximal interval of existence of u. This solution satisfies that there exist two functions h and x such that the set

$$K = \{h(t)^{\frac{n}{2}} u(t, h(t)(\cdot - x(t))) : t \in I\}$$

is precompact in L^2. Besides, we can suppose that one of the three following scenarios holds true: (soliton-like solution) there holds $I = \mathbb{R}$ and $h(t) = 1$ for all t; (double high-to-low cascade) there holds $I = \mathbb{R}$, $\limsup_{t\to\pm\infty} h(t) = +\infty$, and $h(t) \geq 1$ for all t; (self-similar solution) there holds $I = (0, +\infty)$ and $h(t) = t^{\frac{1}{4}}$ for all t.

In our last work Pausader [54] we show, in case $n = 8$ and $p = 3$, which corresponds to the energy-critical nonlinearity $p = 2^\sharp - 1$ in this dimension, how we can go beyond the radial restriction in Pausader [52] for the energy-critical equation. Here the viewpoint differs from the one in the two preceding sections. We fix the nonlinearity and study the behavior of the solution when the equation becomes subcritical, critical and supercritical. More precisely, we consider in Pausader [54] the cubic homogeneous fourth-order Schrödinger equation

$$i\partial_t u + \Delta^2 u + |u|^2 u = 0 \tag{1.2.6}$$

in arbitrary space dimension n. As already noticed, the equation is globally wellposed when $n \leq 7$ ($\dot{H}^2$-subcritical regime) and scatters when $n = 5, 6, 7$ (L^2-supercritical regime). In Pausader [54] we prove that when $n \geq 9$ (H^2-supercritical regime), local wellposedness does not hold and that, in the remaining case $n = 8$ ($\dot{H}^2$-critical regime), global wellposedness and scattering hold true in full generality in H^2 i.e. without any radial symmetry assumption. Our main result states as follows:

Theorem 5 ([54], Chapter 6)**.** *In dimensions $n \leq 8$, the fourth order Schrödinger equation* (1.2.6) *is globally wellposed in H^2, and for any t, the mapping $u(0) \mapsto u(t)$, from H^2 into H^2, is analytic. Besides scattering holds true when $5 \leq n \leq 8$, and the scattering operator is analytic. When $n \geq 9$, the mapping $u(0) \mapsto u(t)$ cannot be continuous from H^2 into H^2 for any t.*

In order to prove the failure of local wellposedness, following Christ, Colliander and Tao [12], we use an analysis of the low-dispersion regime. Other references on stronger ill-posedness results for wave and Schrödinger equations are in Lebeau [38], Burq, Gerard and Tzvetkov [9] and Thomann [66, 67, 68]. In what follows, we give a brief description of the proof of the global in time results in the case $n = 8$. We proceed by contradiction and assume that global wellposedness fails. The first step, as in Kenig and Merle [29] is to use the description of the loss of compactness in the Strichartz estimates as carried over in Pausader [53] to prove the existence of some special solution that remain (after translation and dilation) in a compact subset of $\dot{H}^2$. Besides, as in Killip Tao and Visan [33], we show that we can suppose that this solution behaves as in one of three different scenarios. The remaining part is to exclude these scenarios. In case there is a prescribed blow-up, we run rather easily into a contradiction. However, if the solution is global, we have to use frequency localized interaction Morawetz estimates as was done in the analysis of the energy critical Schrödinger equation by Colliander, Keel, Staffilani, Takaoka and Tao [13], Ryckman and Visan [57] and Visan [69]. Note that this estimate is no longer an a priori estimate, and is proved by a delicate bootstrap argument. Then, in order to exclude the last scenario, we prove gain of regularity for these special solutions, namely that they are in H^2 instead of being only in $\dot{H}^2$.

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Chapter 2

The Levandosky-Strauss Conjecture

Abstract

We investigate scattering theory in the energy space for fourth-order nonlinear defocusing wave equations and prove the Levandosky-Strauss conjecture stating that scattering holds true for such equations and arbitrary initial data.

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There has been an increasing activity in recent years on models involving nonlinear fourth-order partial differential equations. We investigate in the sequel scattering theory for nonlinear wave equations of fourth order in $\mathbb{R}^n$, $n \geq 1$. The fourth order nonlinear wave equation we discuss in this paper is often referred to in the mathematics and physics literature as the nonlinear beam equation but also, see, for instance, the book by Peletier and Troy [28], as the Bretherton's equation. It is written as

$$\frac{\partial^2 u}{\partial t^2} + \Delta^2 u + mu = \lambda |u|^{p-1} u, \tag{2.0.1}$$

where $m > 0$ is a positive real number, $\Delta = \mathrm{div}\nabla$ is the classical Laplace operator, and $\lambda \in \mathbb{R}$, $\lambda \neq 0$. The equation is said to be defocusing when $\lambda < 0$ and focusing when $\lambda > 0$. At a first glance, (2.0.1) is a formal fourth-order extension of the classical Klein-Gordon equation, but it also inherits a Schrödinger structure because of the decomposition $\partial_t^2 + \Delta^2 = (\partial_t + i\Delta)(\partial_t - i\Delta)$. However, it can be noted that the equation satisfies neither finite speed propagation nor mass conservation, and this turns out to be a painful source of difficulties. The original Bretherton equation, written down for $n = 1$ by Bretherton [5], arose in the study of weak interactions of dispersive waves. A similar equation for $n = 2$ was proposed in Love [23] for the motion of a clamped plate. The equation was discussed in Levine [20]. Recent developments on (2.0.1) were established by Levandosky [17, 18], and Levandosky and Strauss [19]. We also refer to Berger and Milewski [2], Berloff and Howard [3], Holm and Lynch [11], Lazer and McKenna [16], Lin [21], and McKenna and Walter [24, 25] for closely related references.

As already mentioned, we are concerned in this paper with scattering theory for the fourth order wave equation (2.0.1). A rough definition of scattering is that solutions of the equation can be approximated by solutions of a model equation, in our case the linear equation, when time becomes infinite. A more precise definition is in Section 2.1. Abstract scattering theory, in the semigroup setting, was developed in Strauss [29, 30]. In what follows we let H^2 be the Sobolev space of functions in L^2 with two derivatives in L^2. Also we let $2^\sharp$ be given by

$$2^\sharp = +\infty \text{ if } n \leq 4 \text{ and } 2^\sharp = \frac{2n}{n-4} \text{ if } n \geq 5 .$$

As is well known, $2^\sharp$ is the critical exponent for the embedding of H^2 into Lebesgue's spaces when $n \geq 5$. Scattering for low energy initial data, arbitrary λ, and when $1 + \frac{8}{n} \leq p < 2^\sharp - 1$ was established by Levandosky [18]. Levandosky and Strauss [19] then conjectured that scattering should also hold true for such p and arbitrary initial data in the defocusing case. We prove the Levandosky-Strauss conjecture when $n \geq 5$.

Our paper is organized as follows. We state our result in Section 2.1 and fix notations in Section 2.2. We prove local and global Strichartz estimates in Section 2.3. While local Strichartz estimates can be obtained by exploiting the sole Schrödinger structure of the equation, we get the global estimates by using recent advances in Levandosky [18] and material about oscillatory integrals in Kenig, Ponce and Vega [15]. A general scattering criterion, in the spirit of the one in Tao and Visan [33], is developed in Section 2.4. Frequency localization is proved in Section 2.5. What we refer to as almost finite speed propagation is established in Section 2.6. At last we prove the Levandosky-Strauss conjecture in Section 2.7 by using the material in the preceeding sections and a Morawetz type estimate established in Levandosky and Strauss [19].

2.1 Statement of the result

We let $\mathcal{E} = H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ be the energy space associated with (2.0.1), and for I an interval, we let

$$\mathbb{E}_I = C(I, H^2) \cap C^1(I, L^2) \cap C^2(I, H^{-2}). \tag{2.1.1}$$

We say that u is a solution in I of the nonlinear fourth order equation (2.0.1) if $u \in \mathbb{E}_I$ and u solves (2.0.1) in H^{-2}. The linear equation associated to (2.0.1) is written as

$$\frac{\partial^2 u}{\partial t^2} + \Delta^2 u + mu = 0. \tag{2.1.2}$$

Let $(u_0, u_1) \in \mathcal{E}$. Then there exists a unique solution $\omega \in \mathbb{E}_{\mathbb{R}}$ of (2.1.2) with Cauchy data (u_0, u_1). We let E_0 be the linear energy associated with the linear equation (2.1.2), and E be the energy associated with the nonlinear equation (2.0.1). For $(u, v) \in \mathcal{E}$ we then have that

$$\begin{aligned} E_0(u,v) &= \frac{1}{2}\int_{\mathbb{R}^n} \left(v^2 + (\Delta u)^2 + mu^2\right) dx \text{ , and} \\ E(u,v) &= \frac{1}{2}\int_{\mathbb{R}^n} \left(v^2 + (\Delta u)^2 + mu^2\right) dx - \frac{\lambda}{p+1}\int_{\mathbb{R}^n} |u|^{p+1} dx. \end{aligned} \tag{2.1.3}$$

We equip $\mathcal{E}$ with the scalar product whose polar form is E_0. This gives the usual Hilbert structure on $\mathcal{E}$. In what follows we say that there is scattering in forward time for (u_0, u_1) if the two following conditions hold true:

(i) the solution u of (2.0.1) with Cauchy data (u_0, u_1) is defined on the whole of $\mathbb{R}_+$, and

(ii) there exists a unique couple $(u_0^+, u_1^+) \in \mathcal{E}$ such that

$$\|(u(t), u_t(t)) - (\omega(t), \omega_t(t))\|_{\mathcal{E}} \to 0 \tag{2.1.4}$$

as $t \to +\infty$, where $\omega(t)$ is the solution of the linear equation with Cauchy data (u_0^+, u_1^+).

In the sequel we refer to (u_0^+, u_1^+) as the scattering pair associated to (u_0, u_1). Given a set $\mathfrak{F} \subset \mathcal{E}$ such that scattering in forward time holds true for any initial data in $\mathfrak{F}$, we define the wave operator $W_+ : \mathfrak{F} \to \mathcal{E}$ by

$$W_+(u_0, u_1) = (u_0^+, u_1^+), \tag{2.1.5}$$

where (u_0^+, u_1^+) is such that (2.1.4) holds. Note that W_+ is often referred to in the mathematical literature as W_+^{-1}. Similarly, we say that there is scattering in backward time for (u_0, u_1) if there is scattering in forward time for $(u_0, -u_1)$. At last, we refer to scattering without any specificity when scattering holds true both in backward and forward time. The main result of this paper is concerned with the Levandosky-Strauss conjecture [19]. As already mentioned, the

Levandosky-Strauss conjecture asserts that scattering holds true when (2.0.1) is defocusing, in other words when $\lambda < 0$ in (2.0.1), and when $1 + \frac{8}{n} < p < 2^\sharp - 1$. We prove the conjecture when $n \geq 5$.

Theorem 6. *Let $n \geq 5$, $\lambda < 0$, and $1 + \frac{8}{n} < p < 2^\sharp - 1$. Scattering for (2.0.1) holds true for any initial data $(u, v) \in \mathcal{E}$, and W_+ in (2.1.5) realizes an homeomorphism from $\mathfrak{F}_R$ onto $\mathfrak{B}_R$ for all $R > 0$, where $\mathfrak{F}_R$ consists of the $(u, v) \in \mathcal{E}$ such that $E(u, v) \leq R$, and $\mathfrak{B}_R$ consists of the $(u, v) \in \mathcal{E}$ such that $E_0(u, v) \leq R$.*

The rest of the paper is devoted to the proof of the above theorem. We roughly follow the approach developed by Lin and Strauss [22] for the Schrödinger equation. However, a major difficulty with (2.0.1) is that it does not satisfy mass conservation. It neither satisfies finite speed propagation. Finite speed propagation is traditionally used to prove scattering for the nonlinear Klein-Gordon equation as, for instance, in Brenner [4], and Morawetz and Strauss [27]. We overcome the difficulty by using recent ideas of Tao [31] about frequency localization. A brief sketch of the proof is as follows. We prove local and global in time Strichartz estimates in Section 2.3. We prove in Section 2.4 that, as one would have expected, strong decay implies scattering. A key point we establish in Sections 2.5 and 2.6 is that, in the subcritical case, (2.0.1) satisfies almost finite speed propagation. We prove in Section 2.7 that almost finite speed propagation, combined with the Morawetz type estimates in Levandosky and Strauss [19], provides strong decay of the solutions. Then it remains to remember that, as already mentioned, strong decay of the solutions implies scattering.

2.2 Notations

We introduce notations we use in the sequel. Given $(u_0, u_1) \in \mathcal{E}$, there exists a unique solution $u \in \mathbb{E}_\mathbb{R}$ of (2.1.2) such that $(u(0), u_t(0)) = (u_0, u_1)$. We define $\mathcal{W}(t)$ by $(u(t), u_t(t)) = \mathcal{W}(t)(u_0, u_1)$ for all t. In other words, $\mathcal{W}(t)$ is the isometry semigroup associated to the skew-adjoint operator $(D(A), A)$ with $D(A) = H^4 \times H^2 \subset \mathcal{E}$, $A(u, v) = (v, -\Delta^2 u - mu)$. We let $\pi_1 : \mathcal{E} \to H^2$ and $\pi_2 : \mathcal{E} \to L^2$ be the first and second projections. We let also $\mathcal{F}f = \hat{f}$ be the Fourier transform of f given by

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} f(y) e^{i\langle y, \xi \rangle} dy \tag{2.2.1}$$

for all $\xi \in \mathbb{R}^n$. Let $\psi \in C_c^\infty(\mathbb{R}^n)$ be supported in the ball $B_0(2)$ such that $\psi = 1$ in $B_0(1)$, and $0 \leq \psi \leq 1$. For any dyadic number $N = 2^k, k \in \mathbb{Z}$, we define the following Littlewood-Paley operators:

$$\begin{aligned} \widehat{P_{\leq N} f}(\xi) &= \psi(\xi/N) \hat{f}(\xi) \\ \widehat{P_{> N} f}(\xi) &= (1 - \psi(\xi/N)) \hat{f}(\xi) \\ \widehat{P_N f}(\xi) &= (\psi(\xi/N) - \psi(2\xi/N)) \hat{f}(\xi) \end{aligned} \tag{2.2.2}$$

Similarly we define $P_{<N}$ and $P_{\geq N}$ by the equations $P_{<N} = P_{\leq N} - P_N$ and $P_{\geq N} = P_{>N} + P_N$. We adopt the convention that these operators act on couples of functions by $P_{\leq N}(u, v) = (P_{\leq N} u, P_{\leq N} v)$, and similarly for the other operators $P_{>N}$, P_N, $P_{<N}$, and $P_{\geq N}$. These operators commute one with another. They also commute with derivative operators and with the semigroup $\mathcal{W}(t)$. In addition they are self-adjoint and bounded on L^p for all $1 \leq p \leq \infty$. Moreover, they enjoy the following Bernstein properties:

$$\begin{aligned} &(i) \qquad \|P_{\geq N} f\|_{L^p} \leq CN^{-s} \||\nabla|^s P_{\geq N} f\|_{L^p} \leq CN^{-s} \||\nabla|^s f\|_{L^p} \\ &(ii) \qquad \||\nabla|^s P_{\leq N} f\|_{L^p} \leq CN^{s} \|P_{\leq N} f\|_{L^p} \leq CN^{s} \|f\|_{L^p} \\ &(iii) \qquad \||\nabla|^{\pm s} P_N f\|_{L^p} \leq CN^{\pm s} \|P_N f\|_{L^p} \leq CN^{\pm s} \|f\|_{L^p} \end{aligned} \tag{2.2.3}$$

for all $s \geq 0$, and all $1 \leq p \leq \infty$, where $|\nabla|^s$ is the classical fractional differentiation operator, and $C > 0$ is independent of f, N, and p. When $N = 1$, these estimates follow from straightforward computations on the convolution kernels of the operators. We recover the case of N arbitrary by considering the effect of dilations on these estimates. We refer to Tao [32] for more details. Given $a \geq 1$, we let a' be the conjugate of a, so that $\frac{1}{a} + \frac{1}{a'} = 1$. For short, we adopt the convention that $x^{\mathfrak{p}} = |x|^{p-1}x$.

2.3 Strichartz estimates

We discuss Strichartz estimates for (2.0.1) and start with local in time estimates in Lemma 2.3.1. Global in time estimates are discussed in Lemma 2.3.2. Local in time estimates follow from the sole Schrödinger structure of the equation. Following standard terminology we say that a pair (q, r) is Schrödinger admissible, for short S-admissible, if

$$\frac{2}{q} + \frac{n}{r} = \frac{n}{2} \tag{2.3.1}$$

and r is such that $2 \leq r \leq +\infty$ if $n = 1$, $2 \leq r < +\infty$ if $n = 2$, and $2 \leq r \leq 2^\star$ if $n \geq 3$, where $2^\star = \frac{2n}{n-2}$. Now we introduce various notions of admissible and controlling pairs.

Definition 2.3.1. *For $2 \leq q \leq +\infty$, a pair (q, r) is said to be Bretherton or beam-admissible, for short B-admissible, if $2 \leq r \leq +\infty$ when $n = 1, 2, 3$, $2 \leq r < +\infty$ when $n = 4$, and*

$$\frac{2}{q} + \frac{n}{r} = \frac{n-4}{2} \tag{2.3.2}$$

with $0 < r < +\infty$ when $n \geq 5$. A pair (p, q) is said to be Bretherton or beam low-admissible, for short Bl-admissible, if $p, q \geq 2$,

$$\frac{4}{p} + \frac{n}{q} \leq \frac{n}{2}, \tag{2.3.3}$$

and $(p, q, n) \neq (2, \infty, 4)$. A pair (p, q) is Bretherton or beam controlling, for short B-controlling, if (p, q) is Bl-admissible, $q \neq \infty$, and (p, q) satisfies

$$\frac{2}{p} + \frac{n}{q} = \sigma \tag{2.3.4}$$

for some σ such that $(n - 4)/2 \leq \sigma \leq n/2$.

As a remark, if (q,r) is S-admissible in the sense of (2.3.1) and $2r < n$, then $(q, r^\sharp)$ is B-admissible for $r^\sharp = \frac{nr}{n-2r}$. Note that $s = r^\sharp$ is the critical Sobolev exponent for the embedding of $H^{2,r}$ into L^s, where $H^{2,r}$ stands for the Sobolev space of functions in L^r with two derivatives in L^r. More generally, given $s \in \mathbb{R}$ and $p \geq 1$, we let $H^{s,p} = H^{s,p}(\mathbb{R}^n)$ be the usual fractional Sobolev spaces in $\mathbb{R}^n$. Following standard notations we let also $H^s = H^{s,2}$. Local in time Strichartz estimates for (2.0.1) are as follows.

Lemma 2.3.1. *Let $I \subset \mathbb{R}$ be a bounded interval such that $0 \in I$, $u_0 \in H^2$, $u_1 \in L^2$, and $h \in C(I, H^{-2}) \cap L^{a'}(I, L^{b'})$ for some S-admissible pair (a,b). There exists a unique $u \in \mathbb{E}_I$ which solves the linear equation*

$$\frac{\partial^2 u}{\partial t^2} + \Delta^2 u = h \tag{2.3.5}$$

in $C(I, H^{-2})$ with Cauchy data $u_{|t=0} = u_0$ and $u_{t|t=0} = u_1$. Moreover it holds that $u \in L^q(I, L^r)$ for any B-admissible pair (q,r), and that

$$\begin{aligned} &\|(u,u_t)\|_{C(I,\mathcal{E})} + \|u\|_{L^q(I,L^r)} \\ &\leq C\left(1 + |I|^{3/2}\right)\left(\sqrt{E_0(u_0,u_1)} + \|h\|_{L^{a'}(I,L^{b'})}\right) , \end{aligned} \tag{2.3.6}$$

where $|I|$ is the length of I, E_0 is as in (2.1.3), and $C \geq 1$ does not depend on u_0, u_1, h, and I.

Proof. We let v solve (2.3.5) in $C(I, H^{-4})$ with Cauchy data $(0,0)$. We let also w be such that for all t, $(w(t), w_t(t)) = \mathcal{W}(t)(u_0, u_1)$. Then $v \in C(I, L^2) \cap C^1(I, H^{-2}) \cap C^2(I, H^{-4})$ and $w \in \mathbb{E}_I$. Let $\tilde{v} = -iv_t + \Delta v$ and $\tilde{w} = -iw_t + \Delta w$. We consider the linear Schrödinger equation

$$iu_t + \Delta u = h . \tag{2.3.7}$$

As is easily checked, $\tilde{v}$ solves (2.3.7) in $C(I, H^{-4})$ with Cauchy data $\tilde{v}_{|t=0} = 0$, and $\tilde{w}$ solves (2.3.7) in $C(I, H^{-2})$ when $h \equiv 0$ with Cauchy data $\tilde{w}_{|t=0} = -iu_1 + \Delta u_0$. We may then apply the standard Strichartz estimates for the Schrödinger equation, as stated for instance in Cazenave [6], to $\tilde{v}$ and $\tilde{w}$. We refer also to Keel and Tao [14]. The Strichartz estimates for $\tilde{v}$ give that $\tilde{v} \in C(I, L^2) \cap L^q(I, L^s)$ for any S-admissible pair (q,s), and that the L^qL^s-norm of $\tilde{v}$ is controlled by the $L^{a'}L^{b'}$-norm of h. This includes the choice of (q,s) given by $q = +\infty$ and $s = 2$. In particular, it follows that $v \in \mathbb{E}_I$, and by considering the real and imaginary parts of $\tilde{v}$ we also get that for any S-admissible pair (q,s),

$$\|\Delta v\|_{C(I,L^2)} + \|v_t\|_{C(I,L^2)} + \|\Delta v\|_{L^q(I,L^s)} + \|v_t\|_{L^q(I,L^s)} \leq C\|h\|_{L^{a'}(I,L^{b'})} , \tag{2.3.8}$$

where $C > 0$, independent of I, depends only on n, (a,b), and (q,s). As a remark this implies that v solves (2.3.5) in $C(I, H^{-2})$ and not only in $C(I, H^{-4})$. By the control on the norm of v_t in (2.3.8), we can write that

$$\|v\|_{C(I,H^2)} + \|v_t\|_{C(I,L^2)} + \|v\|_{L^q(I,H^{2,s})} \leq C\left(1 + |I|\right)\|h\|_{L^{a'}(I,L^{b'})} , \tag{2.3.9}$$

where $C > 0$, independent of I, depends only on n, (a,b), and (q,s). Let (q,r) be a B-admissible pair as in the statement of Lemma 2.3.1. When $n \leq 4$, by the Sobolev embedding theorem,

$$\|v\|_{L^q(I,L^r)} \leq C|I|^{1/q}\|v\|_{C(I,H^2)} \leq C\left(1 + |I|^{1/2}\right)\|v\|_{C(I,H^2)} , \tag{2.3.10}$$

where $C > 0$ depends only on n and (q,r). When $n \geq 5$, we let s be given by $s = nr/(n+2r)$. Then (q,s) is S-admissible and $s^\sharp = r$. Combining (2.3.9) and the Sobolev embedding theorem, we get that

$$\|v\|_{C(I,H^2)} + \|v_t\|_{C(I,L^2)} + \|v\|_{L^q(I,L^r)} \leq C\left(1+|I|^{3/2}\right)\|h\|_{L^{a'}(I,L^{b'})}, \quad (2.3.11)$$

where $C > 0$, independent of I, depends only on n, (a,b), and (q,r). Similarly, the Strichartz's estimates for $\tilde{w}$ give that

$$\begin{aligned}&\|w\|_{C(I,H^2)} + \|w_t\|_{C(I,L^2)} + \|w\|_{L^q(I,L^r)}\\ &\leq C\left(1+|I|^{3/2}\right)\left(\|u_1\|_{L^2} + \|u_0\|_{L^2} + \|\Delta u_0\|_{L^2}\right)\\ &\leq C\left(1+|I|^{3/2}\right)\sqrt{E_0(u_0,u_1)}\,,\end{aligned} \quad (2.3.12)$$

where $C \geq 1$, independent of I, depends only on n, m, and (q,r). By (2.3.11) and (2.3.12), letting $u = v + w$, we get a solution of (2.3.5) in $C(I,H^{-2})$ with Cauchy data $u_{|t=0} = u_0$ and $u_{t|t=0} = u_1$ which satisfies (2.3.6) for any B-admissible pair (q,r). Uniqueness of u follows from the remark that if u_1 and u_2 are two such solutions, then $\tilde{u} = u_2 - u_1$ solves (2.3.5) with $h = 0$ and Cauchy data $\tilde{u}_{|t=0} = 0$ and $\tilde{u}_{t|t=0} = 0$ so that $\tilde{u} = 0$. This proves Lemma 2.3.1. □

As a remark, the proof of Lemma 2.3.1 also gives that $u_t \in L^q(I,L^s)$ for any S-admissible pair (q,s). Since $2 \leq s \leq 2^\sharp$ for such pairs, and $u \in C(I,H^2)$, we also get from the Sobolev embedding theorem that $u \in L^q(I,L^s)$.

Local well-posedness in the energy-subcritical and in the energy-critical case for (2.0.1) follows from Lemma 2.3.1 by the standard methods developed for semilinear Schrödinger equations by Ginibre and Velo [10], Kato [12, 13], and Cazenave and Weissler [8, 9]. Unconditional uniqueness also holds true for (2.0.1). We refer to Cazenave [6] for an excellent exposition in book form on such methods. Let p be such that $1 \leq p \leq 2^\sharp - 1$ if $n \geq 5$, any $1 \leq p < \infty$ if $n \leq 4$. With only slight and obvious modifications with respect to the proofs in Cazenave [6], it follows from the estimates in Lemma 2.3.1 that for any $(u_0,u_1) \in \mathcal{E}$, there exists a unique solution $u \in \mathbb{E}_I$ of (2.0.1) defined on some maximal interval $I = (-T^-, T^+)$. For any B-admissible pair (q,r), $u \in L^q_{loc}(I,L^r)$, and the solution satisfies conservation of the energy:

$$E(u(t),u_t(t)) = E(u_0,u_1) \quad (2.3.13)$$

for all $t \in I$. Moreover, we also have that if $T^+ \neq \infty$ and $p < 2^\sharp - 1$, then $\|u(t)\|_{H^2} \to \infty$ as $t \to T^+$, while if $n \geq 5$ and $p = 2^\sharp - 1$, then the blow-up arises in mixed norms and

$$\|u\|_{L^{2\frac{n+2}{n-4}}([0,T^+)\times\mathbb{R}^n)} = +\infty. \quad (2.3.14)$$

Similar statements hold true for T^-. At last, well-posedness holds true in the sense that if (u_0^k,u_1^k) is a sequence in $\mathcal{E}$ that converges to (u_0,u_1) in $\mathcal{E}$, and if u^k denotes the corresponding solution of (2.0.1) with maximal interval $(-T^{-,k},T^{+,k})$, then $\liminf T^{+,k} \geq T^+$, $\liminf T^{-,k} \geq T^-$, and for any finite interval $I' \subset (-T^-,T^+)$, and any B-admissible pair (q,r),

$$u^k \to u \text{ in } C(I',H^2) \cap C^1(I',L^2) \cap L^q(I',L^r) \quad (2.3.15)$$

as $k \to +\infty$. As a remark, local well-posedness has already been established by Levandosky [18] in the energy-subcritical case of (2.0.1). The approach in Levandosky [18] was based on the system representation of (2.0.1).

Local in time Strichartz estimates, as in Lemma 2.3.1, are powerful enough to deal with local existence. Scattering requires global in time estimates. We prove such global in time estimates in what follows. In order to do this we need to deal with a degenerate critical point in the low frequency mode. The critical point is responsible for slow decay as time goes to infinity. We overcome the difficulty thanks to a powerful estimate in Levandosky [18] for the Fourier transform of radial functions. A similar idea for fourth order Schrödinger equations was later on used in Ben-Artzi, Koch, and Saut [1]. High frequencies are treated via standard stationary phase estimates from Kenig, Ponce, and Vega [15]. For $h \in C(I, H^{-2})$ we consider the linear equation with forcing term

$$\frac{\partial^2 u}{\partial t^2} + \Delta^2 u + mu = h. \tag{2.3.16}$$

The global in time Strichartz estimates we prove state as follows.

Lemma 2.3.2. *Let $I \subset \mathbb{R}$ be an interval such that $0 \in I$. Let (p, q) be any B-controlling pair, and (a, b) be any Bl-admissible pair as in (2.3.3) and (2.3.4). Let also (c, d) be any S-admissible pair, $(u_0, u_1) \in \mathcal{E}$, and $h \in C(I, H^{-2}) \cap L^{a'}(I, L^{b'}) \cap L^{c'}(I, L^{d'})$. There exists a unique $u \in \mathbb{E}_I$ such that u solves the linear equation (2.3.16) with Cauchy data (u_0, u_1), and*

$$\begin{aligned} &\|(u, u_t)\|_{C(I,\mathcal{E})} + \|u\|_{L^p(I,L^q)} \\ &\leq C\left(\|(u_0, u_1)\|_{\mathcal{E}} + \|h\|_{L^{a'}(I,L^{b'})} + \|h\|_{L^{c'}(I,L^{d'})}\right), \end{aligned} \tag{2.3.17}$$

where C is independent of u_0, u_1, and h. Moreover, for any $\alpha \geq 2$, if $u_0 \in L^{\alpha'}$ and $u_1 \in H^{-2,\alpha'}$, then

$$\|u\|_{L^\alpha} \leq C\left(|t|^{-\frac{n}{2}(1-\frac{2}{\alpha})} + |t|^{-\frac{n}{4}(1-\frac{2}{\alpha})}\right)\left(\|u_0\|_{L^{\alpha'}} + \|(1+\Delta^2)^{-1/2} u_1\|_{L^{\alpha'}}\right) \tag{2.3.18}$$

for all $t \neq 0$ when $h = 0$, where C is independent of u_0 and u_1.

Proof. In order to prove the lemma, we define a "half-wave" operator $u \mapsto T_t u$ for u in $L^1 + L^2$ by

$$\mathcal{F}(T_t u)(\xi) = \exp(it\sqrt{1+|\xi|^4})\mathcal{F}(u)(\xi) \tag{2.3.19}$$

for $\xi \in \mathbb{R}^n$, and $t \in \mathbb{R}$. Also we define T_t^l and T_t^h, the low and high frequency parts of T_t, by

$$T_t^l u = P_{\leq 2} T_t u \text{ and } T_t^h u = P_{>1/2} T_t u. \tag{2.3.20}$$

As is easily checked, $T_t = P_{\leq 1} T_t^l + P_{>1} T_t^h$ for all t. Now we claim that there exists $C > 0$ depending only on n such that for any $\alpha \geq 2$, and any $u \in L^{\alpha'}$,

$$\|T_t^l u\|_{L^\alpha} \leq C(1+|t|)^{-\frac{n}{4}(1-\frac{2}{\alpha})}\|u\|_{L^{\alpha'}} \tag{2.3.21}$$

for all $t \in \mathbb{R}$. We prove (2.3.21) in what follows.

Let $u \in C_c^\infty(\mathbb{R}^n)$ be a smooth function with compact support. By a crude estimate, we see that

$$\begin{aligned} |T_t^l u(x)| &\le C \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\langle y-x,\xi\rangle} e^{it\sqrt{1+|\xi|^4}} \psi(\xi/2) u(y) d\xi dy \right| \\ &\le C\|u\|_{L^1}, \end{aligned} \tag{2.3.22}$$

where ψ is as in (2.2.2). It is clear that

$$T_t^l u = (2\pi)^{-\frac{n}{2}} \left(T_t \mathcal{F}^{-1}\psi(\cdot/2)\right) * u \tag{2.3.23}$$

for all t and all u. By Levandosky [18, Lemma 2.3], combined with (2.3.23), we then get that for t such that $|t| \ge 1$,

$$\|T_t^l u\|_{L^\infty} \le C|t|^{-\frac{n}{4}} \|u\|_{L^1}. \tag{2.3.24}$$

where $C > 0$ is independent of t and u. Independently, Plancherel's theorem asserts that T_t^l is bounded $L^2 \to L^2$. Hence T_t^l extends to an operator $L^1 + L^2 \to L^2 + L^\infty$ and, by (2.3.22) and (2.3.24), we then get that

$$\begin{aligned} &\|T_t^l\|_{L^1 \to L^\infty} \le C(1+|t|)^{-\frac{n}{4}}\ , \text{ and} \\ &\|T_t^l\|_{L^2 \to L^2} \le C \end{aligned} \tag{2.3.25}$$

for all t, where $C > 0$ is independent of t. Then (2.3.21) follows from (2.3.25) by the Riesz-Thorin theorem. This proves the above claim that (2.3.21) holds true.

Now that (2.3.21) is proved we continue with the proof of the lemma. Let (p, q) and (a, b) be Bl-admissible pairs as in (2.3.3). By the definition of $P_{\le N}$ in (2.2.2), and the definition of T_t in (2.3.19), we can write that $T_s^l T_t^{l\star} = P_{\le 2} T_{s-t}^l$ and also that $P_{\le 1} T_s^l T_t^{l\star} = P_{\le 1} T_{s-t}$. Since $P_{\le N}$ is bounded on L^p for $1 \le p \le \infty$, we get with (2.3.25) and the $TT^\star$-method of Keel and Tao [14] that there exists $C > 0$, independent of u, such that

$$\|P_{\le 1} T_t u\|_{L^p(\mathbb{R}, L^q)} \le C\|u\|_{L^2} \tag{2.3.26}$$

for all $u \in L^2$, and that

$$\begin{aligned} &\left\| \int_0^t P_{\le 1} T_{t-s} u(s) ds \right\|_{L^p(\mathbb{R}, L^q)} \le C\|u\|_{L^{a'}(\mathbb{R}, L^{b'})}, \\ &\left\| \int_{\mathbb{R}} P_{\le 1} T_{-s} u(s) ds \right\|_{L^2} \le C\|u\|_{L^{a'}(\mathbb{R}, L^{b'})} \end{aligned} \tag{2.3.27}$$

for all $u \in L^{a'}(\mathbb{R}, L^{b'})$. For the reader's convenience we briefly recall the result in Keel and Tao [14]. Let H be an Hilbert space and $U(t) : H \to L^2$ be such that for any s, t, and any $f \in L^1$,

$$\|U(t)\|_{H \to L^2} \le C\ , \tag{2.3.28}$$

and one of the two following decay estimates holds true

$$\begin{aligned} &\|U(s)U(t)^\star f\|_{L^\infty} \le C\,|t-s|^{-\sigma}\,\|f\|_{L^1}\ , \text{ or} \\ &\|U(s)U(t)^\star f\|_{L^\infty} \le C\,(1+|t-s|)^{-\sigma}\,\|f\|_{L^1}\ , \end{aligned} \tag{2.3.29}$$

where $C > 0$ and $\sigma > 0$ do not depend on s, t, and f. Following Keel and Tao [14], define σ-admissible pairs (q, r) by the relations $q, r \geq 2$, $(q, r, \sigma) \neq (2, \infty, 1)$, and

$$\frac{1}{q} + \frac{\sigma}{r} \leq \frac{\sigma}{2} \ , \tag{2.3.30}$$

and say that the pair is sharp σ-admissible if equality holds in (2.3.30). The result in Keel and Tao [14] then states that for any $f \in H$ and any $F \in L^{q'}(\mathbb{R}, L^{r'})$,

$$\begin{aligned} &\|U(t)f\|_{L^q(\mathbb{R},L^r)} \leq C\|f\|_H \ , \\ &\left\|\int_{\mathbb{R}} U(s)^\star F(s)ds\right\|_H \leq C\|F\|_{L^{q'}(\mathbb{R},L^{r'})} \ , \text{ and} \\ &\left\|\int_{s<t} U(t)U(s)^\star F(s)ds\right\|_{L^{\tilde{q}}(\mathbb{R},L^{\tilde{r}})} \leq C\|F\|_{L^{q'}(\mathbb{R},L^{r'})} \end{aligned} \tag{2.3.31}$$

for all sharp σ-admissible pairs (q, r) and $(\tilde{q}, \tilde{r})$, where $C > 0$ does not depend on f and F, and for all σ-admissible pairs (q, r) and $(\tilde{q}, \tilde{r})$ if the second condition in (2.3.29) holds true. In our case we let $H = L^2$, $U(t) = T^l_t$, and $\sigma = n/4$. Then (2.3.28) and the second equation in (2.3.29) follow from (2.3.25), the boundedness of $P_{\leq N}$, and the identity $T^l_s T^{l\star}_t = P_{\leq 2} T^l_{s-t}$. Then (2.3.26) follows from the first equation in (2.3.31), and by noting that $P_{\leq 1} P_{\leq 2} = P_{\leq 1}$. The second equation in (2.3.27) follows from the second equation in (2.3.31) and again by noting that $P_{\leq 1} P_{\leq 2} = P_{\leq 1}$. The first equation in (2.3.27), when the $L^p L^q$-norm is restricted to $\mathbb{R}_+$, follows from the third equation in (2.3.31) that we apply to $F = \mathbf{1}_{\mathbb{R}_+} P_{\leq 1} u$, where $\mathbf{1}_{\mathbb{R}_+}$ is the characteristic function of $\mathbb{R}_+$, and from the identity $P_{\leq 1} T^l_s T^{l\star}_t = P_{\leq 1} T_{s-t}$. Then we get the global $L^p L^q$-norm, and so the first equation in (2.3.27), by writing that for $t < 0$,

$$\int_0^t P_{\leq 1} T_{t-s} u(s) ds = \int_{s<t} P_{\leq 1} T_{t-s} u(s) ds - T_t \int_{\mathbb{R}} P_{\leq 1} T_{-s} \mathbf{1}_{\mathbb{R}_-} u(s) ds$$

and thus, thanks to the three equations in (2.3.31), that

$$\begin{aligned} &\left\|\int_0^t P_{\leq 1} T_{t-s} u(s) ds\right\|_{L^p(\mathbb{R}_-,L^q)} \\ &\leq \left\|\int_{s<t} P_{\leq 1} T_{t-s} u(s) ds\right\|_{L^p(\mathbb{R},L^q)} + \left\|T_t \int_{\mathbb{R}} P_{\leq 1} T_{-s} \mathbf{1}_{\mathbb{R}_-} u(s) ds\right\|_{L^p(\mathbb{R},L^q)} \\ &\leq C \left\|P_{\leq 1} u\right\|_{L^{a'}(\mathbb{R},L^{b'})} + C \left\|\int_{\mathbb{R}} P_{\leq 2} T_{-s} \mathbf{1}_{\mathbb{R}_-} u(s) ds\right\|_{L^2} \\ &\leq C \left\|u\right\|_{L^{a'}(\mathbb{R},L^{b'})} \ . \end{aligned}$$

This proves (2.3.26) and (2.3.27).

In parallel to (2.3.21), we claim now that there exists $C > 0$ depending only on n such that for $\alpha \geq 2$,

$$\|T^h_t u\|_{L^\alpha} \leq C|t|^{-\frac{n}{2}(1-\frac{2}{\alpha})}\|u\|_{L^{\alpha'}} \tag{2.3.32}$$

for all $t \in \mathbb{R}\backslash\{0\}$. We prove (2.3.32) in what follows.

Let $u \in C_c^\infty(\mathbb{R}^n)$ be a smooth function with compact support. We clearly have that

$$T_t^h u(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} u(y)\int_{\mathbb{R}^n}\frac{1-\psi(2\xi)}{\sqrt{H_\varphi(\xi)}}\sqrt{H_\varphi(\xi)}e^{it\varphi(\xi)-i\langle x-y,\xi\rangle}d\xi dy \quad (2.3.33)$$

for all $t \in \mathbb{R}$, and all $x \in \mathbb{R}^n$, where $\varphi(\xi) = \sqrt{1+|\xi|^4}$ and $H_\varphi(\xi) = |\det(\partial_{ij}^2\varphi)|$. The phase function φ in (2.3.33) satisfies the assumptions of Kenig, Ponce, and Vega [15, Lemma 3.4]. With respect to the notation in Kenig, Ponce, and Vega [15], $m = 2$ and Ω is the complement of the ball of radius $1/2$. This gives that

$$\|T_t^h u\|_{L^\infty} \leq |t|^{-\frac{n}{2}}\|u\|_{L^1} \quad (2.3.34)$$

for all $t \in \mathbb{R}\backslash\{0\}$. By Plancherel's theorem, we also have

$$\|T_t^h u\|_{L^2} \leq \|u\|_{L^2} \quad (2.3.35)$$

for all t. We get (2.3.32) from (2.3.34) and (2.3.35) by the Riesz-Thorin theorem. This proves the above claim that (2.3.32) holds true.

We continue with the proof of the lemma. Let (p,q) and (a,b) be S-admissible pairs. By noting that $T_s^h T_t^{h\star} = P_{>1/2}T_{s-t}^h$ and $P_{>1}T_s^h T_t^{h\star} = P_{>1}T_{s-t}$, and since $P_{>N}$ is bounded on L^p for $1 \leq p \leq \infty$, we get with (2.3.34), (2.3.35), and the $TT^\star$-method of Keel and Tao [14], that there exists $C > 0$, independent of u, such that

$$\|P_{>1}T_t u\|_{L^p(\mathbb{R},L^q)} \leq C\|u\|_{L^2} \quad (2.3.36)$$

for all $u \in L^2$, and that

$$\begin{aligned} &\left\|\int_0^t P_{>1}T_{t-s}u(s)ds\right\|_{L^p(\mathbb{R},L^q)} \leq C\|u\|_{L^{a'}(\mathbb{R},L^{b'})}, \\ &\left\|\int_{\mathbb{R}} P_{>1}T_{-s}u(s)ds\right\|_{L^2} \leq C\|u\|_{L^{a'}(\mathbb{R},L^{b'})} \end{aligned} \quad (2.3.37)$$

for all $u \in L^{a'}(\mathbb{R}, L^{b'})$. Here we proceed as above, when proving (2.3.26) and (2.3.27), with the slight differences that we only have the first equation in (2.3.29), that σ needs to be changed into $\sigma = n/2$, and that we have to restrict ourselves to sharp σ-admissible pairs in the sense of Keel and Tao [14].

Now we enter more specifically into the proof of Lemma 2.3.2. The existence and uniqueness of the solution u follow from straightforward semigroup techniques (see e.g. Cazenave and Haraux [7]). For the moment we assume that $m = 1$ and prove (2.3.17) and (2.3.18). In order to do this we use the explicit representation formula for solutions of (2.3.16). We compute

$$\hat{u}(t) = \frac{e^{it\rho}+e^{-it\rho}}{2}\hat{u}_0 + \frac{e^{it\rho}-e^{-it\rho}}{2i}\frac{\hat{u}_1}{\rho} + \int_0^t \frac{e^{i(t-s)\rho}-e^{-i(t-s)\rho}}{2i}\frac{\hat{h}(s)}{\rho}ds \text{ , and}$$

$$\partial_t\hat{u}(t) = -\frac{e^{it\rho}-e^{-it\rho}}{2i}\rho\hat{u}_0 + \frac{e^{it\rho}+e^{-it\rho}}{2}\hat{u}_1 + \int_0^t \frac{e^{i(t-s)\rho}+e^{-i(t-s)\rho}}{2}\hat{h}(s)ds,$$

where $\rho = \sqrt{1+|\xi|^4}$. As a consequence,

$$\begin{aligned} u(t) &= \frac{1}{2}\left(T_t + T_{-t}\right)u_0 + \frac{1}{2i}(1+\Delta^2)^{-1/2}\left(T_t - T_{-t}\right)u_1 \\ &+ \frac{1}{2i}(1+\Delta^2)^{-1/2}\int_0^t \left(T_{t-s} - T_{s-t}\right)h(s)ds \end{aligned} \quad (2.3.38)$$

and

$$u_t(t) = -\frac{T_t - T_{-t}}{2i}(1+\Delta^2)^{1/2}u_0 + \frac{T_t + T_{-t}}{2}u_1 + \int_0^t \frac{T_{t-s} + T_{s-t}}{2}h(s)ds \quad (2.3.39)$$

for all t, where T_t is as in (2.3.19). By the decay estimates (2.3.21) and (2.3.32) we get from (2.3.20) and (2.3.38) that, in case $h = 0$, and for $\alpha \geq 2$,

$$\begin{aligned} \|u(t)\|_{L^\alpha} &\leq \|P_{\leq 1}u(t)\|_{L^\alpha} + \|P_{>1}u(t)\|_{L^\alpha} \\ &\leq C\left(|t|^{-\frac{n}{2}(1-\frac{2}{\alpha})} + |t|^{-\frac{n}{4}(1-\frac{2}{\alpha})}\right)\left(\|u_0\|_{L^{\alpha'}} + \|(1+\Delta^2)^{-1/2}u_1\|_{L^{\alpha'}}\right). \end{aligned}$$

This proves (2.3.18).

By (2.3.26), (2.3.27), and (2.3.38)–(2.3.39), we then get that for any Bl-admissible pairs (p, q) and (a, b),

$$\begin{aligned} &\|P_{\leq 1}(u, u_t)\|_{L^p(I,L^q)} \\ &= \|P_{\leq 2}P_{\leq 1}(u, u_t)\|_{L^p(I,L^q)} \\ &\leq C\left(\|(u_0, u_1)\|_{\mathcal{E}} + \|(1+\Delta^2)^{-1/2}P_{\leq 2}h\|_{L^{a'}(I,L^{b'})} + \|P_{\leq 2}h\|_{L^{a'}(I,L^{b'})}\right) \\ &\leq C\left(\|(u_0, u_1)\|_{\mathcal{E}} + \|h\|_{L^{a'}(I,L^{b'})}\right). \end{aligned} \quad (2.3.40)$$

We used in (2.3.40) that $P_{\leq 2}P_{\leq 1} = P_{\leq 1}$ and that the kernels of the operators $P_{\leq 2}$ and $(1+\Delta^2)^{-1/2}P_{\leq 2}$ lie in L^1. Similarly, by (2.3.36), (2.3.37), and (2.3.38)–(2.3.39), we get that for any S-admissible pairs (p, r) and (c, d),

$$\begin{aligned} &\|P_{>1}((1+\Delta^2)^{1/2}u, u_t)\|_{L^p(I,L^r)} \\ &\leq C\left(\|(1+\Delta^2)^{1/2}u_0\|_{L^2} + \|u_1\|_{L^2} + \|h\|_{L^{c'}(I,L^{d'})}\right). \end{aligned} \quad (2.3.41)$$

Now, we just remark that if (p, q) is B-controlling, then there exists $r \leq q$ such that (p, r) is S-admissible, and $H^{2,r} \subset L^q$. Since $\frac{1-\Delta}{\sqrt{1+\Delta^2}}$ is bounded $L^p \to L^p$ for $1 < p < \infty$, we get from Bessel's potential theory that

$$\|P_{>1}u\|_{L^p(I,L^q)} \leq C\|(1-\Delta)P_{>1}u\|_{L^p(I,L^r)} \leq C\|(1+\Delta^2)^{1/2}P_{>1}u\|_{L^p(I,L^r)}. \quad (2.3.42)$$

By (2.2.3), equations (i) and (ii), and (2.3.40)–(2.3.42), we get that (2.3.17) holds true. At this stage we proved (2.3.17) and (2.3.18) when $m = 1$.

In case $m \neq 1$, we remark that if u solves (2.3.16) with Cauchy data (u_0, u_1), then $v(t, x) = u(\lambda^2 t, \lambda x)$ solves (2.3.16) with $\lambda^4 m$ in place of m and $\tilde{h}$ in place of h, where $\tilde{h}(t, x) = \lambda^4 h(\lambda^2 t, \lambda x)$. Moreover v satisfies the Cauchy data $(v(0), v_t(0)) = (\tilde{u}_0, \lambda^2\tilde{u}_1)$, where $\tilde{u}_0(x) = u_0(\lambda x)$ and $\tilde{u}_1(x) = u_1(\lambda x)$. This ends the proof of the lemma. □

As a remark, combining the second inequality in (2.3.27), the second inequality in (2.3.37), and the explicit formula for $\mathcal{W}(t)$ in (2.3.38) and (2.3.39), we get the estimate that for any S-admissible pair (a, b), any Bl-admissible pair (c, d), and any $u \in L^{a'}(\mathbb{R}, L^{b'}) \cap L^{c'}(\mathbb{R}, L^{d'})$,

$$\left\|\int_{\mathbb{R}} \mathcal{W}(-t)(0, u(t))dt\right\|_{\mathcal{E}} \leq C\left(\|u\|_{L^{a'}(\mathbb{R},L^{b'})} + \|u\|_{L^{c'}(\mathbb{R},L^{d'})}\right), \quad (2.3.43)$$

where $C > 0$ does not depend on u. Indeed,

$$\begin{aligned}\left\|\int_{\mathbb{R}} \mathcal{W}(-t)(0,u(t))dt\right\|_{\mathcal{E}}^2 &= \left\|\int_{\mathbb{R}} \frac{1}{2i}\left(1+\Delta\right)^{-1/2}\left(T_t - T_{-t}\right)u(t)dt\right\|_{H^2}^2 \\ &+ \left\|\int_{\mathbb{R}} \frac{T_t + T_{-t}}{2} u(t)dt\right\|_{L^2}^2 \\ &\le \left\|\int_{\mathbb{R}} T_t u(t)dt\right\|_{L^2}^2 + \left\|\int_{\mathbb{R}} T_{-t} u(t)dt\right\|_{L^2}^2 \\ &\le C\left(\|u\|_{L^{a'}(\mathbb{R},L^{b'})} + \|u\|_{L^{c'}(\mathbb{R},L^{d'})}\right)^2 .\end{aligned}$$

Also we get that for any S-admissible pairs (a,b) and (c,d), and for any $u \in L^{c'}(\mathbb{R}, L^{d'})$,

$$\left\|\int_{0<s<t} \pi_2 P_{>1}\mathcal{W}(t-s)(0,u(s))ds\right\|_{L^a(\mathbb{R},L^b)} \le C\|u\|_{L^{c'}(\mathbb{R},L^{d'})}, \tag{2.3.44}$$

and, when $q \ge 2$, that

$$\begin{aligned}&(i) \quad \|\pi_2\mathcal{W}(t)P_{\le 1}(u,v)\|_{L^q} \le C|t|^{-\frac{n}{4}(1-\frac{2}{q})}\left(\|(1+\Delta^2)^{1/2}u\|_{L^{q'}} + \|v\|_{L^{q'}}\right), \\ &(ii) \quad \|\pi_2\mathcal{W}(t)P_{>1}(u,v)\|_{L^q} \le C|t|^{-\frac{n}{2}(1-\frac{2}{q})}\left(\|(1+\Delta^2)^{1/2}u\|_{L^{q'}} + \|v\|_{L^{q'}}\right)\end{aligned} \tag{2.3.45}$$

for all t, all $u \in C_c^\infty$, and all $v \in L^{q'}$, where C depends only on n. Moreover, for $N \ge 8$, since $P_{\ge N}P_{>1} = P_{\ge N}$ and since $P_{\ge N}$ is bounded on L^p, we can change $P_{>1}$ into $P_{\ge N}$ in (2.3.44) and (2.3.45), equation (ii). From (2.3.18), (2.3.45), equations (i) and (ii), and since $\pi_2\mathcal{W} = \partial_t\pi_1\mathcal{W}$, we have that, when $2 \le q \le 2^\sharp$,

$$\|\pi_1\mathcal{W}(t)(0,v)\|_{L^q} \le C\min(|t|^{-\frac{n}{4}(1-\frac{2}{q})}, |t|^{1-\frac{n}{2}(1-\frac{2}{q})})\|v\|_{L^{q'}}. \tag{2.3.46}$$

We mainly use the first bound in the right hand side of (2.3.46) for t large, and the second bound in the right hand side of (2.3.46) for t small. The function of t in the second bound is integrable around 0 when $q < 2^\sharp$. As a remark, $q = p+1$ is an important example, where p is the exponent in (2.0.1). At last we mention that (2.3.38) can be rewritten as

$$(u(t), u_t(t)) = \mathcal{W}(t)(u_0,u_1) + \int_0^t \mathcal{W}(t-s)(0,h(s))ds \tag{2.3.47}$$

for all t, and all solution u of (2.3.16). Equation (2.3.47) is referred to as the Duhamel formula for (2.3.16).

2.4 A general criterion for Scattering

We prove a general result for scattering in the spirit of the one in Tao and Visan [33] concerning the Schrödinger equation. As one can check, by our assumptions on p, the pairs

$$(2\frac{n+4}{n+8}p, 2\frac{n+4}{n+8}p) \text{ and } (2\frac{n+2}{n+4}p, 2\frac{n+2}{n+4}p)$$

are B-controlling in the sense of Definition 2.3.1. Our result is stated as follows.

Lemma 2.4.1. *Let $u \in \mathbb{E}_{\mathbb{R}_+}$ be a strong solution of (2.0.1) with $1 + \frac{8}{n} \le p \le 2^\sharp - 1$ when $n \ge 5$, and $1 + \frac{8}{n} \le p < \infty$ when $n \le 4$. Suppose that*

$$u \in L^{2\frac{n+4}{n+8}p}(\mathbb{R}_+ \times \mathbb{R}^n) \cap L^{2\frac{n+2}{n+4}p}(\mathbb{R}_+ \times \mathbb{R}^n). \tag{2.4.1}$$

Then there is scattering in forward time for $(u_0, u_1) = (u(0), u_t(0))$ and

$$E(u(0), u_t(0)) = E_0(u_0^+, u_1^+), \tag{2.4.2}$$

where (u_0^+, u_1^+) is the scattering pair associated to $(u(0), u_t(0))$ as in (2.1.4). Furthermore, W_+, as defined in (2.1.5), is continuous at (u_0, u_1) in the sense that if u^k is the solution of the nonlinear problem (2.0.1) corresponding to an initial data (u_0^k, u_1^k) such that $(u_0^k, u_1^k) \to (u_0, u_1)$ in $\mathcal{E}$ as $k \to +\infty$, then u^k is defined on $\mathbb{R}_+$ for k sufficiently large, and there is scattering in forward time for (u_0^k, u_1^k) with scattering associated pair $(u_0^{+,k}, u_1^{+,k})$ satisfying that $(u_0^{+,k}, u_1^{+,k}) \to (u_0^+, u_1^+)$ in $\mathcal{E}$ as $k \to +\infty$.

Proof. First, we prove that if u solves (2.0.1) with $1 + \frac{8}{n} \le p \le 2^\sharp - 1$ and (2.4.1) holds true, then there exists a couple $(u_0^+, u_1^+) \in \mathcal{E}$ such that

$$\|(u(t), u_t(t)) - \mathcal{W}(t)(u_0^+, u_1^+)\|_{\mathcal{E}} \to 0 \text{ as } t \to +\infty, \tag{2.4.3}$$

where (u_0^+, u_1^+) is uniquely defined by

$$(u_0^+, u_1^+) = (u_0, u_1) + \lambda \int_0^\infty \mathcal{W}(-s)(0, u^{\mathfrak{p}}(s))ds, \tag{2.4.4}$$

and $u^{\mathfrak{p}} = |u|^{p-1}u$ is as defined in Section 2.2. We prove (2.4.3) and (2.4.4) in what follows. Let

$$\overline{v}(t) = (v_0(t), v_1(t)) = \mathcal{W}(-t)(u(t), u_t(t)) \tag{2.4.5}$$

be the value at time $-t$ of the solution v of the Cauchy problem (2.1.2) with initial data $(v(0), v_t(0)) = (u(t), u_t(t))$. In order to prove (2.4.3) it suffices to prove that $(v_0(t), v_1(t))$ converges in $\mathcal{E}$ as $t \to +\infty$. It follows from Duhamel's formula (2.3.47) and the semigroup property that

$$\begin{aligned} (v_0(t), v_1(t)) &= \mathcal{W}(-t)\left(\mathcal{W}(t)(u_0, u_1) + \lambda \int_0^t \mathcal{W}(t-s)(0, u^{\mathfrak{p}}(s))ds\right) \\ &= (u_0, u_1) + \lambda \int_0^t \mathcal{W}(-s)(0, u^{\mathfrak{p}}(s))ds. \end{aligned} \tag{2.4.6}$$

Hence

$$\overline{v}(t+s) - \overline{v}(t) = \lambda \int_t^{t+s} \mathcal{W}(-t')(0, u^{\mathfrak{p}}(t'))dt',$$

where $\overline{v}$ is as in (2.4.5), and if $s \ge 0$, by the Strichartz estimates (2.3.43) with $(a, b) = (2(n+2)/n, 2(n+2)/n)$ and $(c, d) = (2(n+4)/n, 2(n+4)/n)$, we get that

$$\|\overline{v}(t+s) - \overline{v}(t)\|_{\mathcal{E}} \le C\left(\|u^{\mathfrak{p}}\|_{L^{a'}([t,t+s]\times\mathbb{R}^n)} + \|u^{\mathfrak{p}}\|_{L^{c'}([t,t+s]\times\mathbb{R}^n)}\right). \tag{2.4.7}$$

By (2.4.1), given $\epsilon > 0$, there exists t_0 sufficiently large such that

$$\|u\|_{L^{2\frac{n+4}{n+8}p}([t_0,\infty)\times\mathbb{R}^n)} + \|u\|_{L^{2\frac{n+2}{n+4}p}([t_0,\infty)\times\mathbb{R}^n)} \leq \epsilon.$$

As a consequence, by (2.4.7), for $t \geq t_0$ and $s \geq 0$,

$$\|(v_0(t+s), v_1(t+s)) - (v_0(t), v_1(t))\|_{\mathcal{E}} \leq C\epsilon$$

and we get that $\overline{v}(t)$ converges to some limit $\overline{u}_s^+ = (u_0^+, u_1^+)$ as $t \to +\infty$. Since $\mathcal{W}(t)$ is a unitary operator,

$$\|(u(t), u_t(t)) - \mathcal{W}(t)(u_0^+, u_1^+)\|_{\mathcal{E}} = \|\mathcal{W}(-t)(u(t), u_t(t)) - (u_0^+, u_1^+)\|_{\mathcal{E}} \to 0$$

as $t \to +\infty$. By Duhamel's formula we then get that

$$(u_0^+, u_1^+) = (u_0, u_1) + \lambda \int_0^t \mathcal{W}(-s)(0, u^{\mathfrak{p}}(s))ds + o(1) , \tag{2.4.8}$$

where $\|o(1)\|_{\mathcal{E}} \to 0$ as $t \to +\infty$, and letting $t \to +\infty$ in (2.4.8), we get (2.4.4). This ends the proof of (2.4.3) and (2.4.4). In what follows, we let

$$\mathcal{W}(t)(u_0^+, u_1^+) = (u^+(t), u_t^+(t)) \tag{2.4.9}$$

for $t \geq 0$, and we note that by the Strichartz estimates (2.3.17), we have that $u^+ \in L^{p+1}(\mathbb{R}, L^{p+1})$. Here we use (2.3.17) with $h = 0$ and the B-controlling pair $(p+1, p+1)$ which turns out to be B-controlling because of the assumptions on p. In particular, there exists a sequence of positive times $t_k \to \infty$ such that

$$\|u^+(t_k)\|_{L^{p+1}} \to 0. \tag{2.4.10}$$

By conservation of the energy for u and of the linear energy for u^+, and since $\|u^+(t_k) - u(t_k)\|_{H^2} + \|u_t^+(t_k) - u_t(t_k)\|_{L^2} \to 0$ by (2.4.3), we can write with (2.4.10) that

$$\begin{aligned} E(u_0, u_1) &= E(u(t_k), u_t(t_k)) \\ &= E(u^+(t_k), u_t^+(t_k)) + o(1) \\ &= E_0(u^+(t_k), u_t^+(t_k)) - \frac{\lambda}{p+1}\int_{\mathbb{R}^n} |u^+(t_k)|^{p+1} + o(1) \\ &= E_0(u_0^+, u_1^+) + o(1). \end{aligned}$$

Letting $k \to +\infty$, it follows that $E(u_0, u_1) = E_0(u_0^+, u_1^+)$. This proves (2.4.2). In order to end the proof of Lemma 2.4.1 it remains to prove the continuity of W_+ as defined in the lemma. Let $(u_0^k, u_1^k) \in \mathcal{E}$ be such that $(u_0^k, u_1^k) \to (u_0, u_1)$ in $\mathcal{E}$ as $k \to +\infty$. Let u^k be the solution of the nonlinear problem (2.0.1) associated to the Cauchy data (u_0^k, u_1^k) and, when it exists, $\overline{u}_s^{+,k} = (u_0^{+,k}, u_1^{+,k})$ be the associated scattering pair. Let $w = u - u^k$. Then w solves the equation

$$\frac{\partial^2 w}{\partial t^2} + \Delta^2 w + mw = \lambda u^{\mathfrak{p}} - \lambda(u-w)^{\mathfrak{p}} \tag{2.4.11}$$

with Cauchy data $(w(0), w_t(0)) = (u_0 - u_0^k, u_1 - u_1^k)$. Let $T > 0$ be such that

$$\|u\|_{L^{2\frac{n+4}{n+8}p}([T,\infty)\times\mathbb{R}^n)} + \|u\|_{L^{2\frac{n+2}{n+4}p}([T,\infty)\times\mathbb{R}^n)} < \epsilon, \tag{2.4.12}$$

where $\epsilon > 0$ is to be chosen later on. We know by the local theory, see the discussion after Lemma 2.3.1, that $w \to 0$ in $C([0,T],H^2) \cap C^1([0,T],L^2) \cap L^{2\frac{n+2}{n+4}p}([0,T]\times\mathbb{R}^n)$. For $t \geq T$, we let

$$g(t) = \|w\|_{L^{2\frac{n+4}{n+8}p}([T,t]\times\mathbb{R}^n)} + \|w\|_{L^{2\frac{n+2}{n+4}p}([T,t]\times\mathbb{R}^n)} + \|(w,w_t)\|_{C([T,t],\mathcal{E})}. \quad (2.4.13)$$

By the Strichartz estimates (2.3.17) that we consider for (2.4.11), we get that

$$\begin{aligned}
g(t) &\leq C\left(\sqrt{E_0(w(T),w_t(T))} + \sum_\rho \|u^{\mathfrak{p}} - (u-w)^{\mathfrak{p}}\|_{L^\rho([T,t]\times\mathbb{R}^n)}\right)\\
&\leq C\left(\sqrt{E_0(w(T),w_t(T))} + \sum_\rho \||u|^{p-1}|w| + |w|^p\|_{L^\rho([T,t]\times\mathbb{R}^n)}\right)\\
&\leq C\Bigg(\sqrt{E_0(w(T),w_t(T))} + \sum_\rho \Big(\|u\|^{p-1}_{L^{\rho p}([T,t]\times\mathbb{R}^n)}\|w\|_{L^{\rho p}([T,t]\times\mathbb{R}^n)}\\
&\qquad + \|w\|^p_{L^{\rho p}([T,t]\times\mathbb{R}^n)}\Big)\Bigg)\\
&\leq C\left(\sqrt{E_0(w(T),w_t(T))} + \sum_\rho \left(\epsilon^{p-1}h(t) + h(t)^p\right)\right),
\end{aligned}$$

where ϵ and T are as in (2.4.12), g is as in (2.4.13), and $\sum_\rho$ stands for the summation over the two values $\rho = 2(n+4)/(n+8)$ and $\rho = 2(n+2)/(n+4)$. Now we let $\epsilon \in (0,1)$ be such that $4C\epsilon^{\frac{8}{n}} < 1$ and we choose k sufficiently large such that

$$C\sqrt{E_0(w(T),w_t(T))} \leq \min\left(\frac{1}{6(24C)^{\frac{n}{4}}}, \frac{1}{6}\right).$$

Then

$$g(t) \leq 4C\sqrt{E_0(w(T),w_t(T))} \to 0 \quad (2.4.14)$$

as $k \to +\infty$, where w is as in (2.4.11). In particular, for k sufficiently large, u^k exists globally. Indeed, the u^k's are bounded in $\mathcal{E}$ by (2.4.14). As already mentioned, this ensures global existence when $p < 2^\sharp - 1$. By noting that the u^k's are also bounded in $L^{2\frac{n+2}{n-4}}(\mathbb{R}_+\times\mathbb{R}^n)$ when $p = 2^\sharp - 1$ and $n \geq 5$, we get global existence in that case from (2.3.14). Still by (2.4.14), now with $t = +\infty$, we get that $u^k \to u$ in $L^{2\frac{n+4}{n+8}p}(\mathbb{R}_+\times\mathbb{R}^n) \cap L^{2\frac{n+2}{n+4}p}(\mathbb{R}_+\times\mathbb{R}^n)$ as $k \to +\infty$. By (2.4.3) there is scattering in forward time for u^k and by (2.4.4), the convergence of u^k, and Strichartz estimates (2.3.43), we get that

$$\|\overline{u}_s^+ - \overline{u}_s^{+,k}\|_{\mathcal{E}} = |\lambda|\left\|\int_0^\infty \mathcal{W}(-s)(0, u^{\mathfrak{p}}(s) - (u(s)+w(s))^{\mathfrak{p}})ds\right\|_{\mathcal{E}} \to 0$$

as $k \to +\infty$. This ends the proof of Lemma 2.4.1. □

The following result is a useful corollary to Lemma 2.4.1. It will be used in the proof of our theorem in Section 2.1.

Corollary 2.4.1. *Let $n \geq 5$ and $u \in \mathbb{E}_{\mathbb{R}_+}$ be a strong solution of equation (2.0.1) with $1 + 8/n < p < \frac{n+4}{n-4}$. Assume $(u, u_t) \in \mathcal{E}$ is uniformly bounded with respect to t and that for some $\gamma \geq 1$,*

$$\|u(t)\|_{L^\gamma} \to 0 \tag{2.4.15}$$

as $t \to +\infty$. Then there is scattering in forward time for $(u(0), u_t(0))$, (2.4.1) holds true, and the conclusion of Lemma 2.4.1 also holds true.

Proof. By assumption u is uniformly bounded in $L^2 \cap L^{2^\sharp}$. By (2.4.15) and Hölder's inequality we then get that u converges to 0 in L^q at least for $2 < q < 2^\sharp$. In view of Lemma 2.4.1, and since, by the local theory discussed after Lemma 2.3.1,

$$u \in C(\mathbb{R}_+, H^2) \cap L_{loc}^{2\frac{n+2}{n+4}p}(\mathbb{R}_+, L^{2\frac{n+2}{n+4}p}),$$

the corollary reduces to proving that there exists $T_0 \geq 0$ such that

$$\|u\|_{L^{2\frac{n+4}{n+8}p}([T_0,\infty)\times\mathbb{R}^n)} + \|u\|_{L^{2\frac{n+2}{n+4}p}([T_0,\infty)\times\mathbb{R}^n)} \leq C \tag{2.4.16}$$

for some constant $C > 0$. Let $2 < r = 2np/(n+8), \rho = 2np/(n+4) < 2^\sharp$, and $\epsilon > 0$ be some positive constant to be chosen later on. Let $T_0 > 0$ be such that

$$\sup_{t \geq T_0} \left(\|u(t)\|_{L^r} + \|u(t)\|_{L^\rho}\right) \leq \epsilon \tag{2.4.17}$$

and, for $t \geq T_0$, let

$$g(t) = \max\left(\|u\|_{L^{2\frac{n+4}{n+8}p}([T_0,t)\times\mathbb{R}^n)}, \|u\|_{L^{2\frac{n+2}{n+4}p}([T_0,t)\times\mathbb{R}^n)}\right).$$

By Duhamel's formula (2.3.47),

$$(u(t), u_t(t)) = \mathcal{W}(t - T_0)(u(T_0), u_t(T_0)) + \lambda \int_{T_0}^t \mathcal{W}(t-s)\,(0, u^{\mathfrak{p}}(s))\,ds$$

for all $t \geq T_0$. By the Strichartz estimates (2.3.17) in Lemma 2.3.2, and (2.4.17), using Hölders' inequalities, and since $(2\frac{n+4}{n+8}p, 2\frac{n+4}{n+8}p)$ and $(2\frac{n+2}{n+4}p, 2\frac{n+2}{n+4}p)$ are B-controlling pairs, we then get that

$$\begin{aligned}
g(t) &\leq C\sqrt{E_0(u(T_0), u_t(T_0))} + C\left(\|u^{\mathfrak{p}}\|_{L^2([T_0,t],L^{\frac{2n}{n+2}})} + \|u^{\mathfrak{p}}\|_{L^2([T_0,t],L^{\frac{2n}{n+4}})}\right)\\
&\leq C\sqrt{E_0(u(T_0), u_t(T_0))} + C\Bigg(\|u\|_{L^\infty([T_0,t],L^\rho)}^{\frac{2p}{n+4}}\|u\|_{L^{2\frac{n+2}{n+4}p}([T_0,t]\times\mathbb{R}^n)}^{\frac{(n+2)p}{n+4}}\\
&\quad + \|u\|_{L^\infty([T_0,t],L^r)}^{\frac{4p}{n+8}}\|u\|_{L^{2\frac{n+4}{n+8}p}([T_0,t]\times\mathbb{R}^n)}^{\frac{(n+4)p}{n+8}}\Bigg)\\
&\leq C\left(\sqrt{E_0(u(T_0), u_t(T_0))} + \epsilon^{\frac{2p}{n+4}} h(t)^{\frac{(n+2)p}{n+4}} + \epsilon^{\frac{4p}{n+8}} h(t)^{\frac{(n+4)p}{n+8}}\right).
\end{aligned} \tag{2.4.18}$$

It can be noted here that $(2, 2^\star)$ is S-admissible and that $(2, 2^\sharp)$ is Bl-admissible. The first inequality in (2.4.18) is by (2.3.17), the second inequality is by Hölder's inequality, and the third inequality is by (2.4.17). Now we remark that g is continuous, that $g(T_0) = 0$, and that for any $t > T_0$,

$$g(t) \leq C' + \epsilon' \left(g(t)^{\frac{(n+2)p}{n+4}} + g(t)^{\frac{(n+4)p}{n+8}} \right), \tag{2.4.19}$$

where $C' = C\sqrt{E(u_0, u_1)}$ does not depend on t, and $\epsilon' = C(\epsilon^{\frac{2p}{n+4}} + \epsilon^{\frac{4p}{n+8}})$ can be made as small as we want when ϵ is sufficiently small. In particular, we can choose ϵ such that

$$\epsilon' < \frac{C'}{(2C')^{\frac{(n+2)p}{n+4}} + (2C')^{\frac{(n+4)p}{n+8}}}.$$

Since the two powers in (2.4.19) are greater than 1 by our assumptions on p, we get that $g(t) \leq 2C'$ for all $t \geq T_0$. This proves (2.4.16), and thus also the corollary. □

By standard arguments the counterpart to Lemma 2.4.1 holds true. To make a precise statement, it follows from standard arguments that when $1 + \frac{8}{n} \leq p \leq 2^\sharp - 1$ (respectively $1 + \frac{8}{n} \leq p < \infty$ when $n \leq 4$), given any solution of the linear equation (2.1.2), written as $(\omega, \omega_t) = \mathcal{W}(\cdot)(u_0^+, u_1^+)$, there exists T and a unique solution u of the nonlinear equation (2.0.1), defined on $[T, \infty)$, such that (2.1.4) holds true and

$$u \in L^{2\frac{n+2}{n+4}p}([T, \infty) \times \mathbb{R}^n) \cap L^{2\frac{n+4}{n+8}p}([T, \infty) \times \mathbb{R}^n).$$

Furthermore, one has a continuity property in the sense that if

$$(u_0^{+,k}, u_1^{+,k}) \to (u_0^+, u_1^+)$$

in $\mathcal{E}$, and u^k is the associated solution to the nonlinear equation (2.0.1), then, for k sufficiently large, u^k can be defined on $[T, \infty)$ and u^k converges to u in $C([T, \infty), \mathcal{E})$ as $k \to +\infty$. Besides, if $E_0(u_0^+, u_1^+)$ is sufficiently small, or if $\lambda < 0$ and $1 + \frac{8}{n} \leq p < 2^\sharp - 1$, then u extends to a global solution and then $(u^k(0), u_t^k(0)) \to (u(0), u_t(0))$ in $\mathcal{E}$. One can prove such a counterpart by following the proof for the Schrödinger equation in Cazenave [6]. The counterpart to Lemma 2.4.1 provides the surjectivity of W_+ as well as the continuity of its inverse mapping in our theorem, where W_+ is as in (2.1.5). By time reversibility, the results in this section, and the remark we just made, hold true for $t \to -\infty$.

As a final remark in this section we mention that small data scattering in all dimensions, in the defocusing as well as in the focusing case, and for the energy-subcritical as well as for the energy-critical case of (2.0.1), easily follows from the estimates in Lemma 2.3.2 and from Lemma 2.4.1. Let n be arbitrary, $\lambda \neq 0$ be arbitrary, and p be such that $1 + \frac{8}{n} \leq p \leq 2^\sharp - 1$. Thanks to the local theory we discussed after Lemma 2.3.1, the estimates in Lemma 2.3.2, and Lemma 2.4.1, we can prove, following standard schemes, that there exists $\epsilon_0 > 0$ such that scattering for (2.0.1) holds true for any initial data $(u, v) \in \mathcal{E}$ of energy $E_0(u, v) \leq \varepsilon_0$. Moreover, $E \geq 0$ for such initial data, and W_+ in (2.1.5) realizes an homeomorphism from $\mathfrak{F}'_\varepsilon$ onto $\mathfrak{B}_\varepsilon$ for all $\varepsilon \in (0, \varepsilon_0]$, where $\mathfrak{F}'_\varepsilon$ consists of the $(u, v) \in \mathcal{E}$ such that $E_0(u, v) \leq \varepsilon_0$ and $E(u, v) \leq \varepsilon$, and $\mathfrak{B}_\varepsilon$ consists of the $(u, v) \in \mathcal{E}$ such that $E_0(u, v) \leq \varepsilon$. The case $p < 2^\sharp - 1$ in this statement was proved by Levandosky [18], as well as it was proved by Levandosky [18] that the equation possesses travelling waves of arbitrarily low energy when $\lambda > 0$ and $p < 1+\frac{4}{n}$. Travelling waves cannot scatter since their $L^q(\mathbb{R}^n)$-norms, $2 \le q \le 2^\sharp$, are constant, whereas, by Strichartz estimates, solutions of the linear equations have powers of their L^q-norm integrable in time. If we accept complex valued fonctions, then, based on material in Levandosky [17], we can construct standing waves with arbitrarily small energy when $p < 1 + \frac{8}{n}$, contradicting once again scattering in the small energy setting.

2.5 Frequency localization

We prove frequency localization for solutions of the nonlinear equation (2.0.1). We assume in what follows that p is such that

$$1 + \frac{8}{n} < p < 2^\sharp - 1, \tag{2.5.1}$$

and that $\lambda < 0$. We prove the following frequency localization result in this section, using ideas recently introduced by Tao [31] for the Schrödinger equation.

Lemma 2.5.1. *Let $n \ge 5$, and $u \in \mathbb{E}_{\mathbb{R}_+}$ be a forward global solution of the nonlinear equation (2.0.1) with $\lambda < 0$ and p such that (2.5.1) holds true. There exists a couple $(u_0^+, u_1^+) \in \mathcal{E}$, $\eta > 0$, and a function $w \in \mathbb{E}_{\mathbb{R}_+}$ such that*

$$\begin{aligned}
&(u, u_t) = \mathcal{W}(\cdot)(u_0^+, u_1^+) + (w, w_t)\ , \\
&\mathcal{W}(-t)(w(t), w_t(t)) \rightharpoonup (0,0) \text{ in } \mathcal{E} \text{ as } t \to +\infty\ , \text{ and} \\
&\sup_{N\ge 1} \limsup_{t\to\infty} N^\eta E_0\left(P_{\ge N}(w(t), w_t(t))\right) \le C\ ,
\end{aligned} \tag{2.5.2}$$

where $C > 0$ depends only on $E\left(u(0), u_t(0)\right)$, m, λ, and n.

As a consequence of this lemma we get that the following corollary holds true. We prove the corollary in what follows and then prove the lemma in several steps.

Corollary 2.5.1. *Let $n \ge 5$, $u \in \mathbb{E}_{\mathbb{R}_+}$ be a forward global solution of the nonlinear equation (2.0.1) with $\lambda < 0$, and p such that (2.5.1) holds true, and $\epsilon > 0$. There exists t_0 and N such that*

$$E_0(P_{\ge N}(u(t), u_t(t))) \le \epsilon^2 \tag{2.5.3}$$

for all time $t \ge t_0$.

Proof of Corollary 2.5.1. Since $(u_0^+, u_1^+) \in \mathcal{E}$, there exists N_0 such that

$$E_0(P_{\ge N_0}(u_0^+, u_1^+)) \le \frac{\epsilon^2}{4}. \tag{2.5.4}$$

Since $\mathcal{W}$ is a unitary operator and since $\mathcal{W}$ commutes with $P_{\ge N}$ for any N, we get by (2.5.4) that for any time t, and for any $N > N_0$,

$$\begin{aligned}
E_0(P_{\ge N}\mathcal{W}(t)(u_0^+, u_1^+)) &= E_0(\mathcal{W}(t)P_{\ge N}(u_0^+, u_1^+)) \\
&= E_0(P_{\ge N}(u_0^+, u_1^+)) \\
&= E_0(P_{\ge N}P_{\ge N_0}(u_0^+, u_1^+)) \le \frac{\epsilon^2}{4}.
\end{aligned} \tag{2.5.5}$$

Independently, by (2.5.2), there exists N_1 such that

$$E_0(P_{\geq N}(w(t), w_t(t))) \leq \frac{\epsilon^2}{4} \tag{2.5.6}$$

for all $N \geq N_1$, and all $t \geq t_N$, where t_N depends only on N. Let $N > \max(N_0, N_1)$, and $t \geq t_N$. By (2.5.2), (2.5.5), and (2.5.6) we then get that

$$\begin{aligned} E_0(P_{\geq N}(u(t), u_t(t))) &\leq 2\left(E_0(P_{\geq N}(w(t), w_t(t))) + E_0(P_{\geq N}\mathcal{W}(t)(u_0^+, u_1^+))\right) \\ &\leq \epsilon^2. \end{aligned}$$

This proves the corollary. □

Now it remains to prove Lemma 2.5.1. We proceed in several steps. As a first remark, we note that, when p satisfies (2.5.1), there always exist an S-admissible pair (a, b), $d \geq 2$, $\kappa \in (0,1)$,

$$\frac{2}{p} < \alpha < \frac{2n}{n+4}, \tag{2.5.7}$$

α close to $2n/(n+4)$, and $\theta \in (0,1)$ such that $a > 2$ and

$$\begin{aligned} &(i) && \frac{1}{b'} = \frac{p-\kappa}{d} + \frac{\kappa}{2}, \\ &(ii) && a'(p-\kappa) \geq 2, \\ &(iii) && \frac{n-4}{2} < \frac{2}{a'(p-\kappa)} + \frac{n}{d} < \frac{n}{2}, \\ &(iv) && \frac{1}{\alpha p} = \frac{1-\theta}{2} + \frac{\theta}{\alpha'}, \text{ and } p\theta > 1. \end{aligned} \tag{2.5.8}$$

Now Step 2.5.1 states as follows. Without loss of generality, we assume in the sequel that $m = 1$ and $\lambda = -1$.

Step 2.5.1. *Let $I \subset \mathbb{R}$ be an interval, and $u \in \mathbb{E}_I$ be a solution of (2.0.1) with $\lambda = -1$ and p such that (2.5.1) holds true. Let also $E > 0$ be such that $E(u, u_t) \leq E$. For any B-admissible pair (q, r),*

$$\|u\|_{L^q(I, L^r)} \leq C(1 + |I|)^{\frac{1}{q}} \tag{2.5.9}$$

where C depends only on E, q, and n.

Proof of Step 2.5.1. Step 2.5.1 follows from the Strichartz estimates in Lemma 2.3.1. First we assume that $|I| \leq 1$ is small enough, $I = [t_0, t_1]$. We write (2.0.1) as a superposition of two linear beam equations as in (2.3.5), with forcing term $h_1 = -u^{\mathfrak{p}}$ and $h_2 = -u$. Suppose first that $p > (n+2)/(n-4)$, then there exists $\delta > 0$ such that $(2p + \delta, \frac{2np}{n+2}) = (\gamma, \rho)$ is B-admissible. Let $\mu > 0$ be such that $\frac{1}{2p} = \frac{1}{2p+\delta} + \frac{1}{\mu}$. By the Strichartz estimates (2.3.6) in Lemma 2.3.1, that we apply to the two linear beam equations with forcing terms h_1 and h_2,

$$\begin{aligned} \|u\|_{L^\gamma(I, L^\rho)} &\leq C\left(\sqrt{E(u, u_t)} + \|u^{\mathfrak{p}}\|_{L^2(I, L^{\frac{2n}{n+2}})} + \|u\|_{L^1(I, L^2)}\right) \\ &\leq C\left(\sqrt{E(u, u_t)} + \|u\|^p_{L^{2p}(I, L^{\frac{2np}{n+2}})}\right) \\ &\leq C\left(\sqrt{E(u, u_t)} + |I|^{\frac{p}{\mu}} \|u\|^p_{L^\gamma(I, L^\rho)}\right), \end{aligned}$$

where $u^{\mathfrak{p}} = |u|^{p-1}u$. Besides, $h(t) = \|u\|_{L^\gamma([t_0,t],L^\rho)}$ is continuous and $h(0) = 0$. It follows that if $|I| \le \varepsilon_0$ is sufficiently small, then

$$\|u\|_{L^\gamma(I,L^\rho)} \le 2C\sqrt{E(u,u_t)}. \tag{2.5.10}$$

Applying the Strichartz estimates (2.3.6), with (2.3.13) if $p \le (n+2)/(n-4)$, or (2.5.10) if $p > (n+2)/(n-4)$, since (q,r) is B-admissible and $(2,2^\star)$ is S-admissible, we get that

$$\|u\|_{L^q(I,L^r)} \le C\left(\sqrt{E(u,u_t)} + \|u^{\mathfrak{p}}\|_{L^2(I,L^{\frac{2n}{n+2}})}\right) \le C'. \tag{2.5.11}$$

Now, if I is of arbitrary length, we decompose $I = \cup_{j=1}^k I_j$ with the I_j's such that their interiors are disjoint and such that $|I_j| = \varepsilon_0$, except maybe for the last interval which can be of a smaller length. Then $k \le \frac{|I|}{\varepsilon_0} + 1$ and

$$\|u\|^q_{L^q(I,L^r)} = \sum_{j=1}^k \|u\|^q_{L^q(I_j,L^r)} \le C\left(|I|+1\right).$$

This ends the proof of Step 2.5.1. □

The next step in the proof of Lemma 2.5.1 is stated as follows.

Step 2.5.2. *Let $u \in \mathbb{E}_I$ be a forward solution of (2.0.1) with $\lambda = -1$ and p such that (2.5.1) holds true. For (a,b) an S-admissible pair like in (2.5.8), there exist $\eta > 0$, and $C > 0$ depending only on n and $E = E\left(u(0),u_t(0)\right)$, such that*

$$\|P_{\ge N}u^{\mathfrak{p}}\|_{L^{a'}(I,L^{b'})} \le CN^{-\eta}(1+|I|)^{\frac{1}{a'}} \tag{2.5.12}$$

for all finite interval $I \subset \mathbb{R}_+$.

Proof of Step 2.5.2. Again we may assume that $|I| \le 1$. The case of intervals of arbitrary length follows from the case $|I| \le 1$ as in the proof of Step 2.5.1. Let $u_h = P_{\ge N}u$ and $u_l = u - u_h$. Then

$$|u^{\mathfrak{p}} - u_l^{\mathfrak{p}}| \le C|u_h|(|u|^{p-1} + |u_l|^{p-1}),$$

and we get with Hölder's inequality, (2.2.3), (2.3.13), (2.5.8) equation (i), and (2.5.9), that

$$\begin{aligned}
&\|P_{\ge N}(u^{\mathfrak{p}} - u_l^{\mathfrak{p}})\|_{L^{a'}(I,L^{b'})} \le C\||u_h|^\kappa|u_h|^{1-\kappa}(|u|^{p-1} + |u_l|^{p-1})\|_{L^{a'}(I,L^{b'})}\\
&\le C\|u_h\|^\kappa_{L^\infty(I,L^2)}\||u_h|^{1-\kappa}(|u|^{p-1}+|u_l|^{p-1})\|_{L^{a'}(I,L^{\frac{d}{p-\kappa}})}\\
&\le C\|u_h\|^\kappa_{L^\infty(I,L^2)}\|u_h\|^{1-\kappa}_{L^{a'(p-\kappa)}(I,L^d)}\left(\|u\|^{p-1}_{L^{a'(p-\kappa)}(I,L^d)} + \|u_l\|^{p-1}_{L^{a'(p-\kappa)}(I,L^d)}\right)\\
&\le CN^{-2\kappa}\|u_h\|_{L^\infty(I,H^2)} \le CN^{-2\kappa},
\end{aligned} \tag{2.5.13}$$

where C depends only on n and E, where $\kappa > 0$ by (2.5.8), and where we used the fact that the norm of u with respect to the pair $(a'(p-\kappa),d)$ can be controlled thanks to (2.3.13) and Step 2.5.1. The middle inequalities in (2.5.13) are because of Hölder's inequality and (2.5.8) equation (i). The last inequality in (2.5.13) is by (2.2.3) equation (i), and (2.5.9). Independently, still by (2.2.3), (2.3.13), (2.5.8), and (2.5.9), we can write that

$$\begin{aligned}
&\|P_{\geq N}u_l^{\mathfrak{p}}\|_{L^{a'}(I,L^{b'})} \leq CN^{-1}\||\nabla|u_l^{\mathfrak{p}}\|_{L^{a'}(I,L^{b'})}\\
&\leq CN^{-1}\|\nabla u_l^{\mathfrak{p}}\|_{L^{a'}(I,L^{b'})}\\
&\leq CN^{-1}\||\nabla u_l|^{\kappa}|\nabla u_l|^{1-\kappa}|u_l|^{p-1}\|_{L^{a'}(I,L^{b'})}\\
&\leq CN^{-1}\|\nabla u_l\|^{\kappa}_{L^\infty(I,L^2)}\|\nabla u_l\|^{1-\kappa}_{L^{a'(p-\kappa)}(I,L^d)}\|u_l\|^{p-1}_{L^{a'(p-\kappa)}(I,L^d)}\\
&\leq CN^{-1}\|u\|^{\kappa}_{L^\infty(I,H^1)}N^{1-\kappa}\|u\|^{1-\kappa}_{L^{a'(p-\kappa)}(I,L^d)}\|u\|^{p-1}_{L^{a'(p-\kappa)}(I,L^d)} \leq CN^{-\kappa}.
\end{aligned} \tag{2.5.14}$$

The first inequality in (2.5.14) is by (2.2.3) equation (i). The second inequality is by boundedness of Riesz transforms. The third inequality is by direct computations. The fourth inequality is by (2.5.8) equation (i). The last inequality in (2.5.14) is by (2.2.3) equation (ii). By letting $\eta = \kappa$, (2.5.12) in Step 2.5.2 follows from (2.5.13) and (2.5.14) when $|I| \leq 1$. As already mentioned, this ends the proof of Step 2.5.2. □

The last step before the proof of Lemma 2.5.1 is stated as follows.

Step 2.5.3. *Let $u \in \mathbb{E}_{\mathbb{R}_+}$ be a forward global solution of (2.0.1) with $\lambda = -1$ and p such that (2.5.1) holds true. Let also $E > 0$ be such that $E(u, u_t) \leq E$. Then, there exist a couple $\overline{u}_s = (u_0^+, u_1^+) \in \mathcal{E}$, and a function $w \in \mathbb{E}_{\mathbb{R}_+}$, such that*

$$\begin{aligned}
&(u(t), u_t(t)) = \mathcal{W}(t)\overline{u}_s + (w(t), w_t(t))\ ,\\
&E_0(\overline{u}_s) \leq E\ ,\ E_0(w(t), w_t(t)) \leq 4E\ ,\ \text{and}\\
&\mathcal{W}(-t)(w(t), w_t(t)) \rightharpoonup (0,0)\ \text{in}\ \mathcal{E}
\end{aligned} \tag{2.5.15}$$

as $t \to +\infty$, where the first two equations hold true for all $t \geq 0$. Furthermore,

$$\begin{aligned}
(w(t), w_t(t)) &= \mathcal{W}(t)(u_0 - u_0^+, u_1 - u_1^+) - \int_0^t \mathcal{W}(t-s)(0, u^{\mathfrak{p}}(s))ds\\
&= w\text{–}\lim_{T\to\infty}\int_t^T \mathcal{W}(t-s)(0, u^{\mathfrak{p}}(s))ds
\end{aligned} \tag{2.5.16}$$

for all $t \geq 0$, where the notation w–lim stands for the weak limit.

Proof of Step 2.5.3. By conservation of the energy (2.3.13), and since $\mathcal{W}$ is a unitary operator, we get that $\overline{v}(t)$ is uniformly bounded in $\mathcal{E}$, where for any time $t \geq 0$,

$$\overline{v}(t) = \mathcal{W}(-t)(u(t), u_t(t)). \tag{2.5.17}$$

Hence, up to a subsequence, $\overline{v}(t)$ converges weakly in $\mathcal{E}$ as $t \to +\infty$.

We claim that the limit is unique. In order to prove the claim it suffices to prove that

$$\lim_{t_1,t_2\to+\infty}\langle\overline{v}(t_1) - \overline{v}(t_2), \overline{\phi}\rangle_{\mathcal{E}} = 0 \tag{2.5.18}$$

for all $\phi_0, \phi_1 \in C_c^\infty(\mathbb{R}^n)$, where $\overline{\phi} = (\phi_0, \phi_1)$, and $\langle\cdot,\cdot\rangle_{\mathcal{E}}$ stands for the scalar product in $\mathcal{E}$. Let $t_2 \leq t_1 \in \mathbb{R}_+$, and $\phi_0, \phi_1 \in C_c^\infty(\mathbb{R}^n)$. By Duhamel's formula

(2.3.47), the semigroup property of $\mathcal{W}$, and since $\mathcal{W}$ is a unitary operator, we have

$$\begin{aligned}\left|\left\langle \overline{v}(t_1)-\overline{v}(t_2),\overline{\phi}\right\rangle_{\mathcal{E}}\right| &= \left|\left\langle \int_{t_2}^{t_1} \mathcal{W}(-s)(0,u^{\mathfrak{p}}(s))ds,\overline{\phi}\right\rangle_{\mathcal{E}}\right| \\ &\leq \int_{t_2}^{t_1} \left|\left\langle (0,u^{\mathfrak{p}}(s)),\mathcal{W}(s)\overline{\phi}\right\rangle_{\mathcal{E}}\right| ds \\ &\leq \int_{t_2}^{t_1} \|u^{\mathfrak{p}}(s)\|_{L^{\alpha}} \|\pi_2\mathcal{W}(s)\overline{\phi}\|_{L^{\alpha'}} ds \\ &\leq C\|u\|^p_{L^\infty(\mathbb{R},H^2)} \int_{t_2}^{t_1} \|\pi_2\mathcal{W}(s)\overline{\phi}\|_{L^{\alpha'}} ds,\end{aligned} \tag{2.5.19}$$

where α is as in (2.5.7), so that $H^2 \subset L^{\alpha p}$. Now, since $\alpha' > 2^\sharp$, by (2.3.45) equation (i) we get that there exists $\delta > 0$ and $C > 0$ such that for any $s > 0$,

$$\|\pi_2\mathcal{W}(s)\overline{\phi}\|_{L^{\alpha'}} \leq Cs^{-1-\delta} \tag{2.5.20}$$

and from (2.5.19), (2.5.20), we deduce that (2.5.18) holds true. This implies uniqueness and the above claim.

By (2.5.18) we also get that there exists a pair $(u_0^+, u_1^+) \in \mathcal{E}$ such that

$$\overline{v}(t) \rightharpoonup (u_0^+, u_1^+) \tag{2.5.21}$$

weakly in $\mathcal{E}$ as $t \to +\infty$. Besides, since $\mathcal{W}$ is a unitary operator, and by conservation of the energy as in (2.3.13), we have that

$$\|\overline{v}(t)\|_{\mathcal{E}} = \|(u(t), u_t(t))\|_{\mathcal{E}} \leq \sqrt{E} \tag{2.5.22}$$

while, by weak lower semicontinuity of the norm, we get from (2.5.22) that

$$\|(u_0^+, u_1^+)\|_{\mathcal{E}} \leq \sqrt{E}. \tag{2.5.23}$$

In what follows we let

$$(w(t), w_t(t)) = (u(t), u_t(t)) - \mathcal{W}(t)(u_0^+, u_1^+). \tag{2.5.24}$$

Then the first equation in (2.5.15) holds true. By conservation of the energy (2.3.13), and (2.5.23), we can write that $\|(w, w_t)\|_{\mathcal{E}} \leq 2\sqrt{E}$. Together with (2.5.21), (2.5.23), and (2.5.24), this proves that the second and third equations in (2.5.15) also hold true. Now it remains to prove (2.5.16). By Duhamel's formula (2.3.47), we have

$$(w(t), w_t(t)) = \mathcal{W}(t)(u_0 - u_0^+, u_1 - u_1^+) - \int_0^t \mathcal{W}(t-s)(0, u^{\mathfrak{p}}(s))ds. \tag{2.5.25}$$

This proves the first equation in (2.5.16). We fix $T > 0$. By Duhamel's formula (2.3.47) with initial time T,

$$(u(t), u_t(t)) = \mathcal{W}(t)\mathcal{W}(-T)(u(T), u_t(T)) + \int_t^T \mathcal{W}(t-s)(0, u^{\mathfrak{p}}(s))ds.$$

As a consequence,

$$
\begin{aligned}
(w(t), w_t(t)) =& \mathcal{W}(t)\left(\mathcal{W}(-T)(u(T), u_t(T)) - (u_0^+, u_1^+)\right) \\
&+ \int_t^T \mathcal{W}(t-s)(0, u^{\mathfrak{p}}(s))ds
\end{aligned}
\tag{2.5.26}
$$

for all $t \leq T$. Using (2.5.21), and letting $T \to +\infty$ in (2.5.26), we obtain that the second equation in (2.5.16) holds true. This ends the proof of Step 2.5.3. □

Thanks to Steps 2.5.1–2.5.3 we are in position to prove our frequency localization Lemma 2.5.1. We prove the lemma in the sequel.

Proof of Lemma 2.5.1. We suppose $N \geq 8$. We let $\epsilon = N^{-\eta_0} > 0$ where η_0 is to be defined later on. By density of smooth functions in the energy space, we can find an element $\overline{\phi} = (\phi_0, \phi_1) \in C_c^\infty(\mathbb{R}^n) \times C_c^\infty(\mathbb{R}^n)$ such that

$$
\|(u_0 - u_0^+, u_1 - u_1^+) - \overline{\phi}\|_{\mathcal{E}} \leq \epsilon,
\tag{2.5.27}
$$

where $u_0 = u(0)$, and $u_1 = u_t(0)$. Applying $P_{\geq N}$ to the two equations in (2.5.16), we get

$$
\begin{aligned}
P_{\geq N}(w(t), w_t(t)) &= \mathcal{W}(t) P_{\geq N}\left(\overline{\phi} + \overline{e}\right) - \int_0^t \mathcal{W}(t-s)(0, P_{\geq N} u^{\mathfrak{p}}(s))ds \\
&= w\text{-}\lim_{T\to\infty} \int_t^T \mathcal{W}(t-s)(0, P_{\geq N} u^{\mathfrak{p}}(s))ds,
\end{aligned}
\tag{2.5.28}
$$

where $\overline{e} = (u_0 - u_0^+, u_1 - u_1^+) - \overline{\phi}$. By step 2.5.3, $E_0(\overline{w}) \leq 4E$, where $\overline{w} = (w, w_t)$. Then, with (2.5.27) and (2.5.28), since $\mathcal{W}$ is a unitary operator and $P_{\geq N}$ is bounded on $\mathcal{E}$, we get that for $t \geq 0$,

$$
\begin{aligned}
&E_0(P_{\geq N}\overline{w}) = \left|\langle P_{\geq N}\overline{w}, P_{\geq N}\overline{w}\rangle_{\mathcal{E}}\right| \\
&\leq \left|\left\langle P_{\geq N}\overline{w}, \mathcal{W}(t)P_{\geq N}\overline{\phi} - \int_0^t \mathcal{W}(t-s)(0, P_{\geq N}u^{\mathfrak{p}}(s))ds\right\rangle_{\mathcal{E}}\right| + 2\sqrt{E}\epsilon \\
&\leq \left|\left\langle w\text{-}\lim_T \int_t^T \mathcal{W}(t-t')(0, P_{\geq N}u^{\mathfrak{p}}(t'))dt', \mathcal{W}(t)P_{\geq N}\overline{\phi}\right\rangle_{\mathcal{E}}\right| + C\epsilon \\
&\quad + \left|\left\langle w\text{-}\lim_T \int_t^T \mathcal{W}(t-t')(0, P_{\geq N}u^{\mathfrak{p}}(t'))dt', \int_0^t \mathcal{W}(t-s)(0, P_{\geq N}u^{\mathfrak{p}}(s))ds\Big)\right\rangle_{\mathcal{E}}\right|,
\end{aligned}
$$

where $\langle \cdot, \cdot \rangle_{\mathcal{E}}$ stands for the scalar product in $\mathcal{E}$. Then we get that

$$
E_0(P_{\geq N}\overline{w}) \leq \int_t^\infty U_N(t')dt' + \left|\int_t^T \int_0^t V_N(s, t')dsdt'\right| + C\epsilon,
\tag{2.5.29}
$$

where, by semigroup property, and since $\mathcal{W}$ is a unitary operator,

$$
\begin{aligned}
U_N(t') &= \left|\langle \mathcal{W}(t-t')(0, P_{\geq N}u^{\mathfrak{p}}(t')), \mathcal{W}(t)P_{\geq N}\overline{\phi}\rangle_{\mathcal{E}}\right| \\
&= \left|\langle (0, P_{\geq N}u^{\mathfrak{p}}(t')), \mathcal{W}(t')P_{\geq N}\overline{\phi}\rangle_{\mathcal{E}}\right|, \\
V_N(s, t') &= \langle \mathcal{W}(t-t')\left(0, P_{\geq N}u^{\mathfrak{p}}(t')\right), \mathcal{W}(t-s)\left(0, P_{\geq N}u^{\mathfrak{p}}(s)\right)\rangle_{\mathcal{E}} \\
&= \langle (0, P_{\geq N}u^{\mathfrak{p}}(t')), \mathcal{W}(t'-s)(0, P_{\geq N}u^{\mathfrak{p}}(s))\rangle_{\mathcal{E}},
\end{aligned}
\tag{2.5.30}
$$

and $T = T(t)$ is taken sufficiently large. Now we estimate each term in (2.5.29) and split the integral into several parts. First, using the fact that $H^2 \subset L^{\alpha p}$, which follows from (2.5.7), conservation of the energy, (2.2.3) equation (i), and the fast decay of $P_{\geq N}\mathcal{W}$ as in (2.3.45) equation (ii), we observe that

$$\begin{aligned} U_N(t') &= \left|\left\langle (0, P_{\geq N} u^{\mathfrak{p}}(t')), \mathcal{W}(t') P_{\geq N} \overline{\phi} \right\rangle_{\mathcal{E}}\right| \\ &\leq ||u^{\mathfrak{p}}||_{L^\infty(\mathbb{R}_+, L^\alpha)} \left|\left|\pi_2 \mathcal{W}(t') P_{\geq N}\overline{\phi}\right|\right|_{L^{\alpha'}} \leq Ct'^{-2-\delta} \end{aligned} \tag{2.5.31}$$

for $\delta = \frac{n}{2}\left(1 - \frac{2}{\alpha'}\right) - 2$. It turns out that $\delta > 0$ since, by (2.5.7), we have $\alpha < \frac{2n}{n+4}$. It follows from (2.5.31) that

$$\int_t^\infty U_N(t')dt' \leq \epsilon \tag{2.5.32}$$

for $t > 0$ sufficiently large. Now, from (2.2.3) equation (i), and (2.3.45) equation (ii), we observe that

$$\begin{aligned} |V_N(s,t')| &\leq \|P_{\geq N} u^{\mathfrak{p}}(t')\|_{L^\alpha} \|\pi_2 \mathcal{W}(t'-s)(0, P_{\geq N} u^{\mathfrak{p}}(s))\|_{L^{\alpha'}} \\ &\leq C|t'-s|^{-2-\delta} \|u^{\mathfrak{p}}\|^2_{L^\infty(\mathbb{R}_+, L^\alpha)}. \end{aligned}$$

Hence, by conservation of the energy (2.3.13), for $0 < \eta_1 = \frac{a'}{4}\eta$, where a, η are as in Step 2.5.2, we get that

$$\left| \int_{t' \geq t + N^{\eta_1}} \int_{0 \leq s \leq t} V_N(s,t') ds dt' \right| \leq CN^{-\delta\eta_1}. \tag{2.5.33}$$

Similarly, assuming $t \geq N^{\eta_1}$,

$$\left| \int_{t' \geq t} \int_{0 \leq s \leq t - N^{\eta_1}} V_N(s,t') ds dt' \right| \leq CN^{-\delta\eta_1}. \tag{2.5.34}$$

Writing $I = \{t - N^{\eta_1} \leq s \leq t\}$, and using (2.3.44) and (2.5.30), we get that

$$\begin{aligned} &\left| \int_t^{t+N^{\eta_1}} \int_{t-N^{\eta_1} \leq s \leq t} V_N(s,t') ds dt' \right| \\ &= \left| \int_t^{t+N^{\eta_1}} \left\langle (0, P_{\geq N} u^{\mathfrak{p}}(t')), \int_{0 \leq s \leq t'} \mathcal{W}(t'-s)(0, \mathbf{1}_I(s) P_{\geq N} u^{\mathfrak{p}}(s)) ds \right\rangle_{\mathcal{E}} dt' \right| \\ &\leq \|P_{\geq N} u^{\mathfrak{p}}\|_{L^{a'}([t, t+N^{\eta_1}], L^{b'})} ||\mathbf{1}_I(s) u^{\mathfrak{p}}(s)||_{L^{a'}(\mathbb{R}, L^{b'})}, \end{aligned}$$

where (a, b) is as in (2.5.8) equations (i)-(iii). Hence, using (2.5.9) in Step 2.5.1, and (2.5.12) in Step 2.5.2, we get that

$$\left| \int_t^{t+N^{\eta_1}} \int_{t-N^{\eta_1} \leq s \leq t} V_N(s,t') ds dt' \right| \leq C|I|^{\frac{2}{a'}} N^{-\eta} \leq CN^{\frac{2}{a'}\eta_1} N^{-\eta}. \tag{2.5.35}$$

Now, since $\frac{2}{a'}\eta_1 - \eta = -\frac{1}{2}\eta$, we deduce from (2.5.29) and (2.5.32)–(2.5.35) that

$$E_0(P_{\geq N}\overline{w}) \leq C\epsilon + C\epsilon + CN^{-\delta\eta_1} + CN^{-\frac{\eta}{2}} \tag{2.5.36}$$

for t sufficiently large. The last inequality in (2.5.2) follows from (2.5.36) if we take $\eta_0 < \min(\frac{1}{2}\eta, \delta\eta_1)$. Together with (2.5.15) this ends the proof of Lemma 2.5.1. □

2.6 Almost finite speed propagation

We prove what we referred to as almost finite speed propagation in the introduction. Equation (2.6.1) in Lemma 2.6.1 basically states that solutions almost live in cones like $|x| \leq R(2 + Kt)$ for R sufficiently large. Lemma 2.6.1 states as follows.

Lemma 2.6.1. *Let $E > 0$ and α be as in (2.5.7). We consider (2.0.1) with $\lambda < 0$ and p as in (2.5.1). There exists $\epsilon' > 0$ and $M > 1$ such that for any $N \geq 1$, $t_0 \geq 0$, and $\epsilon \leq \epsilon'$, if $u \in \mathbb{E}_{\mathbb{R}_+}$ is a forward global solution of (2.0.1) of energy less than or equal to E satisfying (2.5.3) as in Corollary 2.5.1, then*

$$\int_{|x| \geq R(2+Kt)} |u(t,x)|^{p\alpha} dx \leq (4M\epsilon)^{p\alpha} \tag{2.6.1}$$

for all $t \geq t_0$, where $R, K \geq 0$ do not depend on t.

A useful corollary to Lemma 2.6.1 is as follows.

Corollary 2.6.1. *Let $n \geq 5$, and let $u \in \mathbb{E}_{\mathbb{R}_+}$ be a forward global solution of (2.0.1) with $\lambda < 0$ and p such that (2.5.1) holds true. Given ϵ, there exist $T > 0$ and $R_1 > 0$ such that*

$$\int_{|x| \geq R_1(1+t)} |u(t,x)|^{p+1} dx \leq \epsilon. \tag{2.6.2}$$

for all $t \geq T$.

Proof of Corollary 2.6.1. Let $E = E(u, u_t)$. For ϵ' as in Lemma 2.6.1 we let also $\epsilon_0 \leq \epsilon'$ to be chosen later on. By Corollary 2.5.1 there exist $N > 0$ and $T > 0$ such that for $t \geq T$, $E_0(P_{\geq N}(u(t), u_t(t))) \leq \epsilon_0$. We may then apply lemma 2.6.1, and we see that there exist $R, K \geq 0$ such that for $t \geq T$, (2.6.1) holds true with ϵ_0 in place of ϵ and T in place of t_0. Independently, by conservation of the energy as in (2.3.13), and the Sobolev embedding theorem, we know that

$$\int_{\mathbb{R}^n} |u(t,x)|^{2^\sharp} dx \leq C\sqrt{E} \tag{2.6.3}$$

for all t. Then, by Hölder's inequality, choosing ϵ_0 to be sufficiently small, depending only on E and ϵ, we get from (2.6.3) that

$$\begin{aligned} \int_{|x| \geq R(2+Kt)} |u(t)|^{p+1} &\leq \left(\int_{|x| \geq R(2+Kt)} |u(t)|^{p\alpha} \right)^{\frac{2^\sharp - (p+1)}{2^\sharp - p\alpha}} \left(\int_{\mathbb{R}^n} |u(t)|^{2^\sharp} \right)^{\frac{p+1-p\alpha}{2^\sharp - p\alpha}} \\ &\leq (4M\epsilon_0)^{p\alpha \frac{2^\sharp - p+1}{2^\sharp - p\alpha}} \left(C\sqrt{E} \right)^{\frac{p+1-p\alpha}{2^\sharp - p\alpha}} \\ &\leq \epsilon. \end{aligned}$$

This proves (2.6.2), and thus the corollary, with $R_1 = (2 + K)R$. □

Now we prove Lemma 2.6.1 by splitting u into several parts as in (2.6.25) and (2.6.39). In view of time translation invariance, we can suppose $t_0 = 0$. Without loss of generality, we may also assume that $m = 1$ and $\lambda = -1$. We proceed in several steps. We let α be as in (2.5.7) and let $M > 0$ be the sharp constant for the embedding of H^2 into $L^{p\alpha}$. Then

$$\|v\|_{L^{p\alpha}} \leq M\|v\|_{H^2} \tag{2.6.4}$$

for all $v \in H^2$. Let u solve (2.0.1) and p be as in (2.5.1). We set $u_0 = u(0)$, $u_1 = u_t(0)$, and define ω by

$$(\omega(t), \omega_t(t)) = \mathcal{W}(t)(u_0, u_1), \tag{2.6.5}$$

where $\mathcal{W}(t)$ is the isometry semigroup in Section 2.2. We let φ be given by

$$\varphi(t, \xi, x) = t\sqrt{1 + |\xi|^4} - \langle x, \xi\rangle \tag{2.6.6}$$

for all $t \in \mathbb{R}$, and all $\xi, x \in \mathbb{R}^n$. We also define

$$K = \sup_{\xi \in B_0(N)} \frac{2|\xi|^3}{\sqrt{1 + |\xi|^4}}. \tag{2.6.7}$$

Given $e \in \mathbb{R}^n$, the notation $\partial_e\varphi$ refers to $\langle \nabla_\xi \varphi, e\rangle$. As a remark, for any $i \geq 2$, there exists $M_i > 0$ such that $\left|\partial^i\varphi\right| \leq M_i t$, where ∂^i stands for iterations of length i of the derivatives $\partial_e\varphi$ for e in the canonical basis of $\mathbb{R}^n$.

Step 2.6.1. *Let $\epsilon > 0$ and $N \geq 1$. There exists $R_0 > 0$ depending on ϵ, N, n, p, u_0, and u_1, such that for any $R \geq R_0$ and any $t \geq 0$,*

$$\|r_2(t)\|_{L^{p\alpha}} \leq M\epsilon, \tag{2.6.8}$$

where $r_2(t) = \mathbb{1}_{S_t} P_{<N}\omega(t)$, $S_t = \{|x| \geq R(2 + Kt)\}$, K is as in (2.6.7), $\mathbb{1}_{S_t}$ is the characteristic function of S_t, ω is as in (2.6.5), and M is an in (2.6.4).

Proof of Step 2.6.1. In order to prove this step, we cut off the initial data at infinity and use a high-frequency cut-off to estimate the solution in the exterior of a cone. First, by density, we may find $\overline{\phi} = (\phi_0, \phi_1)$ for $\phi_0, \phi_1 \in C_c^\infty(\mathbb{R}^n)$, such that

$$E_0(u_0 - \phi_0, u_1 - \phi_1) \leq \epsilon^2/16. \tag{2.6.9}$$

We let

$$(w_c(t), \partial_t w_c(t)) = \mathcal{W}(t)\overline{\phi}. \tag{2.6.10}$$

We let $R_0 \geq 2$ be such that $\mathrm{supp}\phi_i \subset B_0(R_0)$ for $i = 1, 2$. From now on we assume that $R \geq R_0$. Then, by (2.6.4), the boundedness of $P_{<N}$ on $\mathcal{E}$, unitarity of $\mathcal{W}$, and (2.6.9), we get that for any $t \geq 0$,

$$\|P_{<N}(\omega(t) - w_c(t))\|_{L^{p\alpha}} \leq \frac{M}{2}\epsilon. \tag{2.6.11}$$

Now we estimate the norm of $\mathbb{1}_{S_t} P_{<N} w_c(t)$. We do it through nonstationary phase estimates. We know from the explicit formula (2.3.38) that w_c will be a linear combination of terms like

$$\Phi(x) = \mathbb{1}_{S_t}(x) \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\varphi(t,\xi,x-y)}\tilde{s}(\xi)\tilde{\phi}(y)dyd\xi, \tag{2.6.12}$$

where S_t is as in (2.6.8),

$$\tilde{s}(\xi) = \psi(\frac{2\xi}{N}) \text{ or } \tilde{s}(\xi) = \frac{\psi(\frac{2\xi}{N})}{\sqrt{1+|\xi|^4}},$$

where ψ is as in (2.2.2), and $\tilde{\phi}(y) = \phi_j(y)$, $j = 0, 1$. Now we remark that, since $R \geq R_0$, given $x \in S_t$, the expression in the integrand in (2.6.12) vanishes when $|x-y| \leq \frac{R}{2} + KRt$, and when this is not the case, letting $e = \frac{x-y}{\|x-y\|}$, we get by (2.6.7) that for any $t \geq 0$ and any $\xi \in \mathbb{R}^n$,

$$\begin{aligned} |\partial_e \varphi(t,\xi,x-y)| &= \left| \frac{2t|\xi|^2 \langle \xi, e \rangle}{\sqrt{1+|\xi|^4}} - \|x-y\| \right| \\ &\geq \frac{R}{2} + Kt(R-1) \geq (R-1)\left(\frac{1}{2} + Kt\right). \end{aligned} \tag{2.6.13}$$

For x, y such that $|x-y| > \frac{R}{2} + KRt$, we consider the operator $L_{x,y}$ given by

$$L_{x,y}(h) = \frac{1}{\partial_e \varphi(t,\cdot,x-y)} \partial_e h \tag{2.6.14}$$

for $h \in C^\infty(\mathbb{R}^n)$. Integrating by parts n times we get that

$$\left| \int_{\mathbb{R}^n} e^{i\varphi(t,\xi,x-y)} \tilde{s}(\xi) d\xi \right| = \left| \int_{\mathbb{R}^n} e^{i\varphi(t,\xi,x-y)} (L^\star_{x,y})^n \tilde{s} d\xi \right|, \tag{2.6.15}$$

where $L^\star_{x,y}$, the adjoint operator of $L_{x,y}$ in (2.6.14), is defined for all $h \in C^\infty(\mathbb{R}^n)$ by the formula $L^\star_{x,y} h = -\partial_e \frac{h}{\partial_e \varphi}$. Now, (2.6.13) gives that for any $\xi \in \mathbb{R}^n$,

$$\left| (L^\star_{x,y})^n \tilde{s}(\xi) \right| \leq C \frac{\|\tilde{s}\|_{C^n}}{R^n} \leq C R^{-n}, \tag{2.6.16}$$

where C does not depend on $R \geq R_0$, N, t, x, and y such that $|x-y| > \frac{R}{2} + KRt$. Hence, by (2.6.12), (2.6.15), and (2.6.16), we get that

$$\|\Phi\|_{L^\infty} \leq C \frac{|B_0(N)|}{R^n} \|\tilde{\phi}\|_{L^1} \leq C \frac{N^n}{R^{\frac{n}{2}}} \|\tilde{\phi}\|_{L^2}, \tag{2.6.17}$$

where Φ is as in (2.6.12). On the other hand, it is clear from (2.6.12) and Parseval's theorem that

$$\|\Phi\|_{L^2} \leq C \|\psi\|_{L^\infty} \|\tilde{\phi}\|_{L^2}, \tag{2.6.18}$$

where C depends only on n. Combining (2.6.17) and (2.6.18), we deduce by Hölder's inequality that

$$\|\Phi\|_{L^{p\alpha}} \leq \|\Phi\|_{L^\infty}^{1-\frac{2}{p\alpha}} \|\Phi\|_{L^2}^{\frac{2}{p\alpha}} \leq C (N^n R^{-\frac{n}{2}})^{1-\frac{2}{p\alpha}}, \tag{2.6.19}$$

where C is independent of R and t. In particular, we see with (2.6.19) that for $R \geq R_0$ sufficiently large, depending only on u_0, u_1, N, and ϵ, for any $t \geq 0$,

$$\|\mathbb{1}_{S_t} P_{<N} w_c(t)\|_{L^{p\alpha}} \leq \frac{M}{2} \epsilon \tag{2.6.20}$$

and consequently, by combining (2.6.11) and (2.6.20), we see that (2.6.8) holds true. This ends the proof of Step 2.6.1 □

Now, we want to estimate the contribution of the forcing term. For any $0 \le t_1 \le t$ we let

$$\begin{aligned} r_3(t,t_1) &= -\mathbb{1}_{S_t}\int_{t-t_1}^{t} \pi_1\mathcal{W}(t-s)(0,P_{<N}u^{\mathfrak{p}}(s))ds \ , \text{ and} \\ r_3'(t,t_1) &= -\int_{t-t_1}^{t} \pi_1\mathcal{W}(t-s)(0,u^{\mathfrak{p}}(s))ds, \end{aligned} \tag{2.6.21}$$

where S_t is as in (2.6.8). The next step in the proof of Lemma 2.6.1 states as follows.

Step 2.6.2. *There exists $t_2 > 0$, depending only on E and ϵ, such that*

$$\|r_3(t,t_1)\|_{L^{p\alpha}} \le M\epsilon \tag{2.6.22}$$

for all $t \ge 0$, where $t_1 = \min(t_2,t)$.

Proof of Step 2.6.2. Since $p < 2^\sharp - 1$, we have that $\frac{4(p+1)-n(p-1)}{2(p+1)} > 0$. Then by (2.3.46), and the Sobolev embedding theorem,

$$\begin{aligned} \|r_3'(t,t_1)\|_{L^{p+1}} &\le \int_{t-t_1}^{t} \|\pi_1\mathcal{W}(t-t')(0,u^{\mathfrak{p}}(t'))\|_{L^{p+1}}\,dt' \\ &\le C\int_{t-t_1}^{t} (t-t')^{1-\frac{n}{2}\frac{p-1}{p+1}}\|u^{\mathfrak{p}}\|_{L^\infty([t-t_1,t],L^{\frac{p+1}{p}})}dt' \\ &\le C|t_1|^{\frac{4(p+1)-n(p-1)}{2(p+1)}} \\ &\le \epsilon_0 \end{aligned} \tag{2.6.23}$$

for all $t \ge 0$ and all $t_2 \le 1$ sufficiently small, depending only on n, p, E, ϵ_0, where ϵ_0 is some small parameter to be chosen later on. Besides, for any $t \ge 0$ and any $t_1 \in [0,t]$,

$$r_3'(t,t_1) = u(t) - u(t-t_1) - \omega(t) + \omega(t-t_1).$$

Hence, by conservation of the energy as in (2.3.13), r_3' is bounded in L^2 uniformly in t, t_1, so that for any $t \ge 0$, and any $t_1 \in (0,t_2)$, since $P_{<N}$ is bounded on L^{p+1},

$$\begin{aligned} &\|r_3(t,t_1)\|_{L^{p+1}} \le \|P_{<N}r_3'(t,t_1)\|_{L^{p+1}} \le C\|r_3'\|_{L^{p+1}} \le C\epsilon_0 \ , \text{ and} \\ &\|r_3(t,t_1)\|_{L^2} \le \|r_3'(t,t_1)\|_{L^2} \le 4\sqrt{E}. \end{aligned} \tag{2.6.24}$$

By Hölder's inequality and (2.6.24), for ϵ_0 correctly chosen depending only on ϵ, E, and n, we get that (2.6.22) holds true. This ends the proof of Step 2.6.2. □

Now, for any $t' \ge 0$, we split $u(t')$ into

$$u(t') = \mathbb{1}_{S_{t'}^c}u(t') + \mathbb{1}_{S_{t'}}u(t') = u_c(t') + u_f(t'), \tag{2.6.25}$$

where $S_{t'}^c$ stands for the complement of $S_{t'}$. The forcing term also splits as $u^{\mathfrak{p}} = u_c^{\mathfrak{p}} + u_f^{\mathfrak{p}}$. In what follows we estimate the contribution from u_c. For any time $t \ge 0$, and any interval I, we let

$$r_4(t,I) = -\mathbb{1}_{S_t}\int_I \pi_1\mathcal{W}(t-t')(0,P_{<N}u_c^{\mathfrak{p}}(t'))dt', \tag{2.6.26}$$

where $S_t = \{|x| \ge R(2+Kt)\}$ is as in (2.6.8). A third step in the proof of Lemma 2.6.1 is as follows.

Step 2.6.3. *Let $\epsilon > 0$. There exists $R > 0$ such that*

$$\|r_4(t,I)\|_{L^{\alpha'}} \leq \epsilon^p \tag{2.6.27}$$

for all $t \geq 0$ and all $I \subset [0, t-t_1]$, where t_1 is as in Step 2.6.2, and r_4 is as in (2.6.26).

Proof of Step 2.6.3. For any t and t', we define the operator $\mathcal{V}_{t,t'}$ on L^1 by

$$\mathcal{V}_{t,t'}h(x) = \mathbb{1}_{S_t}(x)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\varphi(t-t',\xi,x-y)}\tilde{s}(\xi)\mathbb{1}_{S_{t'}^c}(y)h(y)dyd\xi, \tag{2.6.28}$$

where $h \in L^1$ and φ is as in (2.6.6). Also, for any $\xi \in \mathbb{R}^n$, we let

$$\tilde{s}(\xi) = \psi(\frac{2\xi}{N})/\sqrt{1+|\xi|^4},$$

where ψ is as in (2.2.2). We claim that this operator satisfies that for any $q \geq 2$, there exists C independent of K, N, ϵ, and $R \geq 2$ such that for any for any $t \geq t' \geq 0$, and any function $h \in L^1 \cap L^{q'}$,

$$\|\mathcal{V}_{t,t'}h\|_{L^q} \leq C\left((t-t')(R-1)\right)^{-n(1-\frac{2}{q})}\|h\|_{L^{q'}}. \tag{2.6.29}$$

We prove (2.6.29) in what follows.

First, we note that when $t \geq t'$, $x \in S_t$, and $y \in S_{t'}^c$, then $|x-y| \geq KR(t-t')$ and under these conditions, if $e = (x-y)/\|x-y\|$, we get that

$$|\partial_e\varphi(t-t',\xi,x-y)| \geq K(t-t')(R-1). \tag{2.6.30}$$

Then, again, let $L_{x,y}$ be as in (2.6.14). When $x \in S_t$, after n integrations by parts in the ξ variable, we get that

$$\begin{aligned}|\mathcal{V}_{t,t'}h(x)| &\leq \int_{\mathbb{R}^n}\mathbb{1}_{S_{t'}^c}(y)\left|h(y)\int_{\mathbb{R}^n}e^{i\varphi(t-t',\xi,x-y)}(L_{x,y}^\star)^n\tilde{s}d\xi\right|dy \\ &\leq C|\text{supp}\tilde{s}|\left(K(R-1)(t-t')\right)^{-n}\|h\|_{L^1},\end{aligned} \tag{2.6.31}$$

where C does not depend on h, N, K, R, t', t. Furthermore, by Parseval's theorem,

$$\|\mathcal{V}_{t,t'}h\|_{L^2} \leq C\|h\|_{L^2}, \tag{2.6.32}$$

where C is independent of h, N, t, t', R, and K. By the Riesz-Thorin theorem, we deduce from (2.6.31) and (2.6.32) that for any $q \geq 2$,

$$\|\mathcal{V}_{t,t'}h\|_{L^q} \leq C\left(K(R-1)(t-t')\right)^{-n(1-\frac{2}{q})}N^{n(1-\frac{2}{q})}\|h\|_{L^{q'}}. \tag{2.6.33}$$

As is easily checked, (2.6.33) implies (2.6.29) since by (2.6.7), $N/K \leq 1$.

Now we prove (2.6.27). We let $t \geq 0$ and t_1 be as in Step 2.6.2. Using (2.6.4), (2.6.26), (2.6.29), and since $n(1-2/\alpha') > 1$, we get that, for R sufficiently large, depending only on p, E, ϵ, and t_2, for any $t \geq 0$, and any $I \subset [0, t-t_1]$,

$$\begin{aligned}
\|r_4(t,I)\|_{L^{\alpha'}} &= \left\| \mathbb{1}_{S_t} \int_I \pi_1 \mathcal{W}(t-t')(0, P_{<N} u_c^{\mathfrak{p}}(t'))dt' \right\|_{L^{\alpha'}} \\
&\leq \frac{1}{(2\pi)^n} \int_0^{t-t_1} \|\mathcal{V}_{t,t'} u_c^{\mathfrak{p}}(t')\|_{L^{\alpha'}} dt' \\
&\leq C \int_0^{t-t_1} (R(t-t'))^{-n(1-\frac{2}{\alpha'})} \|u_c^{\mathfrak{p}}\|_{L^\infty(\mathbb{R}_+, L^\alpha)} dt' \\
&\leq C R^{-n(1-\frac{2}{\alpha'})} \|u\|^p_{L^\infty(\mathbb{R}_+, H^2)} \\
&\leq \epsilon^p.
\end{aligned}$$

This proves (2.6.27) and thus Step 2.6.3. □

Using conservation of energy, we also note that for any $t \geq 0$ and any $I \subset [0, t-t_1]$ such that $|I| \leq 1$, we have that

$$\begin{aligned}
\|r_4(t,I)\|_{L^2} &\leq \int_I \|\pi_1 \mathcal{W}(t-t')(0, P_{<N} u_c^{\mathfrak{p}}(t'))\|_{L^2} dt' \\
&\leq \sup_I \|(1+\Delta^2)^{-\frac{1}{2}} u_c^{\mathfrak{p}}\|_{L^2} \leq C\sqrt{E},
\end{aligned} \tag{2.6.34}$$

where $C > 0$ depends only on n. Now, for $t_3 = \min(t-1, 0)$, we define

$$r_5'(t) = -\mathbb{1}_{S_t} \int_{t_3}^{t-t_2} \pi_1 \mathcal{W}(t-t') \left(0, P_{<N} u_f^{\mathfrak{p}}(t')\right) dt', \tag{2.6.35}$$

where t_2 is as in Step 2.6.2 and S_t is as in (2.6.8). The next step in the proof of Lemma 2.6.1 states as follows.

Step 2.6.4. *Suppose we have that*

$$\|u_f\|_{L^\infty([0,t], L^{p\alpha})} \leq 5M\epsilon,$$

then there holds that

$$\|r_5'(t)\|_{L^{p\alpha}} \leq M\epsilon, \tag{2.6.36}$$

where α is as in (2.5.8), *provided that ϵ is sufficiently small.*

Proof of Step 2.6.4. First, in case $p\frac{2n}{n+4} \leq 2^*$, let α be so close to $2n/(n+4)$ (depending on p) that $H^{2,\alpha p} \hookrightarrow L^{\alpha'}$, then we use the decay estimates for the linear propagator (2.3.18) to get

$$\begin{aligned}
\|r_5'(t)\|_{L^{p\alpha}} &\leq \| \int_{t_3}^{t-t_2} \pi_1 \mathcal{W}((t-t')(0, u_f^{\mathfrak{p}}(t'))dt' \|_{L^{p\alpha}} \\
&\leq C \int_{t_3}^{t-t_2} (t-t')^{-\frac{n}{2}\left(1-\frac{2}{p\alpha}\right)} \|(1+\Delta^2)^{-1/2} u_f^{\mathfrak{p}}\|_{L^{(p\alpha)'}} dt' \\
&\leq C \sup_{[t_3,t]} \||u_f|^p\|_{H^{-2,(p\alpha)'}} \\
&\leq C \sup_{[t_3,t]} \||u_f|^p\|_{L^\alpha} \\
&\leq C\epsilon^p \\
&\leq M\epsilon
\end{aligned}$$

provided ϵ is sufficiently small. In case $2^* < 2np/(n+4) < 2^\sharp$, we choose $\beta < 2^*$ close to 2^*, and let $\alpha < 2n/(n+4)$ sufficiently close to $2n/(n+4)$ such that $\beta < p\alpha < \beta^*$, and such that

$$\kappa = \frac{\frac{1}{2} - \frac{1}{(\beta^*)' p}}{\frac{1}{2} - \frac{1}{p\alpha}}$$

satisfies $\kappa p > 1$. Then we proceed as follow, using the conservation of energy and the decay estimate for the linear propagator (2.3.18):

$$\begin{aligned}
\|r_5'(t)\|_{L^{p\alpha}} &\le C\| \int_{t_3}^{t-t_2} \pi_1 \mathcal{W}((t-t')(0, u_f^{\mathfrak{p}}(t'))dt' \|_{H^{1,\beta}} \\
&\le C\|(1+\Delta^2)^{-\frac{1}{4}} \int_{t_3}^{t-t_2} \sin\left((t-t')\sqrt{1+\Delta^2}\right) u_f^{\mathfrak{p}}(t')dt' \|_{L^\beta} \\
&\le C \int_{t_3}^{t-t_2} (t-t')^{-\frac{n}{2}(1-\frac{2}{\beta})} \|(1+\Delta^2)^{-\frac{1}{4}} u_f^{\mathfrak{p}}(t')\|_{L^\infty([t_3,t],L^{\beta'})} dt' \\
&\le C\| |u_f|^p \|_{L^\infty([t_3,t],L^{(\beta^*)'})} \\
&\le C \left(\|u_f\|^\kappa_{L^\infty([t_3,t],L^{p\alpha})} \|u_f\|^{1-\kappa}_{L^\infty([t_3,t],L^2)} \right)^p \\
&\le C\epsilon^{\kappa p} \\
&\le M\epsilon
\end{aligned}$$

provided again that ϵ is sufficiently small. □

With Steps 2.6.1–2.6.4 we are now in position to prove Lemma 2.6.1.

Proof of lemma 2.6.1. Let $t \ge 0$, and t_3 be as in Step 2.6.4. Let also r_1 and r_5 be given by

$$\begin{aligned}
r_1(t) &= \mathbb{1}_{S_t} P_{\ge N} u(t) \text{ , and} \\
r_5(t) &= -\mathbb{1}_{S_t} \int_0^{t_3} \pi_1 \mathcal{W}(t-s)(0, P_{<N} u_f^{\mathfrak{p}}(s))ds.
\end{aligned} \tag{2.6.37}$$

By (2.5.3) and (2.6.4), we have that

$$\|r_1\|_{L^\infty(\mathbb{R}_+, L^{p\alpha})} \le M\epsilon, \tag{2.6.38}$$

where α is as in (2.5.7). Independently, by Duhamel's formula (2.3.47), we can write that

$$u_f(t) = r_1(t) + r_2(t) + r_3(t,t_1) + r_4(t,[0,t_3]) + r_4(t,[t_3, t-t_1]) + r_5(t) + r_5'(t), \tag{2.6.39}$$

where u_f is as in (2.6.25), and the r_i's are as in (2.6.8), (2.6.21), (2.6.26), (2.6.35) ,and (2.6.37). Besides, for any $t \ge 0$,

$$r_4(t,[0,t_3]) + r_5(t) = \mathbb{1}_{S_t} P_{<N} \pi_1 \left(\mathcal{W}(t-t_3)(u(t_3), u_t(t_3)) - \mathcal{W}(t)(u_0,u_1) \right), \tag{2.6.40}$$

and, by boundedness of $P_{<N}$ on L^2, conservation of the energy as in (2.3.13), and since $\mathcal{W}$ is a unitary operator, we thus get from (2.6.40) that

$$\|r_4(\cdot,[0,t_3]) + r_5\|_{L^2} \le 2\sqrt{E}. \tag{2.6.41}$$

Now, we remark that for $t \leq t_2$, we get that $t_3 = t - t_1 = 0$. As a consequence, we have that

$$\|u_f(t)\|_{L^{p\alpha}} \leq r_1(t) + r_2(t) + r_3(t, t_1) \leq 3M\epsilon \tag{2.6.42}$$

for all $t \leq t_2$. We let

$$t_0 = \sup\{t \geq 0 : \forall s \in [0, t], \|u_f(s)\|_{L^{p\alpha}} \leq 5M\epsilon\} \tag{2.6.43}$$

and we assume that $t_0 < +\infty$. We know from (2.6.42) that $t_0 > t_2$. By continuity we then get that

$$\|u_f(t_0)\|_{L^{p\alpha}} = 5M\epsilon. \tag{2.6.44}$$

However, by the decay estimates (2.3.18), by (2.5.8) equation (*iv*), (2.6.27), (2.6.41), and since $P_{<N}$ is bounded on L^α, we can write that

$$\begin{aligned}
&\|u_f(t_0)\|_{L^{p\alpha}} \\
&\leq \|r_1(t_0)\|_{L^{p\alpha}} + \|r_2(t_0)\|_{L^{p\alpha}} + \|r_3(t_0, t_1)\|_{L^{p\alpha}} + \|r_5'(t_0)\|_{L^{p\alpha}} \\
&+ \|r_4(t_0, [t_3, t_0 - t_1])\|_{L^{p\alpha}} + \|r_4(t_0, [0, t_3]) + r_5(t_0)\|_{L^{p\alpha}} \\
&\leq 4M\epsilon + \|r_4(t_0, [t_3, t_0 - t_1])\|_{L^2}^{1-\theta} \|r_4(t_0, [t_3, t_0 - t_1])\|_{L^{\alpha'}}^{\theta} \\
&+ (2\sqrt{E})^{1-\theta} \left(\epsilon^p + \int_0^{t_3} \|\pi_1 \mathcal{W}(t_0 - t')(0, P_{<N} u_f^{\mathfrak{p}}(t'))\|_{L^{\alpha'}} dt' \right)^{\theta} \\
&\leq 4M\epsilon + \left(C\sqrt{E}\right)^{1-\theta} \epsilon^{p\theta} \\
&+ C \left(\epsilon^p + \int_0^{t_3} (t_0 - t')^{-\frac{n}{4}(1 - \frac{2}{\alpha'})} \|P_{<N} u_f^{\mathfrak{p}}\|_{L^\infty([0,t_0], L^\alpha)} dt' \right)^{\theta} \\
&\leq 4M\epsilon + \tilde{C} \left(2\epsilon^p + \|u_f\|_{L^\infty([0,t_0], L^{p\alpha})}^p \right)^{\theta},
\end{aligned} \tag{2.6.45}$$

where θ is such that $0 < \theta < 1 < p\theta$ as in (2.5.8) equation (*iv*). The first inequality in (2.6.45) is by (2.6.39), the second inequality is by (2.6.8), (2.6.22), (2.6.36), (2.6.38), (2.5.8) equation (*iv*), (2.6.27), the second equation in (2.6.37) and (2.6.41), the third inequality is by (2.6.27), (2.6.34) and (2.3.18), and the fourth inequality is by boundedness of $P_{<N}$ on L^α and the fact that $\frac{n}{4}(1 - \frac{2}{\alpha'}) > 1$ since $\alpha < 2n/(n+4)$. Then, by (2.6.43), we get that

$$\|u_f(t_0)\|_{L^{p\alpha}} < 5M\epsilon \tag{2.6.46}$$

for $\epsilon \leq \epsilon'$, where ϵ' is chosen sufficiently small such that

$$\epsilon'^{p\theta - 1} \tilde{C}(2 + (5M)^p)^\theta < M,$$

and $\tilde{C}$ depends only on E, n, and p. Clearly, (2.6.46) is in contradiction with (2.6.44). Hence $t_0 = +\infty$, where t_0 is given by (2.6.43). This proves (2.6.1) and Lemma 2.6.1. □

2.7 Proof of the Theorem

We prove our theorem in this section. In addition to the material developed in the preceding sections, a key ingredient we need in the proof is a Morawetz estimate [26] obtained by Levandosky and Strauss [19]. We refer also to Lin [21]. Let $n \geq 5$ and $u \in \mathbb{E}_{\mathbb{R}_+}$ be a forward global solution of the nonlinear equation (2.0.1) with $1 + \frac{8}{n} \leq p \leq 2^\sharp - 1$ and $\lambda < 0$. Then, as proved in Levandosky and Strauss [19], it holds that

$$\int_0^\infty \int_{\mathbb{R}^n} \frac{|u(t,x)|^{p+1}}{|x|} dtdx \leq C, \tag{2.7.1}$$

where $C > 0$ depends on n and u only through the energy. We prove our theorem in what follows, using the method developed by Lin and Strauss [22] and Morawetz and Strauss [27] for the Schrödinger and Klein-Gordon equations.

Proof of the theorem. As above, we may assume that $m = 1$ and $\lambda = -1$. Let $n \geq 5$ and u be a solution of (2.0.1) with p such that (2.5.1) holds true. By Corollary 2.4.1 in Section 2.4 it suffices to prove that

$$\|u(t)\|_{L^{p+1}} \to 0 \tag{2.7.2}$$

as $t \to +\infty$. The surjectivity of W_+ and the continuity of its inverse, come from the remark after Corollary 2.4.1. In order to prove (2.7.2) we claim that for any $\epsilon > 0$, $t_0 \geq 0$, and $t_1 > 0$, there exists $t_2 > t_0$ such that

$$\sup_{t' \in [t_2 - t_1, t_2]} \|u(t')\|_{L^{p+1}} \leq \epsilon. \tag{2.7.3}$$

We prove (2.7.3) in what follows. Applying Corollary 2.6.1, we get that there exist $T, R > 0$ such that

$$\int_{|x| \geq R(1+t)} |u(t,x)|^{p+1} dx \leq \epsilon_1 \tag{2.7.4}$$

for all $t \geq T$, where ϵ_1, depending only on n, E, and p, is to be chosen later on. We let $t_0' = \max(T, t_0)$. Given $\epsilon_0 > 0$ and $\tau > 0$, there exists $\tilde{t} > t_0' + 2\tau$ such that

$$\int_{\tilde{t}-2\tau}^{\tilde{t}} \int_{|x| \leq R(1+t')} |u(t',x)|^{p+1} dxdt' \leq \epsilon_0. \tag{2.7.5}$$

Indeed, by the Morawetz estimate (2.7.1), we can write that

$$\begin{aligned} \infty &> \int_{t_0'}^\infty \frac{1}{R(1+t')} \int_{|x| \leq R(1+t')} |u(t',x)|^{p+1} dxdt' \\ &\geq \sum_{k=0}^\infty \frac{1}{R(1 + (t_0' + 2(k+1)\tau))} \int_{t_0'+2k\tau}^{t_0'+2(k+1)\tau} \int_{|x| \leq R(1+t')} |u(t',x)|^{p+1} dxdt'. \end{aligned}$$

Since

$$\sum_k \frac{1}{R(1 + (t_0' + 2(k+1)\tau))} = \infty,$$

there exists $k_0 > 0$ such that

$$\int_{t_0'+2k_0\tau}^{t_0'+2(k_0+1)\tau} \int_{|x| \leq R(1+t')} |u(t',x)|^{p+1} dxdt' \leq \epsilon_0. \tag{2.7.6}$$

Letting $\tilde{t} = t'_0 + 2(k_0+1)\tau$, (2.7.6) gives that (2.7.5) holds true. Now that we have (2.7.5), we write with Duhamel's formula (2.3.47) that for any $t \geq \sigma$,

$$\begin{aligned}(u(t), u_t(t)) &= \mathcal{W}(t)(u_0, u_1) - \int_0^{t-\sigma} \mathcal{W}(t-t')(0, u^{\mathrm{p}}(t'))dt' \\ &\quad - \int_{t-\sigma}^{t} \mathcal{W}(t-t')(0, u^{\mathrm{p}}(t'))dt' \\ &= (v(t), v_t(t)) + (w(t,\sigma), w_t(t,\sigma)) + (z(t,\sigma), z_t(t,\sigma)),\end{aligned} \tag{2.7.7}$$

where $\sigma \geq 0$ is to be chosen later on. We observe that

$$\|v(t)\|_{L^{p+1}} \to 0 \tag{2.7.8}$$

as $t \to +\infty$. Indeed, let $\delta > 0$ arbitrary, and let $\phi_0, \phi_1 \in C_c^\infty(\mathbb{R}^n)$ be such that $E_0(u_0 - \phi_0, u_1 - \phi_1) \leq \delta$. Then we define $(\omega(t), \omega_t(t)) = \mathcal{W}(t)(\phi_0, \phi_1)$. By conservation of the energy, the Sobolev embedding theorem, and the decay estimate (2.3.18),

$$\begin{aligned}\|v(t)\|_{L^{p+1}} &\leq C\|\omega(t) - v(t)\|_{H^2} + \|\omega(t)\|_{L^{p+1}} \\ &\leq C\delta + C\left(t^{-\frac{n}{2}(1-\frac{2}{p+1})} + t^{-\frac{n}{4}(1-\frac{2}{p+1})}\right) \leq 2C\delta\end{aligned}$$

for $t \geq t_0$ sufficiently large depending only on n, p, ϕ_0, ϕ_1, and δ. This proves (2.7.8). As a consequence of (2.7.8), we get that there exists t''_0 such that for any time $t' \geq t''_0$,

$$\|v(t')\|_{L^{p+1}} \leq \frac{\epsilon}{4}. \tag{2.7.9}$$

Now, we let $\beta \geq 1$ be such that

$$\beta = \begin{cases} \frac{2}{p} & \text{if } p \leq 2, \\ 1 & \text{otherwise.} \end{cases} \tag{2.7.10}$$

Then, $1 - 2/\beta' = \min(1, p-1)$, and $\beta < 2n/(n+4)$. By the decay estimate (2.3.18) and the Sobolev embedding theorem, for β as in (2.7.10), we get that

$$\begin{aligned}\|w(t,\sigma)\|_{L^{\beta'}} &\leq C\int_0^{t-\sigma} (t-t')^{-\frac{n}{4}(1-2/\beta')}\|u(t')\|_{L^{p\beta}}^p dt' \\ &\leq C\sigma^{\frac{4-n\min(1,p-1)}{4}} \sup_{t'} \|u(t')\|_{H^2}^p,\end{aligned} \tag{2.7.11}$$

where $C > 0$ depends only on n. Independently, we see from (2.7.7) that

$$(w(t,\sigma), w_t(t,\sigma)) = \mathcal{W}(\sigma)(u(t-\sigma), u_t(t-\sigma)) - \mathcal{W}(t)(u_0, u_1). \tag{2.7.12}$$

Since $\mathcal{W}$ is unitary on $\mathcal{E}$, and $E_0(u, u_t)$ remains bounded, we see from (2.7.12) that $w(t,\sigma)$ remains bounded in L^2. Hence, since $2 < p+1 < \beta' \leq +\infty$, by Hölder's inequality, (2.7.11), and the boundedness of $w(t,\sigma)$ in L^2, we get that there exist positive constants C and K, depending only on n, p, and E, such that for any $\sigma > 0$, and any $t \geq \sigma$,

$$\|w(t,\sigma)\|_{L^{p+1}} \leq C\|w(t,\sigma)\|_{L^{\beta'}}^{\frac{1-\frac{2}{p+1}}{1-\frac{2}{\beta'}}} \leq K\sigma^{-\frac{n(p-1)-4\max(1,p-1)}{4(p+1)}}. \tag{2.7.13}$$

As a consequence, there exists σ_0 such that

$$\|w(t,\sigma)\|_{L^{p+1}} \leq \frac{\epsilon}{3} \tag{2.7.14}$$

for all $\sigma \geq \sigma_0$, and all $t \geq \sigma$.

Finally, we remark that since $p < 2^\sharp - 1$, we may find $q \in [1, 4(p+1)/n(p-1))$ such that $pq' \geq p+1$, and, with the decay estimate (2.3.46) and the fact that u remains bounded in L^{p+1}, we get, for $z(t,\sigma)$ as in (2.7.7), that

$$\begin{aligned}
\|z(t,\sigma)\|_{L^{p+1}} &\leq C \int_{t-\sigma}^{t} (t-t')^{-\frac{n(p-1)}{4(p+1)}} \|u(t')\|_{L^{p+1}}^{p} dt' \\
&\leq C \left(\int_{t-\sigma}^{t} (t-t')^{-\frac{n(p-1)q}{4(p+1)}} dt' \right)^{\frac{1}{q}} \left(\int_{t-\sigma}^{t} \|u(t')\|_{L^{p+1}}^{pq'} dt' \right)^{\frac{1}{q'}} \\
&\leq C\sigma^{\delta} \left(\int_{t-\sigma}^{t} \|u(t')\|_{L^{p+1}}^{p+1} dt' \right)^{\frac{1}{q'}}
\end{aligned} \tag{2.7.15}$$

for some $\delta > 0$, where C depends only on n, p, σ, and E but not on t. Note that $\frac{n(p-1)q}{4(p+1)} < 1$ thanks to our assumption on q. Now, for σ_0 as in (2.7.14) and t_0'' as in (2.7.9), we let

$$t_1 = \max(\sigma_0, t_0''), \tag{2.7.16}$$

and we choose $t_2 \geq t_0' + 2t_1$ such that (2.7.5) holds true for $\tau = t_1$, $\tilde{t} = t_2$ and ϵ_0 small in a sense to be made precise below. Since $[t - t_1, t] \subset [t_2 - 2t_1, t_2]$ when $t \in [t_2 - t_1, t_2]$, we get with (2.7.4), (2.7.5), and (2.7.15) that

$$\begin{aligned}
\|z(t,t_1)\|_{L^{p+1}} &\leq Ct_1^{\delta} \Bigg(\int_{t-t_1}^{t} \int_{|x| \leq R(1+t')} |u(t',x)|^{p+1} dx dt' \\
&\qquad + t_1 \sup_{t' \in [t, t-t_1]} \|\mathbb{1}_{S_{t'}} u(t')\|_{L^{p+1}}^{p+1} \Bigg)^{\frac{1}{q'}} \\
&\leq Ct_1^{\delta} \left(\epsilon_0 + t_1 \epsilon_1 \right)^{\frac{1}{q'}} \leq \frac{\epsilon}{3}
\end{aligned} \tag{2.7.17}$$

for ϵ_0 and ϵ_1 sufficiently small depending only on n, p, and t_1. Estimates (2.7.15)-(2.7.17) can be regarded as the key estimates in this section.

By combining (2.7.9), (2.7.14), (2.7.16), and (2.7.17) we get that (2.7.3) holds true. Now that we have (2.7.3), we prove that (2.7.2) also holds true. Given $\epsilon > 0$ sufficiently small, we let σ_ε large be such that

$$K\sigma_\epsilon^{-\frac{n(p-1)-4\max(1,p-1)}{4(p+1)}} = \frac{\epsilon}{4}, \tag{2.7.18}$$

where K is the constant (depending only on E, n, and p) appearing in (2.7.13). By (2.7.7), we can write that $u(t) = v(t) + w(t,\sigma) + z(t,\sigma)$ with $\sigma = \sigma_\varepsilon$. We let t_0'' be such that (2.7.9) holds true for $t' \geq t_0''$. For $t \geq \max(t_0'', \sigma_\epsilon)$,

$$\|u(t)\|_{L^{p+1}} \leq \frac{\epsilon}{2} + \|z(t,\sigma_\epsilon)\|_{L^{p+1}}, \tag{2.7.19}$$

and by the decay estimate (2.3.46) we get that there exist $C, C' > 0$, depending only on p and n, such that

$$\begin{aligned} \|z(t,\sigma_\epsilon)\|_{L^{p+1}} &\leq C\int_{t-\sigma_\epsilon}^{t} (t-t')^{-\frac{n(p-1)}{4(p+1)}} \|u(t')\|_{L^{p+1}}^p dt' \\ &\leq C'\sigma_\epsilon^{1-\frac{n(p-1)}{4(p+1)}} \sup_{[t-\sigma_\epsilon,t]} \|u\|_{L^{p+1}}^p. \end{aligned} \tag{2.7.20}$$

There exists $t_2 \geq \max(t_0'', \sigma_\epsilon)$ such that (2.7.3) holds true with $t_1 = \sigma_\epsilon$. We let

$$t_\epsilon = \sup\{t \geq t_2 : \forall s \in [t_2 - \sigma_\epsilon, t], \|u(s)\|_{L^{p+1}} \leq \epsilon\}. \tag{2.7.21}$$

Assuming that $t_\epsilon \neq \infty$, and since the map $t \mapsto u(t)$ is continuous on L^{p+1}, we get that $\|u(t_\epsilon)\|_{L^{p+1}} = \epsilon$. From this, (2.7.19), and (2.7.20), we see that

$$\epsilon \leq \frac{\epsilon}{2} + C'\sigma_\epsilon^{1-\frac{n(p-1)}{4(p+1)}}\epsilon^p.$$

Hence, $\sigma_\epsilon^{1-\frac{n(p-1)}{4(p+1)}}\epsilon^{p-1} \geq 1/2C'$, and, by (2.7.18), we get that

$$\sigma_\epsilon^\gamma \geq \frac{1}{2C'(4K)^{p-1}}, \tag{2.7.22}$$

where K and C' depend only on n, p, and E, and where

$$\gamma = -\frac{np(p-1) - 4(p+1+(p-1)\max(1,p-1))}{4(p+1)}. \tag{2.7.23}$$

When $p < 2$, we get with (2.7.23) that

$$\gamma = -\frac{np(p-1) - 8p}{4(p+1)} = \frac{2p}{p+1}\left(1 - n\frac{p-1}{8}\right) \tag{2.7.24}$$

and γ is negative when p satisfies (2.5.1), while if $p \geq 2$, we get from (2.7.23) that

$$\gamma = -\frac{n-4}{4(p+1)}\left(p^2 - p - \frac{8}{n-4}\right). \tag{2.7.25}$$

If we let $h(x) = x^2 - x - 8/(n-4)$, then h is increasing for $x \geq 1$ and, hence, $h(p) > h((n+8)/n) > 0$ when $n \geq 8$. Finally, in the two cases (2.7.24) and (2.7.25), we have that $\gamma < 0$. Then, (2.7.22) together with (2.7.18) imply that

$$\epsilon \geq 4K\left(2C'(4K)^{p-1}\right)^{\frac{n(p-1)-4\max(1,p-1)}{4(p+1)\gamma}}, \tag{2.7.26}$$

where the right-hand side depends only on E, p, and n. Letting $\epsilon_0 > 0$ be smaller than the right-hand side in (2.7.26), we get a contradiction for any $\epsilon \leq \epsilon_0$. This proves that for such ϵ's, $t_\epsilon = +\infty$. In particular, for any $\epsilon > 0$ sufficiently small, there exists $T > 0$ such that $\|u(t)\|_{L^{p+1}} \leq \epsilon$ for all $t \geq T$. Replacing the L^{p+1}-norm by a L^q-norm for $q < 2^\sharp - 1$ close to $2^\sharp - 1$, the above argument also gives the result when $5 \leq n \leq 7$. This proves (2.7.2). As already mentioned, this also proves our theorem. □

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Chapter 3

More about the scattering operator

Abstract

For a fourth-order nonlinear wave equation in $\mathbb{R}^n$, we prove that the scattering operator $\mathcal{S}$ is analytic on energy space. Furthermore, $\mathcal{S}$ determines the scatterer uniquely. For small powers, there is no scattering.

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3.1 Introduction

We continue our study of scattering theory for nonlinear wave equations of fourth order in $\mathbb{R}^n$. Scattering of fourth-order nonlinear waves is a subject that has been investigated only recently. The fourth-order nonlinear wave equation that we study in this paper, often referred to in the mathematics and physics literature as the nonlinear beam equation or the Bretherton equation, is

$$\frac{\partial^2 u}{\partial t^2} + \Delta^2 u + mu = \lambda |u|^{p-1} u, \tag{3.1.1}$$

where $m > 0$ and λ are real numbers, Δ is the classical Laplace operator, and Δ^2 is the bi-Laplacian. The equation is said to be *defocusing* if $\lambda < 0$ and *focusing* if $\lambda > 0$. At first glance, (3.1.1) is a formal fourth-order variant of the classical Klein-Gordon equation, but it also has a Schrödinger structure because of the decomposition $\partial_t^2 + \Delta^2 = (\partial_t + i\Delta)(\partial_t - i\Delta)$. It has infinite propagation speed.

In a recent series of papers [6], [7], [8], [11], Levandosky and the authors began a systematic mathematical study of equation (3.1.1). The culminating result in [11] is that every solution of finite energy is a scattering state, provided the nonlinearity is defocusing, $1 + 8/n < p < 1 + 8/(n-4)$ and $n \geq 5$. There is a mathematical literature going back 40 years that treats the global scattering theory of second-order nonlinear waves, but [11] is the first paper that fully succeeds in the fourth-order case. For general background on nonlinear scattering, see [13].

The aim of the current paper is to further develop the scattering theory for equation (3.1.1) in greater detail. First we prove (in Section 3.3) that the scattering operator $\mathcal{S}$ is as smooth as the nonlinearity allows. It is even an *analytic* operator in case p is an odd integer. The analogue of this theorem in the second-order case is due to Kumlin [5]. Our method is also applicable to the second-order case and is much simpler than the method of [5].

Furthermore, $\mathcal{S}$ uniquely determines the nonlinearity p and the coupling constant λ, as we prove in Section 3.4. On the other hand, in Section 3.5 we prove that if $1 < p \leq 1 + 2/n$, then $\mathcal{S}$ cannot exist, even for arbitrarily low energies.

3.2 Notation and Background

We work in the energy space, $\mathcal{E} = H^2 \times L^2$ of pairs of functions. On this space we define the two energy functionals

$$\begin{aligned} E_0(u,v) &= \frac{1}{2}\int_{\mathbb{R}^n} \left(|v|^2 + |\Delta u|^2 + m|u|^2\right) dx, \\ E(u,v) &= E_0(u,v) - \frac{\lambda}{p+1}\int_{\mathbb{R}^n} |u|^{p+1} dx \end{aligned} \tag{3.2.1}$$

and we choose $\sqrt{E_0(u,v)}$ as a norm on $\mathcal{E}$. Some more definitions and notations are as follows. For any interval $I \subset \mathbb{R}$, we define

$$\mathbb{E}_I = C(I, H^2) \cap C^1(I, L^2) \cap L^{q_1}(I \times \mathbb{R}^n) \cap L^{q_2}(I \times \mathbb{R}^n),$$

where

$$q_1 = \frac{2(n+2)p}{n+4}, \qquad q_2 = \frac{2(n+4)p}{n+8}.$$

Given $u \in \mathbb{E}_I$ and $t \in I$, in general we let $\bar{u}(t)$ denote the pair

$$\bar{u}(t) = (u(t), u_t(t)) \in \mathcal{E}.$$

We consider the following norms for functions defined on space and time:

$$\begin{aligned} \|u\|_{Z(I)} &= \sup_{i=1,2} \|u\|_{L^{q_i}(I\times\mathbb{R}^n)} \\ \|u\|_{\mathbb{E}_I} &= \|u\|_{Z(I)} + \|u\|_{L^\infty(I,H^2)} + \|u_t\|_{L^\infty(I,L^2)} \\ \|u\|_{N(I)} &= \|u\|_{L^{q_1/p}(I\times\mathbb{R}^n)} + \|u\|_{L^{q_2/p}(I\times\mathbb{R}^n)} \end{aligned} \tag{3.2.2}$$

We define a *strong solution* of the nonlinear equation (3.1.1) on an interval I to be a function $u \in \mathbb{E}_I$ that satisfies (3.1.1) for every time $t \in I$. Such a solution has constant energy in the sense that

$$\forall t, s \in I, \quad E(\bar{u}(t)) \equiv E(u(t), u_t(t)) = E(u(s), u_t(s)) \equiv E(\bar{u}(s)).$$

Now consider the linear (free) beam equation with $\lambda = 0$. We let $\mathcal{U}_0(t)$ be the propagator for the linear equation on the energy space, defined by

$$\mathcal{U}_0(t)\,(u_0, u_1) = (v(t), v_t(t)) = \left(\mathcal{U}_0^1(t)(u_0,u_1), \mathcal{U}_0^2(t)(u_0,u_1)\right)$$

where v is the unique function in $C(\mathbb{R}, H^2)$ satisfying

$$v_{tt} + \Delta^2 v + v = 0\ ,\ (v(0), v_t(0)) = (u_0, u_1)\,,$$

for $(u_0, u_1) \in \mathcal{E}$. Such a function v is called a *free solution.*

We define the time-reversal operator $J : \mathcal{E} \to \mathcal{E}$ by $J(u,v) = (u, -v)$. Note that $\mathcal{U}_0(t) = J\mathcal{U}_0(-t)J$. We also define the operator $\mathcal{N}(u,v) = (0, |u|^{p-1}u)$ mapping $L^{q_i}(\mathbb{R}^n) \times (*)$ into $\{0\} \times L^{q_i/p}(\mathbb{R}^n)$.

Now we recall the global estimates of Strichartz type in [11].

Lemma 3.2.1. *Let $1 + 8/n \leq p \leq 1 + 8/(n-4)$. There is a constant C independent of $\bar{u}$, h and I such that*

$$\left\| \mathcal{U}_0(t)\bar{u} + \int_0^t \mathcal{U}_0(t-s)\,(0, h(s))\,ds \right\|_{\mathbb{E}_I} \leq C\left(\|\bar{u}\|_{\mathcal{E}} + \|h\|_{N(I)}\right) \tag{3.2.3}$$

for any interval I, all $\bar{u} \in \mathcal{E}$ and all $h \in N(I)$,

We refer to Lemma 3.2 in [11] for the proof.

3.3 Regularity of the scattering operator

The theorem that defines the *scattering operator* $\mathcal{S}^{m,\lambda}\bar{u}^- = \bar{u}^+$ is as follows. It is the main result of [11], to which the reader is referred for the proof.

Theorem 7. *Let $n \geq 5$, $m > 0$, $\lambda < 0$, and $1 + 8/n < p < 1 + 8/(n-4)$. Then for any $\bar{u}^- \in \mathcal{E}$, there exists a unique strong solution $u \in \mathbb{E}_{\mathbb{R}}$ of* (3.1.1) *such that*

$$\|\bar{u}(t) - \mathcal{U}_0(t)\bar{u}^-\|_{\mathcal{E}} \to 0 \tag{3.3.1}$$

as $t \to -\infty$. It follows that

$$E(\bar{u}) = E_0(\bar{u}^-). \tag{3.3.2}$$

Furthermore, there exists a unique $\bar{u}^+ \in \mathcal{E}$ such that (3.3.1) *holds for $t \to +\infty$ with $\bar{u}^+$ instead of $\bar{u}^-$.*

In the sequel, we say that a strong solution u *scatters* or is a *scattering state* if there exist $\bar{u}^-$ and $\bar{u}^+$ satisfying the conditions above. For the reader's convenience we define the C^γ-class of regularity as follows.

Definition 3.3.1. *We define $C^\gamma = C^{[p]}$ if p is not an integer. Also, we define $C^\gamma = C^{p-1,1}$, the space of functions whose derivatives of order $p-1$ are Lipschitz, if p is an even integer, and define C^γ to be the space $\mathcal{A} = C^\omega$ of analytic operators if p is an odd integer.*

Note that the composition of two C^γ functions is again C^γ. Our main result in this section is the following.

Theorem 8. *The mapping $\mathcal{S} : (m, \lambda, \bar{u}^-) \mapsto \bar{u}^+ = \mathcal{S}^{m,\lambda}\bar{u}^-$ from $\mathbb{R}_+^* \times \mathbb{R}_-^* \times \mathcal{E}$ into $\mathcal{E}$ is continuous. Besides, for fixed m, it is of class C^γ. In particular it is analytic if p is an odd integer and of class $C^{p-1,1}$ if p is an even integer.*

The second-order analogue of Theorem 8 was proven by a more complicated method in [5]. Earlier second-order results appeared in [1], [12] and [10]. Our first lemma shows that by scaling we can normalize the two constants.

Lemma 3.3.1. *In the proof of Theorem 8 we may assume that $m = 1 = -\lambda$.*

Proof. Let $T_{\alpha,\beta}$ act on space-time functions by

$$T_{\alpha,\beta}u(t,x) = \beta^{\frac{1}{p-1}} u\left(\frac{t}{\alpha^2}, \frac{x}{\alpha}\right).$$

Let u satisfy (3.1.1) with initial conditions (u_0, u_1). Letting $\alpha^4 = m$, and $\beta = \frac{|\lambda|}{m}$, we see that $T_{\alpha,\beta}u$ satisfies (3.1.1) with $m = 1 = -\lambda$ and with the initial conditions

$$g_{\alpha,\beta}(u_0, u_1) = \left(\left(\frac{|\lambda|}{m}\right)^{\frac{1}{p-1}} u_0\left(\frac{\cdot}{m^{\frac{1}{4}}}\right), \frac{|\lambda|^{\frac{1}{p-1}}}{m^{\frac{p+1}{2(p-1)}}} u_1\left(\frac{\cdot}{m^{\frac{1}{4}}}\right)\right).$$

Thus the dependence of $\mathcal{S}$ on m, λ is explicit through the relation

$$\mathcal{S}^{m,\lambda} = g_{\alpha,\beta}^{-1}\ \mathcal{S}^{1,-1}\, g_{\alpha,\beta}.$$

So we only have to prove the theorem for $m = 1 = -\lambda$. □

For simplicity in the rest of this section we denote $\mathcal{S} = \mathcal{S}^{1,-1}$. The scattering operator can be decomposed in terms of the wave operator and its inverse. We establish that all these operators are C^γ, and then prove regularity of the scattering operator. For this, we need the following lemma.

Lemma 3.3.2. *There exists $\delta > 0$ such that the following statements are true. Let any strong solution u of* (3.1.1) *and any interval I be given such that $\|u\|_{Z(I)} \leq \delta$.*

(i) If $I = [T_1, T_2]$ is finite, consider the mapping $\mathcal{U}_I$ from $\bar{v}_0 \in \mathcal{E}$ at T_1 into the strong solution $v \in \mathbb{E}_I$ of (3.1.1) *such that $\bar{v}(T_1) = \bar{v}_0$. Then this mapping is C^γ in a neighborhood of $\bar{u}(T_1)$.*

(ii) If $I = (-\infty, T_2]$, then there exists $\bar{u}^- \in \mathcal{E}$ that satisfies (3.3.1)*. Consider the wave operator $\mathcal{W}_I$ from $\bar{v}^- \in \mathcal{E}$ at time $-\infty$ into the strong solution $v \in \mathbb{E}_I$ of* (3.1.1) *for which $\|\bar{v}(t) - \mathcal{U}_0(t)\bar{v}^-\|_{\mathcal{E}} \to 0$. Then this mapping is C^γ in a neighborhood of $\bar{u}^-$, with γ as above.*

Proof. We begin with the first statement. Without loss of generality, we can assume $T_1 = 0$. Let $\mathcal{T}$ be the mapping defined on $\mathcal{E} \times \mathbb{E}_I$ by

$$\mathcal{T}(\bar{v}_0, v) = \left(\bar{v}_0,\ \mathcal{U}_0(\cdot)\bar{v}_0 - \int_0^{\cdot} \mathcal{U}_0(\cdot - s)(0, |v(s)|^{p-1}v(s))ds\right). \tag{3.3.3}$$

By (3.1.1), the restriction $u_{|I} \in \mathbb{E}_I$ satisfies $(\bar{u}(0), u) = \mathcal{T}(\bar{u}(0), u)$. By the Strichartz estimate (3.2.3), the second component of $\mathcal{T}$ satisfies

$$\|\mathcal{T}_2(\bar{v}_0, v)\|_{\mathbb{E}_I} \leq C(\|\bar{v}_0\|_{\mathcal{E}} + \||v|^{p-1}v\|_{N(I)}) \leq C(\|\bar{v}_0\|_{\mathcal{E}} + \|v\|^p_{Z(I)}), \tag{3.3.4}$$

so that $\mathcal{T}$ maps $\mathcal{E} \times \mathbb{E}_I$ into $\mathcal{E} \times \mathbb{E}_I$. Note that the function $s \mapsto f(s) = |s|^{p-1}s$ is of class C^γ on $\mathbb{C}$, so that $\mathcal{T}$ is also of class C^γ on $\mathcal{E} \times \mathbb{E}_I$. In order to prove the lemma, we will apply the implicit function theorem to the mapping $\mathcal{T}$ at the given solution $(\bar{u}(0), u)$. Thus we require that the linear map $D_2\,(Id - \mathcal{T})\,(\bar{u}(0), u)$ from $\mathbb{E}_I$ into $\mathbb{E}_I$ be invertible, where Id is the identity mapping on $\mathcal{E} \times \mathbb{E}_I$ and D_2 denotes the derivative with respect to the second argument. It suffices to prove that

$$\|D_2\mathcal{T}(\bar{u}(0), u)\|_{\mathbb{E}_I \to \mathbb{E}_I} < 1. \tag{3.3.5}$$

Now for every $v \in \mathbb{E}_I$, we get

$$\begin{aligned} \|D_2\mathcal{T}\,(\bar{u}(0), u)\,(v)\,\|_{\mathbb{E}_I} &= \left\|\int_0^t \mathcal{U}_0(t-s)\,(0, f'(u(s))v(s))\,ds\right\|_{\mathbb{E}_I} \\ &\leq C\||u|^{p-1}v\|_{N(I)} \leq C\|u\|^{p-1}_{Z(I)}\|v\|_{Z(I)} \\ &\leq C\delta^{p-1}\|v\|_{\mathbb{E}_I}, \end{aligned} \tag{3.3.6}$$

by (3.2.3). So we see that if δ is sufficiently small, then (3.3.6) implies (3.3.5). Thus $D_2\,(Id - \mathcal{T})\,(\bar{u}(0), u)$ is an isomorphism. Hence there exists a neighborhood $\mathcal{O} \times \mathcal{P}$ of $(\bar{u}(0), u)$ in $\mathcal{E} \times \mathbb{E}_I$, and a C^γ function $\phi : \mathcal{O} \to \mathbb{E}_I$ such that for any $\bar{v}_0 \in \mathcal{O}$, $\phi(\bar{v}_0)$ is the unique solution of $(Id - \mathcal{T})\,(\bar{v}_0, \phi(\bar{v}_0)) = 0$. By definition of $\mathcal{T}$, this is equivalent to the statement that $\phi(\bar{v}_0)$ satisfies (3.1.1) and $\overline{\phi(\bar{v}_0)}(0) = \bar{v}_0$. Hence $\phi = \mathcal{U}_I$. This proves the first statement of the lemma. The second statement is proven in exactly the same way if we replace (3.3.3) by

$$\mathcal{T}(\bar{v}^-, v) = \left(\bar{v}^-,\ \mathcal{U}_0(\cdot)\bar{v}^- - \int_{-\infty}^{\cdot} \mathcal{U}_0(\cdot - s)(0, |v(s)|^{p-1}v(s))ds\right).$$

This ends the proof of Lemma 3.3.2. ∎

Now we establish the regularity of the wave operator.

Lemma 3.3.3. *The mapping $\mathcal{W}^- : \bar{u}^- \in \mathcal{E} \mapsto u \in \mathbb{E}_{\mathbb{R}}$, where u is the strong solution of* (3.1.1) *such that* (3.3.1) *holds, is of class C^γ with γ as above.*

Proof. Fix $\bar{u}^- \in \mathcal{E}$. Let u be the strong solution of (3.1.1) defined for all $t \in \mathbb{R}$ that satisfies (3.3.1) and belongs to $\mathbb{E}_{\mathbb{R}}$. In particular, the $Z(\mathbb{R})$-norm of u is finite. We choose times $-\infty < T_0 < T_1 < \cdots < T_{k+1} < \infty$ and divide $\mathbb{R}$ into subintervals $\mathbb{R} = (-\infty, T_0] \cup \bigcup_{i=0}^{k} [T_i, T_{i+1}] \cup [T_{k+1}, +\infty)$, where $\|u\|_{Z(I)} = \delta$ for $I = [T_{i-1}, T_i]$ for $i = 0, .., k$; and furthermore $\|u\|_{Z(I)} \leq \delta$ for $I = (-\infty, T_0]$ and $[T_{k+1}, +\infty)$. First applying Lemma 3.3.2 on the infinite interval $(-\infty, T_0]$, we deduce that the mapping

$$\mathcal{W}^- : \bar{v}^- \mapsto v_{|(-\infty,T_0]}$$

is C^γ from a neighborhood of $\bar{u}^- \in \mathcal{E}$ into $\mathbb{E}_{(-\infty,T_0]}$, where $v \in \mathbb{E}_{(-\infty,T_0]}$ satsfies (3.1.1) and (3.3.1). In particular, the mapping $\bar{v}^- \mapsto \bar{v}(T_0)$ from $\mathcal{E}$ into $\mathcal{E}$ is of class C^γ. Proceeding similarly $k+1$ times, we see that there exists a neighborhood of $\bar{u}^-$, say $\mathcal{O} \subset \mathcal{E}$, such that the mapping

$$\bar{v}^- \mapsto (\bar{v}(T_1), \ldots, \bar{v}(T_{k+1}))$$

is of class C^γ from $\mathcal{O}$ into $(\mathcal{E})^{k+1}$.

Applying Lemma 3.3.2 again on $(-\infty, T_0]$, on each interval $[T_i, T_{i+1}]$ for $i = 0, .., k$, and on $[T_{k+1}, +\infty)$, we deduce that the mapping

$$\bar{v}^- \in \mathcal{O}' \mapsto \big(v_{|(-\infty,T_0]}, v_{|[T_0,T_1]}, \ldots, v_{|[T_{k+1},+\infty)}\big) \in \mathbb{E}_{(-\infty,T_0]} \times \cdots \times \mathbb{E}_{[T_{k+1},+\infty)}$$

is of class C^γ, where $\mathcal{O}' \subset \mathcal{O}$ is a neighborhood of $\bar{u}^-$. Finally we note that the "restriction" map

$$\mathcal{R} : w \in \mathbb{E}_{\mathbb{R}} \mapsto \big(w_{|(-\infty,T_0]}, \ldots, w_{|[T_{k+1},+\infty)}\big) \in \mathbb{E}_{(-\infty,T_0]} \times \cdots \times \mathbb{E}_{[T_{k+1},+\infty)}$$

is obviously an isomorphism onto its range. So by considering $\mathcal{R}^{-1}$, we conclude that the mapping

$$\bar{v}^- \in \mathcal{O}' \mapsto v \in \mathbb{E}_{\mathbb{R}}$$

is also of class C^γ. This completes the proof of Lemma 3.3.3. □

We are now in position to prove the theorem.

Proof of Theorem 8. By (4.4) in [11], we know that if $u = \mathcal{W}^-(\bar{u}^-)$, then

$$\bar{u}^+ = \mathcal{S}(\bar{u}^-) = \bar{u}(0) - \int_0^\infty \mathcal{U}_0(-s)(0, |u(s)|^{p-1}u(s))ds. \tag{3.3.7}$$

The mapping $\bar{u}^- \mapsto u \mapsto \bar{u}(0)$ is of class C^γ from $\mathcal{E}$ to $\mathbb{E}_{\mathbb{R}}$ to $\mathcal{E}$ by Lemma 3.3.3. The mapping

$$u \mapsto \int_0^\infty \mathcal{U}_0(-s)(0, |u(s)|^{p-1}u(s))ds$$

is of class C^γ from $\mathbb{E}_{[0,\infty)}$ to $\mathcal{E}$ as in (3.3.4). By composition, $\mathcal{S}$ is also of class C^γ. By Lemma 3.3.1, the proof is concluded. □

As a remark, by an almost identical proof we prove the regularity of the local flow map even if λ is arbitrary and p is critical, as stated in the following theorem.

Theorem 9. *Let $1 \leq p \leq 1+8/(n-4)$. Let $m > 0$ and $\lambda \in \mathbb{R}$. For any $\bar{u}_0 \in \mathcal{E}$, let $T^\star(m,\lambda,\bar{u}_0)$ be the maximal positive time of existence of the strong solution u of (3.1.1) such that $\bar{u}(0) = \bar{u}_0$. Then for any $T < T^\star(m,\lambda,\bar{u}_0)$, the mapping $\mathcal{U}_{[0,T]} : (m,\lambda,\bar{v}_0) \mapsto v$, where v is the unique strong solution of (3.1.1) such that $\bar{v}(0) = \bar{v}_0$, is continuous from a neighborhood of $(m,\lambda,\bar{u}_0) \in \mathbb{R}_+^* \times \mathbb{R}^* \times \mathcal{E}$ into $\mathbb{E}_{[0,T]}$ if $p \geq 1+8/n$, and into $C([0,T],\mathcal{E})$ if $p < 1+8/n$. Besides, for fixed m, it is of class C^γ.*

3.4 Inverse scattering

In this section, we recover the dynamics from knowledge of $\mathcal{S}$ alone. The question is: can we recover the nonlinear interaction term from the scattering operator $\mathcal{S}$ alone? We give a positive answer to this question.

Theorem 10. *The scattering operator $\mathcal{S}$ determines λ and p uniquely.*

Proof. In fact, we can even limit ourselves to the consideration of small-data scattering. First we will prove that $\mathcal{S}$ determines p. We recall the following fact about the wave operator $\mathcal{W}^-$ that is a standard consequence of estimate (3.2.3) as in [7]. Namely, there exists $\epsilon > 0$ such that, for any $\bar{u}^- \in \mathcal{E}$ with $\|\bar{u}^-\|_{\mathcal{E}} \leq \epsilon$, we know that the strong solution $u = \mathcal{W}_{\mathbb{R}}^-(\bar{u}^-)$ of (3.1.1) given by Theorem 7 satisfies

$$\|\mathcal{W}_{\mathbb{R}}^-(\bar{u}^-)\|_{\mathbb{E}_{\mathbb{R}}} \leq C\|\bar{u}^-\|_{\mathcal{E}} \ , \quad \|\mathcal{W}_{\mathbb{R}}^-(\bar{u}^-) - \mathcal{U}_0^1(\cdot)\bar{u}^-\|_{\mathbb{E}_{\mathbb{R}}} \leq C\|\bar{u}^-\|_{\mathcal{E}}^p \tag{3.4.1}$$

for some $C > 0$ independent of $\bar{u}^-$.

Now fix $\bar{\phi} \in \mathcal{E}$ with energy norm smaller than ϵ. For a small parameter $\theta > 0$, let $(v^\theta, v_t^\theta) = \mathcal{U}_0(t)(\theta\bar{\phi})$, and let $u^\theta = \mathcal{W}_{\mathbb{R}}^-(\theta\bar{\phi}) \in \mathbb{E}_{\mathbb{R}}$. Then we have

$$(\theta\bar{\phi})^+ = \mathcal{S}\left(\theta\bar{\phi}\right) = \theta\bar{\phi} + \lambda \int_{-\infty}^{+\infty} \mathcal{U}_0(-s)(0, |u^\theta|^{p-1}u^\theta(s))ds$$

as in (3.3.7). Consequently, applying the linear estimate (3.2.3) and then (3.4.1), we obtain

$$\begin{aligned}
&\left\|\mathcal{S}\left(\theta\bar{\phi}\right) - \theta\bar{\phi} - \lambda \int_{-\infty}^{+\infty} \mathcal{U}_0(-s)(0, |v^\theta|^{p-1}v^\theta(s))ds\right\|_{\mathcal{E}} \\
&= |\lambda| \left\| \int_{-\infty}^{+\infty} \mathcal{U}_0(-s)\left(0, |u^\theta|^{p-1}u^\theta(s) - |v^\theta|^{p-1}v^\theta(s)\right) ds \right\|_{\mathcal{E}} \\
&\leq C_{m,\lambda} \| |u^\theta - v^\theta| \left(|u^\theta|^{p-1} + |v^\theta|^{p-1}\right) \|_{N(\mathbb{R})} \\
&\leq C_{m,\lambda} \|u^\theta - v^\theta\|_{Z(\mathbb{R})} \left(\|u^\theta\|_{Z(\mathbb{R})}^{p-1} + \|v^\theta\|_{Z(\mathbb{R})}^{p-1}\right).
\end{aligned} \tag{3.4.2}$$

But $\|v^\theta\|_{Z(\mathbb{R})} = \theta\|\mathcal{U}_0^1(\cdot)\bar{\phi}\|_{Z(\mathbb{R})} = C\theta$ and $\|u^\theta\|_{Z(\mathbb{R})} \leq \|\mathcal{W}_{\mathbb{R}}^-(\theta\bar{\phi})\|_{\mathbb{E}_{\mathbb{R}}} \leq C\theta\|\bar{\phi}\|_{\mathcal{E}}$ by (3.4.1). Moreover,

$$\|u^\theta - v^\theta\|_{Z(\mathbb{R})} \leq \|\mathcal{W}_{\mathbb{R}}^-(\theta\bar{\phi}) - \mathcal{U}_0^1(\cdot)\theta\bar{\phi}\|_{\mathbb{E}_{\mathbb{R}}} \leq C\|\theta\bar{\phi}\|_{\mathcal{E}}^p \leq C\theta^p.$$

So from (3.4.2) we have

$$\left\| \mathcal{S}\left(\theta\bar{\phi}\right) - \theta\bar{\phi} - \lambda\theta^{p}\int_{-\infty}^{+\infty}\mathcal{U}_0(-s)(0,|v^1|^{p-1}v^1(s))ds \right\|_{\mathcal{E}} \leq C_{m,\lambda,\bar{\phi}}\ \theta^{2p-1}.$$

Clearly this integral is $O(\theta^p)$. Thus

$$\|\left(\mathcal{S}-Id\right)\left(\theta\bar{\phi}\right)\|_{\mathcal{E}} = C_{m,\lambda,\bar{\phi}}\ \theta^p + o_{m,\lambda,\bar{\phi}}(\theta^p)$$

as $\theta \to 0$. Obviously p is thereby determined uniquely from the scattering operator on small data.

Now that p is determined, let us determine λ. We follow the approach in [10] by considering the following skew-symmetric form on $\mathcal{E}$:

$$\Omega\left((u_0,u_1);(v_0,v_1)\right) = \int_{\mathbb{R}^n}\left[u_0(x)v_1(x) - v_0(x)u_1(x)\right]dx. \tag{3.4.3}$$

This form is an invariant of the free equation; that is, for any $a, b \in \mathcal{E}$, the form $\Omega(\mathcal{U}_0(t)a;\mathcal{U}_0(t)b)$ does not depend on t. Choosing arbitrary $\bar{u}^-, \bar{v}^- \in \mathcal{E}$, we let $u = \mathcal{W}^-(\bar{u}^-)$ and $v = \mathcal{W}^-(\bar{v}^-)$. Since both u and v are solutions of (3.1.1) in $\mathbb{E}_{\mathbb{R}}$, the mapping $t \mapsto \Omega(\bar{u}(t);\bar{v}(t))$ is C^1 and a simple calculation shows that

$$\frac{d}{dt}\Omega(\bar{u};\bar{v}) = \lambda\int_{\mathbb{R}^n}\left[|u|^{p-1}uv - |v|^{p-1}vu\right]dx.$$

Integrating this expression between $-T$ and T, we get

$$\Omega(\bar{u}(T),\bar{v}(T)) - \Omega(\bar{u}(-T),\bar{v}(-T)) = \lambda\int_{-T}^{T}\int_{\mathbb{R}^n}\left[|u|^{p-1}uv - |v|^{p-1}vu\right]dxdt. \tag{3.4.4}$$

Letting $T \to +\infty$ in (3.4.4) and using (3.3.1), we obtain

$$\Omega(\mathcal{S}(\bar{u}^-);\mathcal{S}(\bar{v}^-)) - \Omega(\bar{u}^-;\bar{v}^-) = \lambda\int_{-\infty}^{+\infty}\int_{\mathbb{R}^n}\left[|u|^{p-1}uv - |v|^{p-1}vu\right]dxdt. \tag{3.4.5}$$

Now we fix $\bar{\phi} \in \mathcal{E}$ to have energy smaller than ϵ and we let $\mathcal{U}_0(t)\bar{\phi} = (w(t), w_t(t))$. For $\theta > 0$, we choose $\bar{u}^- = 2\theta\bar{\phi}$ and $\bar{v}^- = \theta\bar{\phi}$, which define u and v as above. For θ sufficiently small, identity (3.4.5) and then estimate (3.4.1) yield the estimate

$$\begin{aligned}
&\Omega(\mathcal{S}(2\theta\bar{\phi});\mathcal{S}(\theta\bar{\phi}))\\
&= \lambda\int_{-\infty}^{+\infty}\int_{\mathbb{R}^n}\left[|u|^{p-1}uv - |v|^{p-1}vu\right]dxdt\\
&= \lambda\int_{-\infty}^{+\infty}\int_{\mathbb{R}^n}\left[|2\theta w|^{p-1}2\theta^2w^2 - |\theta w|^{p-1}2\theta^2w^2\right]dxdt + O(\theta^{2p})\\
&= (2^p-2)\,\theta^{p+1}\lambda\int_{\mathbb{R}^n\times\mathbb{R}}|w|^{p+1}dxdt + O(\theta^{2p}).
\end{aligned} \tag{3.4.6}$$

By (3.4.6) we conclude that

$$\lambda = \lim_{\theta\to 0}\frac{\theta^{-p-1}}{2(2^{p-1}-1)}\left\{\int\int|\mathcal{U}_0^1(t)\bar{\phi}|^{p+1}dxdt\right\}^{-1}\Omega(\mathcal{S}(2\theta\bar{\phi});\mathcal{S}(\theta\bar{\phi})).$$

Thus λ is determined by $\mathcal{S}$ and the proof is concluded. □

3.5 Nonscattering range

In contrast to Theorem 7, we establish that scattering cannot occur if p is near 1. The negative result we prove concerns convergence as $t \to -\infty$. However, since $\mathcal{U}_0(-t) = J\mathcal{U}_0(t)J$, it has a counterpart as $t \to +\infty$.

Theorem 11. *Let $1 < p \leq 1 + \frac{2}{n}$, $m > 0$, $n \geq 2$ and $\lambda < 0$. Then there exists $\bar{u}^- \in \mathcal{E}$ of arbitrarily small energy norm such that there cannot be any strong solution u of* (3.1.1) *satisfying* (3.3.1).

An analogous result for the classical nonlinear Klein Gordon equation is due to Glassey [2]. We first need the following simple lemma concerning the optimal rate of decay of free solutions.

Lemma 3.5.1. *There exists a free solution w of finite energy that satisfies the lower bound*

$$\int_{\mathbb{R}^n} |w(t,x)|^{p+1} dx \geq C_0 |t|^{-\frac{n(p-1)}{2}} \tag{3.5.1}$$

for any $p \geq 1$ and the upper bound

$$\sup_x |w(t,x)| \leq C_1 |t|^{-\frac{n}{2}} \tag{3.5.2}$$

for some positive constants C_0, C_1.

Proof. Let $\psi \in C_c^\infty(\mathbb{R}^n)$ be such that $\psi \geq 0$, $\int \psi = 1$, and $\psi = 0$ for $|\xi| > 2$ and $|\xi| < 1$. We define

$$\hat{w}_0(\xi) = \psi(\xi), \qquad \hat{w}_1(\xi) = i\sqrt{1+|\xi|^4}\ \psi(\xi)\,. \tag{3.5.3}$$

Clearly $\bar{w} = (w_0, w_1)$ has finite energy. We let

$$w(t) = \mathcal{U}_0^1(t)\bar{w} = e^{it\sqrt{1+\Delta^2}}\check{\psi}$$

be the linear solution with initial data $(w_0, w_1) \in \mathcal{E}$, where $\check{\psi}(x) = \hat{\psi}(-x)$. Applying Lemma 3.4 in [4], we immediately obtain the L^∞ decay (3.5.2).

We claim that there exists $C_n > 0$ such that if $|x| \geq 20t$, then

$$|v(t,x)| \leq C_n |x|^{-n-1}. \tag{3.5.4}$$

By spherical symmetry we can assume without loss of generality that $x_1 = r$ and $x_2 = \cdots = x_n = 0$. By $\partial_1, \partial^\alpha$ we denote derivatives with respect to the variable ξ. The phase function is

$$\varphi(t,x,\xi) = \sqrt{1+|\xi|^4} - \frac{x}{t}\cdot\xi$$

so that

$$|\partial_1\varphi(t,x,\xi)| = \frac{r}{t} - 2\xi_1 \geq \frac{1}{2}\frac{r}{t} \quad \text{and} \quad |\partial^\alpha\varphi(t,x,\xi)| \leq C_\alpha$$

for any multi-index α of length $|\alpha| \geq 2$, uniformly for $|\xi| \in [1,2]$ and $r \geq 20t$. Let $\mathcal{L}_{t,x}$ and $\mathcal{L}^*_{t,x}$ be the operators on $C_c^\infty(\mathbb{R}^n)$ given by

$$\mathcal{L}_{t,x}u(\xi) = \frac{1}{\partial_1\varphi(t,x,\xi)}\partial_1 u(\xi) \quad \text{and} \quad \mathcal{L}^*_{t,x}u(\xi) = -\partial_1\left(\frac{1}{\partial_1\varphi(t,x,\xi)}u(\xi)\right).$$

Then for $r \geq 20t$, integrations by parts $n+1$ times yield

$$\begin{aligned}|w(t,x)| &= \frac{1}{(2\pi)^{\frac{n}{2}}} \left| \int_{\mathbb{R}^n} e^{it\varphi(t,x,\xi)} \psi(\xi) d\xi \right| \\ &= \frac{1}{(2\pi)^{\frac{n}{2}} t^{n+1}} \left| \int_{\mathbb{R}^n} \mathcal{L}_{t,x}^{n+1} \left[e^{it\varphi(t,x,\xi)} \right] \psi(\xi) d\xi \right| \\ &= \frac{1}{(2\pi)^{\frac{n}{2}} t^{n+1}} \left| \int_{\mathbb{R}^n} e^{it\varphi(t,x,\xi)} \mathcal{L}_{t,x}^{*(n+1)} \psi(\xi) d\xi \right| \\ &\leq C_n |x|^{-n-1}.\end{aligned}$$

This proves the claim (3.5.4). Now by (3.5.2) and (3.5.4) we easily get that

$$\|w(t,x)\|_{L^1} \leq \int_{|x|\leq 20t} t^{-\frac{n}{2}} dx + C_n \int_{|x|\geq 20t} \frac{dx}{|x|^{n+1}} \leq C t^{\frac{n}{2}} \tag{3.5.5}$$

for some positive constant C. As $\|w(t)\|_{L^2}$ is constant, Hölder's inequality and (3.5.5) yield the lower bound (3.5.1). This concludes the proof. □

We are now in position to prove the theorem.

Proof of Theorem 11. We w be a free solution as in Lemma 3.5.1. Supposing that u is a solution of the nonlinear equation (3.1.1) that satisfies (3.3.1) as $t \to -\infty$ with $\bar{u}^- = \bar{w}$, we will prove a contradiction. Note that the energy $E(u, u_t) = E_0(w, w_t)$ can be made arbitrarily small, simply by multiplying w by a small constant. Let $H(t) = \Omega(\bar{u}(t), \bar{w}(t))$, where Ω is the skew form defined in (3.4.3). As $t \to -\infty$, we have $H(t) \to \Omega(\bar{u}^-, \bar{u}^-) = 0$ by (3.3.1). However, as before,

$$\begin{aligned}H(r)-H(s) &= \lambda \int_s^r \int_{\mathbb{R}^n} |u(x,t)|^{p-1} u(x,t) w(x,t) dx dt = \lambda \int_s^r (I + II) dt \\ &= \lambda \int_s^r \int_{\mathbb{R}^n} |w|^{p+1} dx dt + \lambda \int_s^r \int_{\mathbb{R}^n} \left[|u|^{p-1} u - |w|^{p-1} w \right] w \ dx dt.\end{aligned} \tag{3.5.6}$$

We will find a lower bound for this expression. Estimating I from below by means of (3.5.1), we have

$$I = \int_{\mathbb{R}^n} |w(t,x)|^{p+1} dx \geq C_0 |t|^{-\frac{n(p-1)}{2}}.$$

On the other hand, because $n \geq 2$ so that $p \leq 2$, we also have

$$\left| \left[|u|^{p-1} u - |w|^{p-1} w \right] w \right| \leq C |u - w| \left(|u|^{p-1} + |w|^{p-1} \right) |w|.$$

Hence we can estimate II from above by

$$\begin{aligned}&\int_{\mathbb{R}^n} \left| \left[|u|^{p-1} u - |w|^{p-1} w \right] w \right| \ dx \\ &\leq C \|u - w\|_{L^2} \ \|w\|_{L^\infty}^{p-1} \left\{ \||w|^{2-p} |u|^{p-1}\|_{L^2} + \|w\|_{L^2} \right\}.\end{aligned}$$

Because $\||w|^{2-p}|u|^{p-1}\|_{L^2} \leq \|w\|_{L^2} + \|u\|_{L^2}$ and u and w are bounded in $L^2(\mathbb{R}^n)$, this integral is at most

$$C\|u - w\|_{L^2} \ \|w\|_{L^\infty}^{p-1} \leq C \|u - w\|_{L^2} |t|^{-\frac{n(p-1)}{2}} = o\left(|t|^{-\frac{n(p-1)}{2}} \right),$$

where we have used the decay (3.5.2) of the free solution as well as the assumption that $u - w$ converges to 0 in energy norm. Thus

$$\frac{1}{\lambda}[H(r) - H(2r)] = \int_{2r}^{r} (I + II)dt \geq \int_{2r}^{r} \left(C_0|t|^{-\frac{n(p-1)}{2}} - \frac{1}{2}C_0|t|^{-\frac{n(p-1)}{2}} \right) dt.$$

Now as $r \to -\infty$, the left side tends to zero but the right side is larger than

$$\frac{1}{2}C_0 \int_{2r}^{r} \frac{1}{|t|} dt = \frac{1}{2}C_0 \log 2$$

because $n(p-1)/2 \leq 1$. This contradiction completes the proof. □

The focusing case is quite different from the defocusing case. In either case it is easy to see that if $1 < p \leq 1 + \frac{8}{n-4}$, then there exists a unique local-in-time solution of finite energy [6],[7]. However, the long-time behavior is more subtle.

Indeed, we now consider the *focusing* case $\lambda > 0$. If $1 < p \leq 1 + \frac{8}{n-4}$, then there are solutions that blow up in a finite time. This contrasts with the defocusing case. In fact, following the proof in [9] and [3], it is easily seen in the focusing case that for any initial data $\bar{u}_0$ with negative energy $E(\bar{u}_0) < 0$, the unique strong solution u of (3.1.1) satisfying $\bar{u}(0) = \bar{u}_0$ cannot be extended to all of $\mathbb{R}_+$ as a strong solution because it blows up. By definition, a solution that blows up cannot scatter.

What about small energy solutions? If $1+\frac{8}{n} \leq p \leq 1+\frac{8}{n-4}$, then all solutions of sufficiently small energy exist for all time and they do scatter [7]. This is true for any $\lambda > 0$. Of course, this fact is consistent with our earlier discussion of the defocusing case. On the other hand, for any $1 < p < 1 + 8/n$, there exist solutions of arbitrarily small energy that do *not* scatter. For example we can let $0 \not\equiv \psi \in H^2(\mathbb{R}^n)$ solve the elliptic equation $\Delta^2\psi + m\psi = |u|^{p-1}\psi$ (see [6]). Then for any $0 < \lambda < 1$ the function

$$v_\lambda(t,x) = \lambda^{\frac{4}{p-1}} e^{it\sqrt{m(1-\lambda^4)}}\psi(\lambda x)$$

solves (3.1.1), and

$$E(v_\lambda(0), i\sqrt{(1-\lambda^4)}v_\lambda(0)) \to 0$$

as $\lambda \to 0$. However, it could not approach any free solution in the energy space because it does not decay at all as $|t| \to \infty$.

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Chapter 4

Global well-posedness for energy critical fourth-order Schrödinger equations in the radial case

Abstract

Energy-critical fourth-order Schrödinger equations are investigated. We establish local well-posedness and stability in a general setting, and we prove global well-posedness and scattering in the defocusing case for radially symmetrical initial data.

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4.1 Introduction

Fourth-order Schrödinger equations have been introduced by Karpman [13] and Karpman and Shagalov [14] to take into account the role of small fourth-order dispersion terms in the propagation of intense laser beams in a bulk medium with Kerr nonlinearity. Such fourth-order Schrödinger equations are written as

$$i\partial_t u + \Delta^2 u + \varepsilon\Delta u + f\left(|u|^2\right) u = 0 \tag{4.1.1}$$

where $\varepsilon \in \mathbb{R}$ is essentially given by $\varepsilon = \pm 1$ or $\varepsilon = 0$, and $u : I \times \mathbb{R}^n \to \mathbb{C}$ is a complex-valued function. Sharp dispersive estimates for the biharmonic Schrödinger operator in (4.1.1), namely for the linear group associated to $i\partial_t + \Delta^2 + \varepsilon\Delta$, have recently been obtained in Ben-Artzi, Koch, and Saut [1], while specific nonlinear fourth-order Schrödinger equations as in (4.1.1) have been recently discussed in Fibich, Ilan, and Papanicolaou [5], Guo and Wang [9], Hao, Hsiao, and Wang [10, 11], and Segata [25]. Fibich, Ilan and Papanicolaou [5] describe various properties of the equation in the subcritical regime, with part of their analysis relying on very interesting numerical developments. Guo and Wang [9] prove global well-posedness and scattering in H^s for small data. Hao, Hsiao and Wang [10, 11] discuss the Cauchy problem in a high-regularity setting. Segata [25] proves scattering in the case the space dimension is one. Related equations also appeared in Fibich, Ilan, and Schochet [6], Huo and Jia [12], and Segata [23, 24].

When $n \leq 4$, and f has polynomial growth, equation (4.1.1) is subcritical and its analysis follows from standard developments. When $n \geq 5$ criticality in the energy space appears with the power-type nonlinearity $f(u) = |u|^{2^\sharp - 2}u$, where $2^\sharp = 2n/(n-4)$ is the critical exponent for the embedding of H^2 into Lebesgue's spaces. Following classical notations, we let H^2 be the space of square integrable functions whose first and second derivatives are also square integrable. We concentrate here on (4.1.1) with a pure power-type nonlinearity and aim in proving global existence in the critical defocusing regime of (4.1.1) for arbitrary initial data. Global well-posedness for the classical second order Schrödinger equation in the critical defocusing regime with a pure power-type nonlinearity has been recently established in a series of papers by Bourgain [2], Colliander, Keel, Staffilani, Takaoka, and Tao [4], Grillakis [8], Ryckman and Visan [22], Tao [26], and Visan [29]. We refer also to Kenig and Merle [17] for a similar result in the focusing case for solutions whose energy and kinetic energy are smaller than that of the ground state, and to Killip, Visan, and Zhang [18], where a quadratic potential is added to the classical second order Schrödinger equation. By analogy with second order Schrödinger equations it can be conjectured that global well-posedness holds true for (4.1.1) with a pure power-type nonlinearity, in the critical defocusing regime, for arbitrary initial data. We prove such global well-posedness when the initial data is radially symmetrical.

As already mentioned, the equations we consider in this paper correspond to (4.1.1) when f is a pure power-type nonlinearity. They are written as

$$i\partial_t u + \Delta^2 u + \varepsilon\Delta u + \lambda|u|^{p-1}u = 0, \tag{4.1.2}$$

where $\lambda \in \mathbb{R}$, and $p \in (1, 2^\sharp - 1]$. The energy critical regime in (4.1.2) corresponds to the case $p = 2^\sharp - 1$, and the defocusing regime to the case $\lambda > 0$. As a remark, when $\varepsilon = 0$, (4.1.2) enjoys scaling invariance. The scaling, as expressed in (4.4.1) below, preserves the homogeneous Sobolev space $\dot{H}^2$ when $p = 2^\sharp - 1$. Our main result states as follows.

Theorem 12. *Let $n \geq 5$, $\varepsilon \in \mathbb{R}$, $\lambda > 0$, and $p \in (1, 2^\sharp - 1]$. For any radially symmetrical data $u_0 \in H^2$ there exists a unique global solution $u \in C(\mathbb{R}, H^2)$ of (4.1.2) such that $u(0) = u_0$.*

When p is subcritical, or u_0 has small energy and p is critical, the radially symmetrical assumption on u_0 in Theorem 12 is not needed. We refer to Corollary 4.4.1 in Section 4.4 and Corollary 4.5.1 in Section 4.5 for more details on such assertions. We mainly concentrate in this paper on the critical case of (4.1.2). In this case we also prove stability for all ε, see Proposition 4.6.1 in Section 4.6, and scattering when $\varepsilon \leq 0$, see Proposition 4.9.1 and Proposition 4.9.2 in Section 4.9. Needless to say, scattering and stability are important notions for physical considerations.

Our paper is organized as follows. We fix notations in Section 4.2. Strichartz type estimates, a classical one and one with gain of derivatives, relying on the dispersion estimates in Ben-Artzi, Koch, and Saut [1], and on the Strichartz type estimates of Keel and Tao [16], are proved in Section 4.3. The local theory for (1.2), without the radially symmetrical assumption, and for arbitrary λ's, is established in Sections 4.4 to 4.6. The subcritical case of (4.1.2) is briefly discussed in Section 4.4. Local existence in the critical case is proved in Section 4.5, and stability in the sense of Tao and Visan [28] is discussed in Section 4.6. We prove localized Morawetz estimates and almost local conservation of mass in Section 4.7. Such estimates and conservations laws are crucial for the proof of Theorem 12. The theorem is proved in Section 4.8 following the strategy initiated in Bourgain [2] and developed in Tao [26]. Finally, in Section 4.9, we briefly discuss the scattering assertion we made after Theorem 12.

4.2 Notations

We fix notations we use throughout the paper. In what follows, we denote by C a generic constant that is allowed to depend on the dimension and the nonlinearity through $|\lambda|$ and p. The exact value of that constant may change from line to line. We write $C(a)$, $C(a,b)$ when there is more dependence. More significative constants are often denoted by K, K_1, K_2 to highlight their role. We let $L^q = L^q(\mathbb{R}^n)$ be the usual Lebesgue spaces, and $L^r(I, L^q)$ be the space of measurable functions from an interval $I \subset \mathbb{R}$ to L^q whose $L^r(I, L^q)$ norm is finite, where

$$\|u\|_{L^r(I,L^q)} = \left(\int_I \|u(t)\|_{L^q}^r dt\right)^{\frac{1}{r}}.$$

Two important conserved quantities of equation (4.1.1) are the mass and the energy defined by

$$M(u) = \int_{\mathbb{R}^n} |u(x)|^2 dx \tag{4.2.1}$$

on what concerns the mass, and

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^n} \left(|\Delta u(x)|^2 - \varepsilon|\nabla u(x)|^2 + F(|u(x)|^2)\right) dx \tag{4.2.2}$$

on what concerns the energy, where $F(s) = \int_0^s f(t)dt$. Several norms have to be considered in the analysis of the critical case of (4.1.2). For $I \subset \mathbb{R}$ an interval, they are defined as

$$\begin{aligned} \|u\|_{M(I)} &= \|\Delta u\|_{L^{\frac{2(n+4)}{n-4}}(I, L^{\frac{2n(n+4)}{n^2+16}})}, \\ \|u\|_{W(I)} &= \|\nabla u\|_{L^{\frac{2(n+4)}{n-4}}(I, L^{\frac{2n(n+4)}{n^2-2n+8}})}, \\ \|u\|_{Z(I)} &= \|u\|_{L^{\frac{2(n+4)}{n-4}}(I, L^{\frac{2(n+4)}{n-4}})}, \text{ and} \\ \|u\|_{N(I)} &= \|\nabla u\|_{L^2(I, L^{\frac{2n}{n+2}})}. \end{aligned} \tag{4.2.3}$$

Accordingly, we let $M(\mathbb{R})$ be the completion of $C_c^\infty(\mathbb{R}^{n+1})$ with the norm $\|\cdot\|_{M(\mathbb{R})}$, and $M(I)$ be the set consisting of the restrictions to I of functions in $M(\mathbb{R})$. We adopt similar definitions for W, Z, and N.

An important quantity, which turns out to be closely related to the mass and the energy, is the functional $\mathcal{E}$ defined for $u \in H^2$ by

$$\mathcal{E}(u) = \begin{cases} \int_{\mathbb{R}^n} |\Delta u(x)|^2 dx & \text{if } \varepsilon = 0 \\ \int_{\mathbb{R}^n} \left(|\Delta u(x)|^2 - \varepsilon |\nabla u(x)|^2 + |u(x)|^2\right) dx & \text{if } \varepsilon = \pm 1. \end{cases} \tag{4.2.4}$$

Note that when $\varepsilon = 0$, $\mathcal{E}(u)$ is nothing but $\|u\|_{\dot{H}^2}^2$, while when $\varepsilon = \pm 1$, $\mathcal{E}(u)$ controls the full inhomogeneous norm $\|u\|_{H^2} = \|\Delta u\|_{L^2} + \|u\|_{L^2}$.

In what follows we let $\mathcal{F} f = \hat{f}$ be the Fourier transform of f given by

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} f(y) e^{i\langle y, \xi \rangle} dy$$

for all $\xi \in \mathbb{R}^n$. The biharmonic Schrödinger semigroup is defined for any tempered distribution g by

$$e^{it(\Delta^2 + \varepsilon\Delta)} g = \mathcal{F}^{-1} e^{it(|\xi|^4 - \varepsilon|\xi|^2)} \mathcal{F} g. \tag{4.2.5}$$

Let $\psi \in C_c^\infty(\mathbb{R}^n)$ be supported in the ball $B_0(2)$, and such that $\psi = 1$ in $B_0(1)$. For any dyadic number $N = 2^k$, $k \in \mathbb{Z}$, we define the Littlewood-Paley operators P_N by

$$\widehat{P_N f}(\xi) = \left(\psi(\xi/N) - \psi(2\xi/N)\right) \hat{f}(\xi). \tag{4.2.6}$$

These operators commute one with another. They also commute with derivative operators and with the semigroup $e^{it(\Delta^2+\varepsilon\Delta)}$. In addition they are self-adjoint and bounded on L^p for all $1 \le p \le \infty$. Moreover, they enjoy the following Bernstein property:

$$\||\nabla|^{\pm s} P_N f\|_{L^p} \le C N^{\pm s} \|P_N f\|_{L^p} \le C N^{\pm s} \|f\|_{L^p}, \tag{4.2.7}$$

for all $s \ge 0$, and all $1 \le p \le \infty$, where $|\nabla|^s$ is the classical fractional differentiation operator, and $C > 0$ is independent of f, N, and p. Given $a \ge 1$, we let a' be the conjugate of a, so that $\frac{1}{a} + \frac{1}{a'} = 1$.

4.3 Strichartz-type Estimates

We prove Strichartz type estimates for solutions of the linear equation associated with the biharmonic Schrödinger operator and forcing term $h \in L^1_{loc}(I, H^{-4})$ for $I \subset \mathbb{R}$ an interval. In other words, for

$$u(t) = e^{it(\Delta^2+\varepsilon\Delta)}u_0 + i\int_0^t e^{i(t-s)(\Delta^2+\varepsilon\Delta)}h(s)ds, \tag{4.3.1}$$

where $u_0 \in L^2$. Key estimates in this section are given by the dispersion estimates of Ben-Artzi, Koch, and Saut [1]. Let I_ε be given by

$$I_\varepsilon(t,x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{it\left(|\xi|^4-\varepsilon|\xi|^2\right)-i\langle x,\xi\rangle}d\xi. \tag{4.3.2}$$

Using (4.2.5), one sees that I_ε is the fundamental solution of (4.3.1). Let also $\alpha \in \mathbb{N}^n$. Then, according to Ben-Artzi, Koch, and Saut [1], the following estimates hold true. Namely,

(a) estimates for the homogeneous equation:

$$|D^\alpha I_0(t,x)| \le Ct^{-\frac{n+|\alpha|}{4}}(1+t^{-\frac{1}{4}}|x|)^{\frac{|\alpha|-n}{3}} \tag{4.3.3}$$

for all $t > 0$ and all $x \in \mathbb{R}^n$,

(b) short time estimates for the inhomogeneous equation:

$$|D^\alpha I_\varepsilon(t,x)| \le Ct^{-\frac{n+|\alpha|}{4}}(1+t^{-\frac{1}{4}}|x|)^{\frac{|\alpha|-n}{3}} \tag{4.3.4}$$

for all $0 < t \le 1$ and all $x \in \mathbb{R}^n$, or all $t > 0$ and all $|x| \ge t$,

(c) long time estimates for the inhomogeneous equation:

$$|D^\alpha I_{-1}(t,x)| \le Ct^{-\frac{n+|\alpha|}{2}}(1+t^{-\frac{1}{2}}|x|)^{|\alpha|} \tag{4.3.5}$$

for all $t \ge 1$ and all $|x| \le t$,

where D stands for differentiation in the x variable. Useful consequences of (4.3.3)-(4.3.5) are that

$$|D^\alpha I_\varepsilon(t,x)| \le C|t|^{-\frac{n}{2}} \tag{4.3.6}$$

and that

$$\|e^{it(\Delta^2+\varepsilon\Delta)}g\|_{L^p} \le C|t|^{-\frac{n}{4}(1-\frac{2}{p})}\|g\|_{L^{p'}} \tag{4.3.7}$$

for all α such that $|\alpha| = n$, all $p \in [2,\infty]$, all $g \in L^{p'}$, and all time $t \neq 0$, where p' is the conjugate exponent of p and, if $\varepsilon = 1$, we also require that $|t| \le 1$. Inequality (4.3.6) is a direct consequence of (4.3.3)-(4.3.5). Inequality (4.3.7) follows from the remark that by (4.3.2)-(4.3.5) we can write that

$$\|e^{it(\Delta^2+\varepsilon\Delta)}g\|_{L^\infty} = \|I_\varepsilon(t) * g\|_{L^\infty} \le \|I_\varepsilon\|_{L^\infty}\|g\|_{L^1} \le C|t|^{-\frac{n}{4}}\|g\|_{L^1}$$

while Plancherel's theorem ensures that $e^{it(\Delta^2+\varepsilon\Delta)}$ is bounded $L^2 \to L^2$. By the Riesz-Thorin theorem, interpolation between the L^2 and L^∞ bounds gives (4.3.7).

Following standard notations, we say that a pair (q, r) is Schrödinger admissible, for short S-admissible, if $2 \leq q, r \leq \infty$, $(q, r, n) \neq (2, \infty, 2)$, and

$$\frac{2}{q} + \frac{n}{r} = \frac{n}{2}. \tag{4.3.8}$$

Also we add the terminology that a pair (q, r) is biharmonic admissible, for short B-admissible, if $2 \leq q, r \leq \infty$, $(q, r, n) \neq (2, \infty, 4)$, and

$$\frac{4}{q} + \frac{n}{r} = \frac{n}{2}. \tag{4.3.9}$$

Our Strichartz type estimates for (4.3.1) are stated as follows.

Proposition 4.3.1. *Let $u \in C(I, H^{-4})$ be a solution of* (4.3.1) *with $\varepsilon \in \{-1, 0, 1\}$ on an interval $I = [0, T]$. If $\varepsilon = 1$, suppose also $|I| \leq 1$. For any B-admissible pairs (q, r) and $(\bar{q}, \bar{r})$,*

$$\|u\|_{L^q(I, L^r)} \leq C \left(\|u_0\|_{L^2} + \|h\|_{L^{\bar{q}'}(I, L^{\bar{r}'})} \right) \tag{4.3.10}$$

whenever the right hand side in (4.3.10) *is finite, where C depends only on n, and $\bar{q}'$ and $\bar{r}'$ are the conjugate exponents of $\bar{q}$ and $\bar{r}$. Besides, for any S-admissible pairs (q, r) and (a, b), and any $s \geq 0$,*

$$\||\nabla|^s u\|_{L^q(I, L^r)} \leq C \left(\||\nabla|^{s - \frac{2}{q}} u_0\|_{L^2} + \||\nabla|^{s - \frac{2}{q} - \frac{2}{a}} h\|_{L^{a'}(I, L^{b'})} \right) \tag{4.3.11}$$

whenever the righthand side in (4.3.11) *is finite, where C depends only on n, and a' and b' are the conjugate exponents of a and b.*

It can be noted that, when $n \geq 4$, estimates (4.3.11) implies estimates (4.3.10).

Proof. Estimates (4.3.10) are easy to obtain. They directly follow from the linear estimates of Ben-Artzi, Koch and Saut (4.3.7) with $p = +\infty$ and $p = 2$, and the theorem in Keel and Tao [16] applied to the operator $\mathcal{U}(t)$, where

$$\mathcal{U}(t) = \begin{cases} e^{it(\Delta^2 + \varepsilon\Delta)} & \text{if } \varepsilon \leq 0 \\ e^{it(\Delta^2 + \Delta)} \mathbb{1}_{[-1,1]}(t) & \text{if } \varepsilon = 1. \end{cases} \tag{4.3.12}$$

Now we turn to the proof of (4.3.11). Since the free propagator $e^{it(\Delta^2 + \varepsilon\Delta)}$ commutes with the derivative operator $|\nabla|^s$, it suffice to prove (4.3.11) with $s = 0$. For N a dyadic integer and P_N as in (4.2.6), we let $Q_N = P_{N/2} + P_N + P_{2N}$. The Littlewood-Paley projector Q_N is such that $P_N Q_N = P_N$. For any $\lambda > 0$, let d_λ be the rescaling operator defined on all functions g by $d_\lambda g(x) = \lambda^{\frac{n}{2}} g(\lambda x)$. We consider the family of operators

$$\mathcal{V}_N(t) = d_N Q_N \mathcal{U}(t). \tag{4.3.13}$$

By Plancherel's theorem and since d_N is an isometry on L^2, we get that

$$\|\mathcal{V}_N(t)\|_{L^2 \to L^2} \leq C. \tag{4.3.14}$$

Independently, it is easily seen that

$$Q_N e^{it(\Delta^2+\varepsilon\Delta)}u_0 = e^{it(\Delta^2+\varepsilon\Delta)}Q_N\delta * u_0,$$

where δ denotes the Dirac measure. Then, as a consequence of (4.2.7), of the boundedness of the Riesz transform, and of (4.3.6), we can write that

$$\begin{aligned}\|Q_N e^{it(\Delta^2+\varepsilon\Delta)}u_0\|_{L^\infty} &\le \|e^{it(\Delta^2+\varepsilon\Delta)}Q_N\delta\|_{L^\infty}\|u_0\|_{L^1}\\ &\le N^{-n}\||\nabla|^n e^{it(\Delta^2+\varepsilon\Delta)}Q_N\delta\|_{L^\infty}\|u_0\|_{L^1}\\ &\le CN^{-n}\sup_{|\alpha|=n}\|D^\alpha e^{it(\Delta^2+\varepsilon\Delta)}Q_N\delta\|_{L^\infty}\|u_0\|_{L^1}\\ &\le C|N^2t|^{-\frac{n}{2}}\|u_0\|_{L^1},\end{aligned}\tag{4.3.15}$$

where C does not depend on N, u_0 or t, and, in case $\varepsilon = 1$, we assume $|t| \le 1$. Now, we can use (4.3.13) and (4.3.15) to compute

$$\begin{aligned}\|\mathcal{V}_N(s)\mathcal{V}_N(t)^*g\|_{L^\infty} &= \|d_N Q_N^2\mathcal{U}(s)\mathcal{U}(t)^*d_N^*g\|_{L^\infty}\\ &\le N^{\frac{n}{2}}\|Q_N e^{i(s-t)(\Delta^2+\varepsilon\Delta)}d_N^*g\|_{L^\infty}\\ &\le CN^{-\frac{n}{2}}|t|^{-\frac{n}{2}}\|d_N^*g\|_{L^1}\\ &\le C|t|^{-\frac{n}{2}}\|g\|_{L^1}.\end{aligned}\tag{4.3.16}$$

By (4.3.14) and (4.3.16), we can apply the results of Keel and Tao [16] to the operators $\mathcal{V}_N(t)$. We then get that for any S-admissible pairs (q,r) and (a,b), the following holds true:

$$\begin{aligned}&\|d_N Q_N\mathcal{U}(t)u_0\|_{L^q(\mathbb{R},L^r)} \le C\|u_0\|_{L^2}\text{ , and}\\ &\|d_N\int_{s<t}Q_N^2\mathcal{U}(t-s)h(s)ds\|_{L^q(\mathbb{R},L^r)} \le C\|d_N h\|_{L^{a'}(\mathbb{R},L^{b'})}.\end{aligned}\tag{4.3.17}$$

Now, applying the first inequality of (4.3.17) to $P_N u_0$, the second inequality of (4.3.17) to $P_N h$, and considering the effect of dilations d_N on space norms, we get

$$\begin{aligned}&N^{\frac{n}{2}-\frac{n}{r}}\|P_N\mathcal{U}(t)u_0\|_{L^q(\mathbb{R},L^r)} \le C\|P_N u_0\|_{L^2}\text{ , and}\\ &N^{\frac{n}{2}-\frac{n}{r}}\|\int_{s<t}P_N\mathcal{U}(t-s)h(s)ds\|_{L^q(\mathbb{R},L^r)} \le CN^{\frac{n}{2}-\frac{n}{b'}}\|P_N h\|_{L^{a'}(\mathbb{R},L^{b'})}.\end{aligned}\tag{4.3.18}$$

At this point, using (4.2.7), (4.3.8), (4.3.12), (4.3.18), and the Littlewood-Paley Theorem, we prove (4.3.11) by writing that

$$\begin{aligned}\|u\|_{L^q(I,L^r)} &\le C\|\big(\sum_N|P_N u|^2\big)^{\frac{1}{2}}\|_{L^q(I,L^r)}\\ &\le C\left(\sum_N\|P_N\left(e^{it(\Delta^2+\varepsilon\Delta)}u_0 + i\int_0^t e^{i(t-s)(\Delta^2+\varepsilon\Delta)}h(s)ds\right)\|^2_{L^q(I,L^r)}\right)^{\frac{1}{2}}\\ &\le C\left(\sum_N N^{-\frac{4}{q}}\|P_N u_0\|^2_{L^2} + N^{-\frac{4}{q}-\frac{4}{a}}\|P_N\mathbf{1}_I h\|^2_{L^{a'}(\mathbb{R},L^{b'})}\right)^{\frac{1}{2}}\\ &\le C\left(\sum_N\||\nabla|^{-\frac{2}{q}}P_N u_0\|^2_{L^2}\right)^{\frac{1}{2}} + C\left(\sum_N\||\nabla|^{-\frac{2}{q}-\frac{2}{a}}P_N\mathbf{1}_I h\|^2_{L^{a'}(\mathbb{R},L^{b'})}\right)^{\frac{1}{2}}\\ &\le C\|u_0\|_{\dot{H}^{-\frac{2}{q}}} + \||\nabla|^{-\frac{2}{q}-\frac{2}{a}}h\|_{L^{a'}(I,L^{b'})}\end{aligned}$$

where $\mathbb{1}_I$ stands for the characteristic function of I, and summation in the above inequalities occurs over all dyadic integers N. This ends the proof of (4.3.11) and of Proposition 4.3.1. □

A direct consequence of (4.3.11) and Sobolev's inequality is that when $n \geq 5$, for any B-admissible pair (q, r), if $u \in C(I, H^{-4})$ solves (4.3.1) with $u_0 \in \dot{H}^2$ and $h \in N(I)$, then $u \in C(I, \dot{H}^2) \cap M(I)$, where $N(I)$ and $M(I)$ are defined in (4.2.3), and

$$\|\Delta u\|_{L^q(I,L^r)} \leq C\left(\|\Delta u_0\|_{L^2} + \|\nabla h\|_{L^2(I,L^{\frac{2n}{n+2}})}\right). \tag{4.3.19}$$

A key feature of (4.3.19) is that the second derivative of u in the left hand side of (4.3.19) is estimated using only one derivative of the forcing term h. Note that for classical second order Schrödinger equations, estimates like (4.3.11) and (4.3.19) do not hold true as they would violate Galilean invariance.

Another estimate we need when discussing scattering in Section 4.9 is stated as follows.

Proposition 4.3.2. *Let $\varepsilon = -1$ or $\varepsilon = 0$. Let also (a, b) and (q, r) be S-admissible pairs, $s \geq 0$, and $h \in L^{a'}(\mathbb{R}, \dot{H}^{s-\frac{2}{a},b'})$. Then*

$$\|\int_{\mathbb{R}} e^{-it(\Delta^2+\varepsilon\Delta)}h(t)dt\|_{\dot{H}^s} \leq C\||\nabla|^{s-\frac{2}{a}}h\|_{L^{a'}(\mathbb{R},L^{b'})}, \tag{4.3.20}$$

where $C > 0$ depends only on the dimension, and a', b' are the conjugate exponents of a and b.

Proof. Again, we may assume $s = 0$. For N a dyadic number, we let $\mathcal{V}_N(t)$ be as in (4.3.13). Applying the result in Keel and Tao [16], thanks to (4.3.14) and (4.3.16), we get that

$$\begin{aligned}\|\int_{\mathbb{R}} Q_N e^{-is(\Delta^2+\varepsilon\Delta)}h(s)ds\|_{L^2} &= \|\int_{\mathbb{R}} \mathcal{V}_N(s)^* d_N h(s)ds\|_{L^2} \\ &\leq C\|d_N h\|_{L^{a'}(\mathbb{R},L^{b'})} \\ &= N^{\frac{n}{2}-\frac{n}{b'}}\|h\|_{L^{a'}(\mathbb{R},L^{b'})}.\end{aligned} \tag{4.3.21}$$

Applying estimates (4.3.21) to $P_N h$, using (4.2.7), and the Littlewood-Paley theorem, we finally get

$$\begin{aligned}\|\int_{\mathbb{R}} e^{-is(\Delta^2+\varepsilon\Delta)}h(s)ds\|_{L^2}^2 &\leq C\sum_N \|P_N \int_{\mathbb{R}} e^{-is(\Delta^2+\varepsilon\Delta)}h(s)ds\|_{L^2}^2 \\ &\leq C\sum_N N^{-\frac{2}{a}}\|P_N h\|_{L^{a'}(\mathbb{R},L^{b'})}^2 \\ &\leq C\||\nabla|^{-\frac{2}{a}}h\|_{L^{a'}(\mathbb{R},L^{b'})}^2,\end{aligned} \tag{4.3.22}$$

where the sum in (4.3.22) is over all dyadic integers N. This ends the proof of Proposition 4.3.2. □

4.4 Local and Global Existence in the subcritical case

For the reader's convenience we very briefly discuss the local and global theory for (4.1.2) in the subcritical regime where $1 < p < 2^\sharp$ if $n \geq 5$, and $1 < p < \infty$ if $n \leq 4$. Most of the results in this section go back to Fibich, Ilan, and Papanicolaou [5]. They can be seen as a direct consequence of the straightforward Strichartz estimates (4.3.10) of Proposition 4.3.1. As a preliminary remark, it can be noted that for $\alpha > 0$, the scaling

$$u(t,x) \to \alpha^{\frac{4}{p-1}} u(\alpha^4 t, \alpha x) \tag{4.4.1}$$

preserves (4.1.2) when $\varepsilon = 0$, and that letting $\alpha = |\varepsilon|^{-1/2}$, (4.4.1) transforms a solution of (4.1.2) with $\varepsilon \neq 0$ into a solution of (4.1.2) with $|\varepsilon| = 1$. In particular, we may assume in what follows that $\varepsilon \in \{-1, 0, 1\}$. As another easy remark, it can be noted that equations like (4.1.1) also enjoy time reversal symmetry, and time translation symmetry. Unless otherwise stated, λ and u_0 in this section are arbitrary.

Proposition 4.4.1. *Given any initial data $u_0 \in H^2$, any $p \in (1, 2^\sharp - 1)$ when $n \geq 5$, and any $p > 1$ when $n \leq 4$, there exists $T > 0$ and a unique solution $u \in C([0,T], H^2)$ of (4.1.2) such that $u(0) = u_0$. The solution has conserved mass and energy in the sense that*

$$M(u(t)) = M(u_0) \ \text{ and } \ E(u(t)) = E(u_0) \tag{4.4.2}$$

for all $t \in [0,T]$, where the mass M is defined in (4.2.1), and the energy E is defined in (4.2.2). Besides, if T^* is the maximal time of existence of u, then

$$\lim_{t \to T^*} \|u(t)\|_{H^2} = +\infty \tag{4.4.3}$$

when $T^* < +\infty$, and the solution map $u_0 \to u$ is continuous in the sense that for any $T \in (0, T^*)$, if $u_0^k \in H^2$ is a sequence converging in H^2 to u_0, and if u^k denotes the solution of (4.1.2) with initial data u_0^k, then u^k is defined on $[0,T]$ for sufficiently large k and $u^k \to u$ in $C([0,T], H^2)$.

Proof. Proposition 4.4.1 follows from an easy adaptation of the standard proof for second order Schrödinger equations, as developed, for instance, in Cazenave [3], once the Strichartz estimates (4.3.10) have been established. □

A direct consequence of Proposition 4.4.1 is as follows.

Corollary 4.4.1. *Let $p \in (1, 2^\sharp - 1)$ when $n \geq 5$, $p > 1$ when $n \leq 4$, $u_0 \in H^2$, and u be the solution of (4.1.2) with initial data u_0. Then u can be extended to a solution on the whole of $\mathbb{R}$ in the following cases:*

(a) $\lambda \geq 0$,

(b) $\lambda < 0$ *and* $p < 1 + \frac{8}{n}$,

(c) $\lambda < 0$, $p = 1 + \frac{8}{n}$ *and u_0 is sufficiently small in L^2,*

(d) $\lambda < 0$ *and u_0 is sufficiently small in H^2.*

In particular, when $n \geq 5$, for any $u_0 \in H^2$ there exists a unique global solution $u \in C(\mathbb{R}, H^2)$ of (4.1.2) *such that $u(0) = u_0$ if $\lambda > 0$ and $p \in (1, 2^\sharp - 1)$.*

Proof. The first assertion directly follows from conservation of energy. The second and third assertions follow from Gagliardo-Nirenberg's inequalities and conservation of mass and energy. Indeed,

$$\begin{aligned}\int_{\mathbb{R}^n} |\Delta u(t,x)|^2 dx &\leq E(u_0) + \varepsilon \int_{\mathbb{R}^n} |\nabla u(t,x)|^2 dx - \frac{2\lambda}{p+1} \int_{\mathbb{R}^n} |u(t,x)|^{p+1} dx \\ &\leq E(u_0) + \|u_0\|_{L^2} \|\Delta u\|_{L^2} + C \|u_0\|_{L^2}^{p+1-\frac{n(p-1)}{4}} \|\Delta u\|_{L^2}^{\frac{n(p-1)}{4}}.\end{aligned}$$

and, when $p < 1 + \frac{8}{n}$, or when $p = 1 + \frac{8}{n}$ and $\|u_0\|_{L^2}$ is sufficiently small, we get a contradiction with (4.4.3) if $T^* < +\infty$. The last assertion follows from a Payne and Sattinger [21] type argument similar to the one developed in the proof of Corollary 4.5.1 in Section 4.5. □

Following the strategy in Lin and Strauss [19], see also Cazenave [3], we can prove that scattering holds true in the whole energy space H^2 when $\lambda \geq 0$, $n \geq 5$, $\varepsilon \leq 0$, and $1 + \frac{8}{n} < p < 2^\sharp - 1$. We also refer to Guo and Wang [9] and Wang [30] for similar considerations in the small norm setting.

4.5 Local existence in the critical case

We briefly develop the local theory for (4.1.2) in the energy critical case. Here $p = 2^\sharp - 1$ and $n \geq 5$. As in the preceding section, λ and u_0 can be arbitrary. If $u \in C(I, H^2)$ is a solution of the critical equation (4.1.2) with initial data u_0, then

$$u(t) = e^{it(\Delta^2 + \varepsilon\Delta)} u_0 + i\lambda \int_0^t e^{i(t-s)(\Delta^2+\varepsilon\Delta)} |u(s)|^{\frac{8}{n-4}} u(s) ds. \tag{4.5.1}$$

Conversely, if $u_0 \in H^2$, and $u \in W(I)$ solves (4.5.1), then $|u|^{\frac{8}{n-4}} u \in N(I)$, where $W(I)$ and $N(I)$ are defined in Section 4.2, $u \in C(I, H^2)$ by the Strichartz estimates (4.3.10) and (4.3.19), and u is a solution of the critical (4.1.2) with initial data u_0. Equations like (4.5.1) are often referred to as Duhamel's formula. Here again, because of the scaling invariance (4.4.1) we may assume that $\varepsilon \in \{-1, 0, 1\}$. Local existence is settled by Proposition 4.5.1. Stability, and uniform continuity of the map $u_0 \mapsto u$, are discussed in the following section.

Proposition 4.5.1. *Let $n \geq 5$ and $p = 2^\sharp - 1$. There exists $\delta > 0$ such that for any initial data $u_0 \in H^2$, and any interval $I = [0, T]$ with $T \leq 1$ when $\varepsilon = 1$, if*

$$\|e^{it(\Delta^2+\varepsilon\Delta)} u_0\|_{W(I)} < \delta, \tag{4.5.2}$$

where $W(I)$ is as in (4.2.3) *in Section 4.2, then there exists a unique solution $u \in C(I, H^2)$ of* (4.1.2) *with initial data u_0. This solution has conserved mass and energy in the sense of* (4.4.2)*, and satisfies $u \in M(I) \cap L^{\frac{2(n+4)}{n}}(I \times \mathbb{R}^n)$. Moreover,*

$$\begin{aligned}&\|u\|_{W(I)} \leq 2\delta \text{ , and} \\ &\|u\|_{M(I)} + \|u\|_{L^\infty(I,H^2)} \leq C \left(\|u_0\|_{H^2} + \delta^{\frac{n+4}{n-4}} \right)\end{aligned} \tag{4.5.3}$$

for some $C > 0$ independent of u_0. Besides, the solution depends continuously on the initial data in the sense that there exists δ_0, depending on δ, such that, for any $\delta_1 \in (0, \delta_0)$, if $\|v_0 - u_0\|_{H^2} \le \delta_1$, and if we let v be the local solution of (4.1.2) *with initial data v_0, then v is defined on I and*

$$\|u - v\|_{L^q(I,L^r)} \le C\delta_1, \tag{4.5.4}$$

for any B-admissible pair (q, r) in the sense of (4.3.9)*, where $C > 0$ is independent of u_0 and v_0.*

Proof. The proposition follows from a contraction mapping argument. For $u \in W(I)$, we let $\Phi_{u_0}(u)$ be given by the right hand side in (4.5.1). Thanks to the Strichartz estimates (4.3.10) and (4.3.19) stated after the proof of Proposition 4.3.1, Φ_{u_0} is a contraction on the set

$$X_{M,\delta} = \{v \in M(I); \|v\|_{W(I)} \le 2\delta, \|u\|_{L^{\frac{2(n+4)}{n}}(I,L^{\frac{2(n+4)}{n}})} \le 2M\}$$

for $M = C\|u_0\|_{L^2}$, and $\delta > 0$ sufficiently small, where we equip $X_{M,\delta}$ with the $L^{\frac{2(n+4)}{n}}(I, L^{\frac{2(n+4)}{n}})$ norm. The contraction mapping theorem gives a unique solution u in $X_{M,\delta}$, and a standard variant of the argument gives (4.5.4). The Strichartz estimates (4.3.19) give that $u \in M(I) \cap L^\infty(I, H^2)$ and (4.5.3). A straightforward adaptation of Cazenave [3, Chapter 4] gives uniqueness in $C([0, T], \dot{H}^2)$ and conservation of mass and energy. □

As a remark, for any $u_0 \in H^2$, (4.5.2) holds true for $T > 0$ sufficiently small. Global existence for small data in the energy space, as mentioned in the remark after Theorem 12, is a direct consequence of Proposition 4.5.1.

Corollary 4.5.1. *Let $n \ge 5$ and $p = 2^\sharp - 1$. There exists $\epsilon_0 > 0$ such that for any $u_0 \in H^2$ satisfying $\mathcal{E}(u_0) \le \epsilon_0$, where $\mathcal{E}$ is as in* (4.2.4)*, equation* (4.1.2) *possesses a unique solution $u \in C(\mathbb{R}, H^2)$ with initial data u_0.*

Proof. By the Strichartz estimates (4.3.19), we see that if u exists on $[0, t_0]$, and if the $\dot{H}^2$-norm of $u(t_0)$ is sufficiently small, then we can use (4.5.2) to extend u on $[t_0, t_0 + 1]$. Hence, in order to prove global existence, it suffices to prove that $\|u(t)\|_{\dot{H}^2}$ remains small on the whole interval of existence of u. We prove this now. Let $t > 0$ be such that u is defined on $[0, t]$. By conservation of energy and Sobolev's inequality we can write that

$$E(u(t)) = E(u_0) \le C\left(\mathcal{E}(u_0) + \mathcal{E}(u_0)^{2^\sharp/2}\right). \tag{4.5.5}$$

When $\lambda > 0$, global existence follows from (4.5.5) since $\|u(t)\|^2_{\dot{H}^2} \le E(u(t))$. When $\lambda < 0$, we write with conservation of energy and Sobolev's inequality that

$$\begin{aligned} \|u(t)\|^2_{\dot{H}^2} &\le 2E(u_0) + |\varepsilon|\|u(t)\|_{L^2}\|u\|_{\dot{H}^2} + |\lambda|\frac{n-4}{n}\|u(t)\|^{2^\sharp}_{L^{2^\sharp}} \\ &\le C\left(\mathcal{E}(u_0) + \mathcal{E}(u_0)^{2^\sharp/2}\right) + \mathcal{E}(u_0)^{\frac{1}{2}}\|u\|_{\dot{H}^2} + C\|u(t)\|^{2^\sharp}_{\dot{H}^2} \\ &\le C\left(\mathcal{E}(u_0) + \mathcal{E}(u_0)^{2^\sharp/2}\right) + \mathcal{E}(u_0) + \frac{1}{4}\|u\|^2_{\dot{H}^2} + C\|u(t)\|^{2^\sharp}_{\dot{H}^2}. \end{aligned} \tag{4.5.6}$$

Here again it follows from (4.5.6) that if $\mathcal{E}(u_0)$ is sufficiently small, then u stays small in the $\dot{H}^2$-norm. □

In order to end the section we now discuss a useful criterion for global existence.

Proposition 4.5.2. *Let $n \geq 5$ and $p = 2^\sharp - 1$. Let $u \in C([0,T), H^2)$ be a solution of (4.1.2) such that $\|u\|_{Z([0,T])} < +\infty$. Then there exists $K = K(\|u_0\|_{H^2}, \|u\|_{Z([0,T])})$, respectively $K = K(T, \|u_0\|_{H^2}, \|u\|_{Z([0,T])})$ when $\varepsilon > 0$, such that*

$$\|u\|_{L^{\frac{2(n+4)}{n}}([0,T],L^{\frac{2(n+4)}{n}})} + \|u\|_{L^\infty([0,T],\dot{H}^2)} + \|u\|_{M([0,T])} \leq K, \tag{4.5.7}$$

and u can be extended to a solution $\tilde{u} \in C([0,T'), H^2)$ of (4.1.2) for some $T' > T$.

Proof. Let $\eta > 0$ be small. Let also $B = \|u\|_{Z([0,T])}$. For $x \geq 0$ we let $[x]$ be the largest integer not exceeding x. If $\varepsilon \leq 0$, we split $[0,T]$ into

$$N = [(B/\eta)^{\frac{2(n+4)}{n-4}}] + 1$$

intervals I_j, $j = 1 \dots N$, such that for $1 \leq j \leq N-1$, $\|u\|_{Z(I_j)} = \eta$, and $\|u\|_{Z(I_N)} \leq \eta$. If $\varepsilon = 1$, we split $[0,T]$ into N intervals I_j, $j = 1 \dots N$, for which $\|u\|_{Z(I_j)} \leq \eta$ and $|I_j| \leq 1$, one of these inequalities being an equality if $j < N$. Then,

$$N \leq |T| + 3 + [(B/\eta)^{\frac{2(n+4)}{n-4}}].$$

Applying the Strichartz estimates (4.3.19) in $I_j = [t_j, t_{j+1}]$, we get, for $t \in I_j$,

$$\begin{aligned}
\|u\|_{M([t_j,t])} &\leq C\|u(t_j)\|_{\dot{H}^2} + C\||u|^{\frac{8}{n-4}}u\|_{N([t_j,t])} \\
&\leq C\|u(t_j)\|_{\dot{H}^2} + C\|u\|^{\frac{8}{n-4}}_{Z(I_j)}\|u\|_{M([t_j,t])} \\
&\leq C\|u(t_j)\|_{\dot{H}^2} + C\eta^{\frac{8}{n-4}}\|u\|_{M([t_j,t])},
\end{aligned} \tag{4.5.8}$$

where $C > 0$ depends only on n and λ. Applying the Strichartz estimates (4.3.10) in I_j, and conservation of mass, we get that

$$\begin{aligned}
\|u\|_{L^{\frac{2(n+4)}{n}}([t_j,t],L^{\frac{2(n+4)}{n}})} &\leq C\|u(t_j)\|_{L^2} + C\||u|^{\frac{8}{n-4}}u\|_{L^{\frac{2(n+4)}{n+8}}([t_j,t]\times\mathbb{R}^n)} \\
&\leq C\|u_0\|_{L^2} + C\|u\|^{\frac{8}{n-4}}_{Z([t_j,t])}\|u\|_{L^{\frac{2(n+4)}{n}}([t_j,t],L^{\frac{2(n+4)}{n}})} \\
&\leq C\|u_0\|_{L^2} + C\eta^{\frac{8}{n-4}}\|u\|_{L^{\frac{2(n+4)}{n}}([t_j,t],L^{\frac{2(n+4)}{n}})}.
\end{aligned} \tag{4.5.9}$$

If η is sufficiently small, (4.5.9) implies that

$$\|u\|_{L^{\frac{2(n+4)}{n}}(I_j,L^{\frac{2(n+4)}{n}})} \leq C\|u_0\|_{L^2},$$

while (4.5.8) implies that $\|u\|_{M(I_j)} \leq 2C\|u(t_j)\|_{\dot{H}^2}$. Applying again the Strichartz estimates (4.3.19) this gives $\|u\|_{L^\infty(I_j,\dot{H}^2)} \leq 2C\|u(t_j)\|_{\dot{H}^2}$. In particular, there holds that $\|u(t_{j+1})\|_{\dot{H}^2} \leq 2C\|u(t_j)\|_{\dot{H}^2}$, and finally,

$$\begin{aligned}
&\|u\|_{L^{\frac{2(n+4)}{n}}([0,T),L^{\frac{2(n+4)}{n}})} \leq N^{\frac{n}{2(n+4)}}C\|u_0\|_{L^2} , \\
&\|u\|_{L^\infty([0,T),\dot{H}^2)} \leq (2C)^N\|u_0\|_{\dot{H}^2} < +\infty , \text{ and} \\
&\|u\|_{M(I_j)} \leq (2C)^N\|u_0\|_{\dot{H}^2}
\end{aligned} \tag{4.5.10}$$

for all j. By (4.5.10) we get that (4.5.7) holds true. Now, let $t_0 \in I_N$. Duhamel's formula (4.5.1) gives that

$$u(t) = e^{i(t-t_0)(\Delta^2+\varepsilon\Delta)}u(t_0) + i\lambda\int_{t_0}^{t} e^{i(t-s)(\Delta^2+\varepsilon\Delta)}|u(s)|^{\frac{8}{n-4}}u(s)ds \qquad (4.5.11)$$

for all t, and (4.5.11), Sobolev's inequality, and the Strichartz estimates (4.3.19) give

$$\begin{aligned}\|e^{i(t-t_0)(\Delta^2+\varepsilon\Delta)}u(t_0)\|_{W([t_0,T])} &\le \|u\|_{W([t_0,T])} + C\||u|^{\frac{8}{n-4}}u\|_{N([t_0,T])} \\ &\le \|u\|_{W([t_0,T])} + C\|u\|^{\frac{n+4}{n-4}}_{W([t_0,T])}.\end{aligned} \qquad (4.5.12)$$

Since the $W([0,T])$-norm of u is finite, dominated convergence ensures that the $W([t_0,T])$-norm of u can be made arbitrarily small as $t_0 \to T$, and (4.5.12) shows that the $W([t_0,T])$-norm of the free propagator

$$t \mapsto e^{i(t-t_0)(\Delta^2+\varepsilon\Delta)}u(t_0)$$

is like $o(1)$ as t_0 tends to T. In particular, we can find $t_1 \in (0,T)$ and $T' > T$ such that $u(t_1) \in H^2$ and

$$\|e^{i(t-t_1)(\Delta^2+\varepsilon\Delta)}u(t_1)\|_{W([t_1,T'])} \le \delta. \qquad (4.5.13)$$

Now, it follows from (4.5.13) and Proposition 4.5.1 that there exists a nonlinear solution $v \in C([t_1,T'],H^2)$ such that v solves (4.1.2) with $p = 2^\sharp - 1$ and $v(t_1) = u(t_1)$. By uniqueness, $u = v$ in $[t_1,T)$ and u can be extended in $[0,T']$. This ends the proof of Proposition 4.5.2. □

As a direct consequence of Proposition 4.5.2, if T^* is the maximal time of existence of u, and $T^* < +\infty$, then, necessarily, $\|u\|_{Z([0,T^*])} = +\infty$.

4.6 Stability in the critical case

We briefly discuss stability in the energy critical case of (4.1.2) following the approach developed by Tao and Visan [28] in the case of the energy critical second order Schrödinger equation. Stability is of importance for physical considerations if one keeps in mind that equations like (4.1.2) are often mathematical approximations of more physically relevant equations, as pointed out in Fibich, Ilan, and Papanicolaou [5]. Stability, in its global version, can be stated as follows. As in the preceding sections, λ and u_0 can be made arbitrary, and we may assume that $\varepsilon \in \{-1,0,1\}$.

Proposition 4.6.1. *Let $n \ge 5$ and $p = 2^\sharp - 1$. Let $I \subset \mathbb{R}$ be a compact time interval such that $0 \in I$, and $\tilde{u}$ be an approximate solution of (4.1.2) in the sense that*

$$i\partial_t\tilde{u} + \Delta^2\tilde{u} + \varepsilon\Delta\tilde{u} + \lambda|\tilde{u}|^{\frac{8}{n-4}}\tilde{u} = e \qquad (4.6.1)$$

for some $e \in N(I)$. Assume that $\|\tilde{u}\|_{Z(I)} < +\infty$ and $\|\tilde{u}\|_{L^\infty(I,\dot{H}^2)} < +\infty$. For any $\Lambda > 0$ there exists $\delta_0 > 0$, $\delta_0 = \delta_0(\Lambda, \|\tilde{u}\|_{Z(I)}, \|\tilde{u}\|_{L^\infty(I,\dot{H}^2)})$ if $\varepsilon \le 0$, and

$\delta_0 = \delta_0(\Lambda, \|\tilde{u}\|_{Z(I)}, \|\tilde{u}\|_{L^\infty(I,\dot{H}^2)}, |I|)$ if $\varepsilon > 0$, such that if $\|e\|_{N(I)} \leq \delta$, and $u_0 \in H^2$ satisfies

$$\|\tilde{u}(0) - u_0\|_{\dot{H}^2} \leq \Lambda \text{ and } \|e^{it(\Delta^2+\varepsilon\Delta)}\left(\tilde{u}(0) - u_0\right)\|_{W(I)} \leq \delta \tag{4.6.2}$$

for some $\delta \in (0, \delta_0]$, then there exists $u \in C(I, H^2)$ a solution of (4.1.2) such that $u(0) = u_0$. Moreover, u satisfies

$$\begin{aligned}
&\|u - \tilde{u}\|_{W(I)} \leq C\left(\delta + \delta^{\frac{15}{(n-4)^2}}\right) , \\
&\|u - \tilde{u}\|_{L^q(I,\dot{H}^{2,r})} \leq C\left(\Lambda + \delta + \delta^{\frac{15}{(n-4)^2}}\right) , \text{ and} \\
&\|u\|_{L^q(I,\dot{H}^{2,r})} \leq C
\end{aligned} \tag{4.6.3}$$

for all B-admissible pairs (q, r), where $C = C(\Lambda, \|\tilde{u}\|_{Z(I)}, \|\tilde{u}\|_{L^\infty(I,\dot{H}^2)})$ if $\varepsilon \leq 0$, and $C = C(\Lambda, \|\tilde{u}\|_{Z(I)}, \|\tilde{u}\|_{L^\infty(I,\dot{H}^2)}, |I|)$ if $\varepsilon > 0$, are nondecreasing functions of their arguments.

Letting $e = 0$ in Proposition 4.6.1 provides the uniform continuity of the solution map $u_0 \to u$. On such a statement, recall that by the Strichartz estimates (4.3.19),

$$\|e^{it(\Delta^2+\varepsilon\Delta)}\left(\tilde{u}(0) - u_0\right)\|_{W(I)} \leq C\|\tilde{u}(0) - u_0\|_{\dot{H}^2}.$$

In particular we can take $\delta = C\Lambda$ in (4.6.2) and make δ small when Λ is small. Proposition 4.6.1 is an easy consequence of the Strichartz estimates (4.3.19) when $n \leq 12$, and is more delicate to prove when $n > 12$. We briefly sketch the proof when $n \leq 12$, and refer to Tao and Visan [28] with only very little indications on the proof when $n > 12$. As a first claim, because Proposition 4.6.1 can be localized, we may assume, see Tao and Visan [28] for the argument in the second order case, that $\|\tilde{u}\|_{W(I)} \leq \delta$ and that $|I| \leq 1$ when $\varepsilon > 0$. Let f be given by $f(z) = \lambda|z|^{2^\sharp-2}z$ for $z \in \mathbb{C}$. Assuming that $n \leq 12$, letting $v = u-\tilde{u}$, where u solves (4.1.2) with initial data u_0, and I' be the maximal time interval of existence of u, we can write, using (4.6.1), that

$$i\partial_t v + \Delta^2 v + \varepsilon\Delta v + f(\tilde{u} + v) - f(\tilde{u}) = e \tag{4.6.4}$$

in $I \cap I'$. The Strichartz estimates (4.3.19) then give that for $t \geq 0$ such that $I_t = [0, t] \subset I$,

$$\begin{aligned}
\|v\|_{W(I_t)} &\leq C\|e^{i(t-t_0)(\Delta^2+\varepsilon\Delta)}(\tilde{u}(t_0) - u_0)\|_{W(I_t)} \\
&\quad + C\|f(\tilde{u} + v) - f(\tilde{u})\|_{N(I_t)} + C\|e\|_{N(I_t)} \\
&\leq 2C\delta + C\|\left(f_z(\tilde{u} + v) - f_z(\tilde{u})\right)\nabla\tilde{u}\|_{L^2(I_t,L^{\frac{2n}{n+2}})} \\
&\quad + C\|\left(f_{\bar{z}}(\tilde{u} + v) - f_{\bar{z}}(\tilde{u})\right)\nabla\bar{\tilde{u}}\|_{L^2(I_t,L^{\frac{2n}{n+2}})} \\
&\quad + C\|f_z(\tilde{u} + v)\nabla v\|_{L^2(I_t,L^{\frac{2n}{n+2}})} \\
&\quad + C\|f_{\bar{z}}(\tilde{u} + v)\nabla\bar{v}\|_{L^2(I_t,L^{\frac{2n}{n+2}})} \\
&\leq C\left(\delta + \delta^{\frac{8}{n-4}}\|v\|_{W(I_t)} + \|v\|^{\frac{n+4}{n-4}}_{W(I_t)}\right),
\end{aligned} \tag{4.6.5}$$

where f_z, $f_{\bar z}$ are the usual complex derivatives. Noting that $g(t) = \|v\|_{I_t}$ defines a continuous function such that $g(0) = 0$, and since by (4.6.5),

$$g(t) \le C\delta + C\delta^{\frac{8}{n-4}} g(t) + Cg(t)^{\frac{n+4}{n-4}},$$

we conclude that if $\delta \le \delta_0$ is sufficiently small, depending only on n and λ, then, for all $t \in I \cap I'$, $g(t) \le C\delta$ for some positive constant C. In particular, the $W(I \cap I')$-norm of u is bounded and Proposition 4.5.2 gives that $I \cap I' = I$. Another application of the Strichartz estimates (4.3.19) then gives the control equations (4.6.3). When $n > 12$, the proof of Proposition 4.6.1 becomes very delicate because $\nabla f(u)$ is no longer lipschitz continuous. The solution to this problem, as developed in Tao and Visan [28], is some sort of Exotic Strichartz estimate in order to work with spaces of functions with greater integrability and lesser regularity (in particular, we require that they involve less than $8/(n-4)$ derivatives), but which still remain scale-invariant. To close the argument, we then need a good chain-rule for fractional derivatives as proved in Visan [29]. We briefly discuss the proof now. Let X and Y be the norms defined by

$$\begin{aligned} \|u\|_{X(I)} &= \||\nabla|^{\frac{8n}{n^2-16}} u\|_{L^{\frac{n^2-16}{8}}(I, L^{\frac{2(n+4)}{n}})}, \text{ and} \\ \|u\|_{Y(I)} &= \||\nabla|^{\frac{8n}{n^2-16}} u\|_{L^{\frac{n^2-16}{4(n-2)}}(I, L^{\frac{2(n+4)}{n+8}})}. \end{aligned} \tag{4.6.6}$$

These spaces are involved in the following Exotic Strichartz estimates (4.6.7) which we obtain as a consequence of the result in Foschi [7]. Namely,

$$\| \int_0^t e^{i(t-s)(\Delta^2 + \varepsilon\Delta)} F(s) ds \|_{X(I)} \le C \|F\|_{Y(I)}. \tag{4.6.7}$$

Now, the fact that the Y-norm involves less than $8/(n-4)$ derivative enables us to get the following nonlinear estimate.

Lemma 4.6.1. *Let $n \ge 12$. For any $v \in W(I)$, and any $u \in X(I)$,*

$$\|f_z(v)u\|_{Y(I)} \le C\|v\|_{W(I)}^{\frac{8}{n-4}} \|u\|_{X(I)} \tag{4.6.8}$$

where $C > 0$ depends only on n and λ. A similar estimate holds true for $f_{\bar z}$.

Proof. We use the rule for fractional derivatives of products, see e.g. Kato [15] to estimate

$$\begin{aligned} \||\nabla|^{\frac{8n}{n^2-16}} (f_z(v)u)\|_{L^{\frac{2(n+4)}{n+8}}} &\le C\Big(\||\nabla|^{\frac{8n}{n^2-16}} f_z(v)\|_{L^{\frac{n^2-16}{4(n-2)}}} \|u\|_{L^{\frac{2(n^2-16)}{n^2-4n-16}}} \\ &\quad + \|f_z(v)\|_{L^{\frac{n+4}{4}}} \||\nabla|^{\frac{8n}{n^2-16}} u\|_{L^{\frac{2(n+4)}{n}}} \Big) \\ &\le C\Big(\||\nabla|^{\frac{8n}{n^2-16}} f_z(v)\|_{L^{\frac{n^2-16}{4(n-2)}}} \||\nabla|^{\frac{8n}{n^2-16}} u\|_{L^{\frac{2(n+4)}{n}}} \\ &\quad + \|v\|_{L^{\frac{2(n+4)}{n-4}}}^{\frac{8}{n-4}} \||\nabla|^{\frac{8n}{n^2-16}} u\|_{L^{\frac{2(n+4)}{n}}} \Big). \end{aligned} \tag{4.6.9}$$

Besides, using the chain-rule for fractional derivatives of fractional powers in Visan [29, Appendix A], the boundedness of Riesz transforms, Hölder's and

Sobolev's inequalities, we get

$$\begin{aligned}\||\nabla|^{\frac{8n}{n^2-16}} f_z(v)\|_{L^{\frac{n^2-16}{4(n-2)}}} &\le C\||v|^{\frac{16}{(n+2)(n-4)}}\|_{L^{\frac{(n+2)(n+4)}{8}}} \||\nabla|^{\frac{n+2}{n+4}} v\|^{\frac{8n}{(n+2)(n-4)}}_{L^{\frac{2n(n+4)}{n^2-2n+4}}} \\ &\le C\|v\|^{\frac{16}{(n+2)(n-4)}}_{L^{\frac{2(n+4)}{n-4}}} \|\nabla v\|^{\frac{8n}{(n+2)(n-4)}}_{L^{\frac{2n(n+4)}{n^2-2n+8}}} \\ &\le C\|\nabla v\|^{\frac{8}{n-4}}_{L^{\frac{2n(n+4)}{n^2-2n+8}}} .\end{aligned} \tag{4.6.10}$$

Now, with (4.6.9), (4.6.10), and Sobolev's inequality, we get

$$\||\nabla|^{\frac{8n}{n^2-16}} \left(f_z(v)u\right)\|_{L^{\frac{2(n+4)}{n+8}}} \le C\|\nabla v\|^{\frac{8}{n-4}}_{L^{\frac{2n(n+4)}{n^2-2n+8}}} \||\nabla|^{\frac{8n}{n^2-16}} u\|_{L^{\frac{2(n+4)}{n}}} ,$$

and applying Hölder's inequality, we finally get (4.6.8). This ends the proof of the lemma. □

Noting with (4.6.7) that the X-norm of a solution of the linear equation with zero initial data is controlled by the Y-norm of its forcing term, we get with (4.6.8) that the X-norm of $\tilde{u} - u$ can be controlled in terms of δ and Λ. Once this scale-invariant norm has been controlled, straightforward applications of the Strichartz estimates, as developed in the above mentioned [28], then provide the result.

4.7 Almost conservation Laws

In this section we prove almost conservation of the local mass, and localized Morawetz type estimates. These are important ingredients in the process of proving Theorem 12. As in the preceding sections we may assume that $\varepsilon \in \{-1, 0, 1\}$. First we discuss almost conservation of local mass. Let $\chi \in C_c^\infty(\mathbb{R}^n)$ be a radially symmetrical smooth nonnegative function such that $\chi(r) = 1$ if $r \le 1$, $\chi(r) = 0$ if $r \ge 2$, and $0 \le \chi \le 1$. We define the local mass $M\left(u, B_{x_0}(R)\right)$ over the ball $B_{x_0}(R)$ of a function $u \in L^2$ by

$$M\left(u, B_{x_0}(R)\right) = \int_{\mathbb{R}^n} |u(x)|^2 \chi_R^4\left(x - x_0\right) dx, \tag{4.7.1}$$

where, for ease of exposition, the notation g_R when g is a function stands for $g_R(x) = g(x/R)$. Note that Hölder's and Sobolev's inequalities give that

$$M\left(u, B_{x_0}(R)\right) \le C\|\Delta u\|_{L^2}^2 R^4, \tag{4.7.2}$$

where C depends only on n. Now we claim that the local mass of a solution of (4.1.2) varies slowly in time provided that the radius R is sufficiently large.

Lemma 4.7.1. *Let $p \in (1, 2^\sharp - 1]$ when $n \ge 5$, and $p > 1$ when $n \le 4$. Let $\lambda \in \mathbb{R}$, possibly zero, and $u \in C(I, H^2)$ be a solution of* (4.1.2)*. Then*

$$\left|\partial_t M\left(u(t), B_{x_0}(R)\right)\right| \le C\frac{\mathcal{E}(u)^{\frac{3}{4}}}{R} M\left(u(t), B_{x_0}(R)\right)^{\frac{1}{4}} . \tag{4.7.3}$$

for all $t \in I$, where $C > 0$ does not depend on u and I.

Proof. By translation symmetry, we can suppose $x_0 = 0$. Integrating by parts, using (4.1.2), gives

$$\begin{aligned}\frac{d}{dt}M(u(t),B_0(R)) = &\frac{16}{R}\mathrm{Re}\int_{\mathbb{R}^n} i\Delta u\nabla\overline{u}\left(\nabla\chi\right)_R\chi_R^3dx \\ &+\frac{8}{R^2}\mathrm{Re}\int_{\mathbb{R}^n} i\Delta u\left(\chi_R^3\left(\Delta\chi\right)_R+3\chi_R^2\left(\nabla\chi\right)_R^2\right)\overline{u}dx \\ &-\frac{8\varepsilon}{R}\mathrm{Re}\int_{\mathbb{R}^n} i\nabla u\left(\nabla\chi\right)_R\overline{u}\chi_R^3dx.\end{aligned} \tag{4.7.4}$$

Now we estimate each term in the right hand side of (4.7.4) independently one from another. For the first term in the right hand side of (4.7.4), an application of the Cauchy-Schwartz inequality gives

$$\begin{aligned}&\left|\int_{\mathbb{R}^n}\chi_R^3\Delta u\nabla\overline{u}\left(\nabla\chi\right)_R dx\right| \\ &\le \|\left(\nabla\chi\right)_R\|_{L^\infty}\left(\int_{\mathbb{R}^n}|\Delta u|^2dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^n}|\nabla u|^2\chi_R^6dx\right)^{\frac{1}{2}}.\end{aligned}$$

Integrating by parts,

$$\int_{\mathbb{R}^n}|\nabla u|^2\chi_R^6dx = -\mathrm{Re}\int_{\mathbb{R}^n}\overline{u}\left(\Delta u\chi_R^6+\frac{6}{R}\nabla u\left(\nabla\chi\right)_R\chi_R^5\right)dx$$

and by using Hölder's and Sobolev's inequalities we get that

$$\begin{aligned}&\int_{\mathbb{R}^n}|\nabla u|^2\chi_R^6dx \\ &\le \|\chi_R\|_{L^\infty}^4\left(\int_{\mathbb{R}^n}|u|^2\chi_R^4dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^n}|\Delta u|^2dx\right)^{\frac{1}{2}} \\ &+\frac{6}{R}\left(\int_{\mathbb{R}^n}|u|^2\chi_R^4dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^n}|\nabla u|^{2^*}dx\right)^{\frac{n-2}{2n}}\left(\int_{\mathbb{R}^n}\left(\nabla\chi\right)_R^n\chi_R^{3n}dx\right)^{\frac{1}{n}} \\ &\le C\Big(\|u\|_{\dot{H}^2}M(u(t),B_0(R))^{\frac{1}{2}} \\ &+\frac{1}{R}M(u(t),B_0(R))^{\frac{1}{2}}\|u\|_{\dot{H}^2}R\|\left(\nabla\chi^6\right)_R\|_{L^n}\Big) \\ &\le C\|u\|_{\dot{H}^2}M(u(t),B_0(R))^{\frac{1}{2}}\end{aligned}$$

for some $C > 0$ independent of u. The second term in the right hand side of (4.7.4) is even simpler to estimate. We use the Cauchy-Schwartz inequality and (4.7.2) to get

$$\begin{aligned}&\left|\int_{\mathbb{R}^n}\Delta u\chi_R^2\overline{u}\left(\chi_R\left(\Delta\chi\right)_R+3\left(\nabla\chi\right)_R^2\right)dx\right| \\ &\le C\left(\int_{\mathbb{R}^n}|\Delta u|^2dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^n}|u|^2\chi_R^4dx\right)^{\frac{1}{2}} \\ &\le C\|u\|_{\dot{H}^2}M(u(t),B_0(R))^{\frac{1}{2}} \\ &\le CR\|u\|_{\dot{H}^2}^{\frac{3}{2}}M(u(t),B_0(R))^{\frac{1}{4}},\end{aligned}$$

where again $C > 0$ is independent of u. As for the third term in the right hand side of (4.7.4), we remark that it only has to be considered if $\varepsilon \neq 0$, in which case, $\mathcal{E}$ controls the full norm H^2, and we estimate this third term by writing that

$$\begin{aligned} -2\varepsilon \mathrm{Re} \int_{\mathbb{R}^n} i \left(\frac{4}{R} \bar{u} \chi_R^3 \nabla u \, (\nabla \chi)_R \right) dx &\leq \frac{C}{R} \left(\int_{\mathbb{R}^n} |u|^2 \chi_R^4 dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} |\nabla u|^2 \right)^{\frac{1}{2}} \\ &\leq C \frac{\mathcal{E}(u)^{\frac{3}{4}}}{R} M(u(t), B_0(R))^{\frac{1}{4}}, \end{aligned}$$

where $C > 0$ does not depend on u. Finally, putting all these estimates together, we get (4.7.3). This ends the proof of Lemma 4.7.1. $\square$

Now we discuss localized Morawetz type estimates for (4.1.2).

Proposition 4.7.1. *Let $n \geq 5$, and $p = 2^\sharp - 1$. There exists $C > 0$ such that*

$$\int_I \int_{|x| \leq K|I|^{1/4}} \frac{|u(x)|^{2^\sharp}}{|x|} dx \leq C(K^3 + K^{-1}) \left(\sup_I \hat{\mathcal{E}}(u) \right) |I|^{\frac{3}{4}}. \tag{4.7.5}$$

for all $T > 0$, all solutions $u \in C([0,T], H^2)$ of (4.1.2), all $K > 0$, and all intervals $I \subset [0,T]$ such that $|I| \leq 1/(2K)^4$ when $\varepsilon > 0$, where $\hat{\mathcal{E}}(u) = \mathcal{E}(u) + \mathcal{E}(u)^{2^\sharp/2}$.

Proof. We fix $u_0 \in C_c^\infty(\mathbb{R}^n)$, $h \in C_c^\infty(\mathbb{R}^{n+1})$, and let v solve (4.3.1). We adopt the convention that repeated indices are summed. Also, for f, g two differentiable functions, we let

$$\{f, g\}_p = \mathrm{Re}(f \nabla \bar{g} - g \nabla \bar{f}). \tag{4.7.6}$$

Given a smooth compactly supported real valued function a, we define the Morawetz action centered at 0, M_a^0 by

$$M_a^0(t) = 2 \int_{\mathbb{R}^n} \partial_j a(x) \mathrm{Im}(\bar{v}(t,x) \partial_j v(t,x)) dx. \tag{4.7.7}$$

Integrating by parts, straightforward though lengthy computations that we omit here give that

$$\begin{aligned} \partial_t M_a^0(t) =& 2 \int_{\mathbb{R}^n} \Big(2\partial_j v \partial_k \bar{v} \partial_{jk} \Delta a - \frac{1}{2} \left(\Delta^3 a \right) |v|^2 - 4 \partial_{jk} a \partial_{ik} v \partial_{ij} \bar{v} \\ &+ \Delta^2 a |\nabla v|^2 - \varepsilon \left(2 \partial_{jk} a \partial_j v \partial_k \bar{v} - \frac{1}{2} \Delta^2 a |v|^2 \right) - \partial_j a \{h, v\}_p^j \Big) dx. \end{aligned} \tag{4.7.8}$$

By density, (4.7.8) remains true when $h \in N(I)$, and $v \in C(I, H^2)$. Now we let $u \in C(I, H^2) \cap M(I)$ be a solution of (4.1.2) with $p = 2^\sharp - 1$. In particular, u solves (4.3.1) with

$$h = \lambda |u|^{2^\sharp - 2} u \ , \ h \in N(I).$$

Hence, (4.7.8) holds true for u with h as above. Besides, using (4.7.6), we easily remark that

$$\mathrm{Re} \int_{\mathbb{R}^n} \partial_j a \{h, u\}_p^j dx = \frac{4\lambda}{n} \int_{\mathbb{R}^n} (\Delta a) |u|^{2^\sharp} dx. \tag{4.7.9}$$

Now we let $a(x) =< x >_\delta \chi_R(x)$ in (4.7.8), where

$$< x >_\delta = \left(\delta^2 + |x|^2\right)^{\frac{1}{2}},$$

$R > 0$ is an arbitrary positive real number, and χ_R is as in (4.7.1). We observe that if $\alpha \in \mathbb{N}^n$ is a multi-index, if $R \geq \delta$, and if $R \leq |x| \leq 2R$, then $|D^\alpha a(x)| \leq CR^{1-|\alpha|}$. Consequently, integrating (4.7.8) over I and using (4.7.7) and (4.7.9), we get that

$$\begin{aligned}
&2\int_I \int_{|x|\leq R} \left(\frac{4\sum_i \left(|\nabla \partial_i u|^2 - |\partial_r \partial_i u|^2\right)}{< x >_\delta} + \frac{2(n-1)\left(|\nabla u|^2 - 3|\partial_r u|^2\right)}{< x >_\delta^3} \right) dx \\
&+ 2\int_I \int_{|x|\leq R} \left(\frac{(n-1)(n-3)|\nabla u|^2}{< x >_\delta^3} + \frac{8\lambda(n-1)|u|^{2^\sharp}}{n < x >_\delta} \right) dx + O(\delta) \\
&- \varepsilon \int_I \int_{|x|\leq R} \left(\frac{2\left(|\nabla u|^2 - |\partial_r u|^2\right)}{< x >_\delta} + \frac{(n-1)(n-3)|u|^2}{< x >_\delta^3} \right) dx \\
&\leq C \int_I \int_{R\leq|x|\leq 2R} \left(R^{-3}|\nabla u|^2 + R^{-5}|u|^2 - R^{-1}|\nabla^2 u|^2 + \lambda R^{-1}|u|^{2^\sharp} \right) dx \\
&+ C \int_{|x|\leq 2R} \Big[|u\nabla u|\Big]_{t_1}^{t_2} dx.
\end{aligned} \tag{4.7.10}$$

Letting $\delta \to 0$ in (4.7.10) we get that

$$\begin{aligned}
&2\int_I \int_{|x|\leq R} \left(\frac{4\sum_i \left(|\nabla \partial_i u|^2 - |\partial_r \partial_i u|^2\right)}{|x|} + \frac{2(n-1)\left(|\nabla u|^2 - 3|\partial_r u|^2\right)}{|x|^3} \right) dx \\
&+ 2\int_I \int_{|x|\leq R} \left(\frac{(n-1)(n-3)|\nabla u|^2}{|x|^3} + \frac{8\lambda(n-1)|u|^{2^\sharp}}{n|x|} \right) dx \\
&- \varepsilon \int_I \int_{|x|\leq R} \left(\frac{2\left(|\nabla u|^2 - |\partial_r u|^2\right)}{|x|} + \frac{(n-1)(n-3)|u|^2}{|x|^3} \right) dx \\
&\leq C \int_I \int_{R\leq|x|\leq 2R} \left(R^{-3}|\nabla u|^2 + R^{-5}|u|^2 - R^{-1}|\nabla^2 u|^2 + \lambda R^{-1}|u|^{2^\sharp} \right) dx \\
&+ C \int_{|x|\leq 2R} \Big[|u\nabla u|\Big]_{t_1}^{t_2} dx \\
&\leq C|I|R^{-1} \sup_I \left(\mathcal{E}(u) + \mathcal{E}(u)^{2^\sharp/2} \right) + CR^3 \sup_I \mathcal{E}(u),
\end{aligned} \tag{4.7.11}$$

where C does not depend on I, u, and R. The last inequality in (4.7.11) follows from Hölder's and Sobolev's inequalities and from the fact that, for any $u \in H^2$, the L^2 norm of $\nabla^2 u$ is bounded by some constant times the L^2 norm of Δu. Now we remark that if $u \in H^2(\mathbb{R}^n)$ then, as shown in Levandosky and Strauss [20],

$$\sum_i \left(|\nabla \partial_i u|^2 - |\partial_r \partial_i u|^2\right) \geq \frac{n-1}{|x|^2}|\partial_r u|^2. \tag{4.7.12}$$

By (4.7.12) we see that

$$\int_I \int_{|x|\le R} \frac{4}{|x|} \sum_i \left(|\nabla \partial_i u|^2 - |\partial_r \partial_i u|^2\right) dxdt$$
$$+ \int_I \int_{|x|\le R} \frac{2(n-1)}{|x|^3} \left(|\nabla u|^2 - 3|\partial_r u|^2\right) dxdt \ge 0,$$

and when $\varepsilon \le 0$, letting $R = K|I|^{\frac{1}{4}}$, we get (4.7.5). Now, we turn to the case $\varepsilon = 1$. We assume $R \le 1/2$. The term $(|\nabla u|^2 - |\partial_r u|^2)/|x|$ in (4.7.11) is controlled by the gradient term $|\nabla u|^2/|x|^3$. Independently, when $n \ge 5$, integrating by parts $\int Z \cdot \nabla |u|^2 dx$ for $Z(x) = |x|^{-3}x$, we get that

$$\int_{\mathbb{R}^n} \frac{|u|^2}{|x|^3} dx \le \left(\frac{2}{n-3}\right)^2 \int_{\mathbb{R}^n} \frac{|\partial_r u|^2}{|x|} dx. \tag{4.7.13}$$

Then, using (4.7.13), we see that when $R = K|I|^{\frac{1}{4}} \le 1/2$, the term $|u|^2/|x|^3$ in (4.7.11) is again controlled by the gradient term $|\nabla u|^2/|x|^3$. As a consequence, (4.7.5) also holds true when $\varepsilon = 1$ if $|I| \le (2K)^4$. □

4.8 Global Existence

We prove Theorem 12 in this section. We follow the strategy initiated in Bourgain [2] and developed in Tao [26]. As in the preceding sections we may assume that $\varepsilon \in \{-1, 0, 1\}$. We let H^2_{rad} be the subset of H^2 consisting of radially symmetrical functions. We claim that the following proposition holds true.

Proposition 4.8.1. *Let $n \ge 5$ and $p = 2^\sharp - 1$. Assume $\lambda > 0$. Let $u \in C([t_-, t_+], H^2_{rad})$ be a radially symmetrical solution of (4.1.2) which is such that $\|u\|_{W([t_-,t_+])} < \infty$. Then, there exists $K > 0$ depending only on n, λ, $\mathcal{E} = \sup_t \mathcal{E}(u)$, and $|t_+ - t_-|$ if $\varepsilon > 0$, such that*

$$\|u\|_{Z([t_-,t_+])} \le K. \tag{4.8.1}$$

Besides, in case $\varepsilon \le 0$, one can take $K = \Lambda (1+\mathcal{E})^{\mathcal{E}^\Lambda}$, where $\Lambda >> 1$ is a constant depending only on n and λ.

First we prove that Proposition 4.8.1 implies Theorem 12. Then we prove the proposition.

Proof of Theorem 12. By Corollary 4.4.1 in Section 4.4 me may assume that $n \ge 5$ and $p = 2^\sharp - 1$. Let $u_0 \in H^2_{rad}$ be radially symmetrical. By the Strichartz estimates (4.3.10), there exists $T > 0$ such that (4.5.2) holds true for $I = [0, T]$. Then Proposition 4.5.1 gives that there exists $u \in C(I, H^2)$ which solves (4.1.2) with $p = 2^\sharp - 1$ and such that $u(0) = u_0$. Proposition 4.5.2 allows us to extend u on a maximal interval $[0, T^\star)$ such that $u \in M(I')$ for any compact subinterval $I' \subset [0, T^\star)$, and such that if $T^\star < +\infty$, then the $Z([0, T^\star])$-norm of u is infinite. Besides, it follows from uniqueness that u is spherically symmetrical. Now suppose by contradiction that $T^\star < +\infty$, and let $I' \subset [0, T^\star)$ be a compact subinterval of $[0, T^\star)$. By Proposition 4.8.1, the $Z(I')$-norm of u is bounded by some finite quantity independent of the subinterval I'. Since this contradicts the fact that u must blow-up in the Z-norm, we get that $T^\star = +\infty$ and that u is a global solution of (4.1.2). This proves Theorem 12. □

As a remark the bound (4.8.1) has interest in its own. It is of importance when discussing scattering as in Section 4.9 below. Now we prove Proposition 4.8.1 and split the proof into several steps.

Step 4.8.1. *Let $u \in C([t_1,t_2],H^2_{rad})$ be a radially symmetrical solution of (4.5.1) on $I=[t_1,t_2]$. If $\varepsilon=1$, assume also that $|I|\le 1$. There exists $\eta_0>0$ depending only on n and λ such that if*

$$\frac{1}{4}\eta \le \|u\|_{W(I)} \le \eta \tag{4.8.2}$$

for some $0<\eta\le\eta_0$, then

$$\|u_k\|_{W(I)} \ge \frac{1}{8}\eta, \tag{4.8.3}$$

where $u_k = e^{i(t-t_k)(\Delta^2+\varepsilon\Delta)}u(t_k)$ for $k=1,2$.

Proof of Step 4.8.1. We prove this for u_1, the proof for u_2 being similar. Using Duhamel's formula and (4.3.19), we write

$$\|u_k\|_{W(I)} \ge \|u\|_{W(I)} - C\||u|^{\frac{8}{n-4}}u\|_{N(I)} \ge \frac{1}{4}\eta - C\eta^{\frac{n+4}{n-4}}. \tag{4.8.4}$$

Noting that (4.8.4) gives (4.8.3) provided that $\eta\le\eta_0$ is sufficiently small, this proves Step 4.8.1. □

¿From now on we consider $u\in C([t_-,t_+],H^2_{rad})$ a radially symmetrical solution of (4.5.1) with $\lambda>0$. Besides, in case $\varepsilon=1$, we also assume that $|t_+-t_-|\le 1$. By energy and mass conservation, we have

$$\mathcal{E} = \sup_t \mathcal{E}(u(t)) \le E(u_0) + M(u_0).$$

Moreover, by Proposition 4.5.1 and the Strichartz estimates (4.3.10), we know that there exists $\epsilon_0>0$ such that (4.8.1) holds true if there exists a time t such that $\mathcal{E}(u(t))<\epsilon_0$. Without loss of generality we may then assume that the energy is not too small in the sense that, for any $t\in[t_-,t_+]$, $\mathcal{E}(u(t))\ge\epsilon_0$ for some $\epsilon_0>0$. Let $\eta>0$ be small. We split $[t_-,t_+]$ into N disjoint intervals, $(I_j)_{1\le j\le N}$ such that (4.8.2) holds true on each interval. We let

$$u_\pm(t) = e^{i(t-t_\pm)(\Delta^2+\varepsilon\Delta)}u(t_\pm),$$

and, following the terminology in Tao [26], we call an interval I_j exceptional if one of the following conditions holds true:

$$\begin{aligned} &\|u_+\|_{W(I_j)} > \eta^{K_2}, \text{ or} \\ &\|u_-\|_{W(I_j)} > \eta^{K_2}, \end{aligned} \tag{4.8.5}$$

where $K_2=24n^2$. An interval is said to be unexceptional if it is not exceptional. Using the Strichartz estimates (4.3.10) and Sobolev's inequality, we get an upper bound for the number N_e of exceptional intervals. Namely

$$N_e \le C\left(\|u_0\|_{\dot H^2}\eta^{-K_2}\right)^{\frac{2(n+4)}{n-4}} + 1. \tag{4.8.6}$$

If all intervals are exceptional, (4.8.2) and (4.8.6) give the bound (4.8.1), and this proves the proposition. ¿From now on we assume that there exist unexceptional intervals. A consequence of Lemma 4.8.1 is that the extremal intervals I_1 and I_N are always exceptional, provided that η is sufficiently small. The next step exhibits concentration when dealing with unexceptional intervals.

Step 4.8.2. *Let $u \in C([t_-, t_+], H^2_{rad})$ be a radially symmetrical solution of (4.5.1) and let $I = [t_0, t_1]$ be an unexceptional interval for u such that $|I| \leq 1$ if $\varepsilon = 1$. Then there exists $x_0 \in \mathbb{R}^n$ such that for any $t \in I$,*

$$M\left(u(t), B_{x_0}(2\eta^{-K_1}|I|^{\frac{1}{4}})\right) \geq C\eta^{K_1}\mathcal{E}^{-\frac{n+2}{2}}|I|, \tag{4.8.7}$$

where $K_1 = n^2+6n+4$, $C > 0$ is independent of I, x_0 and u, and we assume that η is sufficiently small in the sense that $\eta < \mathcal{E}^{-5n}\eta_1$ for some $\eta_1 > 0$ depending only on n and λ.

Proof of Step 4.8.2. We consider $I^1 = [t_0, \frac{t_0+t_1}{2}]$ and $I^2 = [\frac{t_0+t_1}{2}, t_1]$. By time reversal and time translation symmetries, and by (4.8.2), we can assume that

$$\|u\|_{W(I^2)} \geq \frac{1}{4}\eta. \tag{4.8.8}$$

Besides, by Duhamel's formula, we get that for any $t \in I^2$,

$$\begin{aligned} u(t) =& u_-(t) + i\lambda\int_{t_-}^{t_0} e^{i(t-s)(\Delta^2+\varepsilon\Delta)}|u(s)|^{\frac{8}{n-4}}u(s)ds \\ &+ i\lambda\int_{t_0}^{t} e^{i(t-s)(\Delta^2+\varepsilon\Delta)}|u(s)|^{\frac{8}{n-4}}u(s)ds. \end{aligned} \tag{4.8.9}$$

Since I is unexceptional, the first term in the right hand side of (4.8.9) is small in the W-Norm. As for the third term in the righthand side of (4.8.9), we use Sobolev's inequality and the Strichartz estimates (4.3.19) to write that

$$\begin{aligned} &\|\int_{t_0}^{t} e^{i(t-s)(\Delta^2+\varepsilon\Delta)}|u(s)|^{\frac{8}{n-4}}u(s)ds\|_{W(I)} \\ &\leq C\|\int_{t_0}^{t} e^{i(t-s)(\Delta^2+\varepsilon\Delta)}|u(s)|^{\frac{8}{n-4}}u(s)ds\|_{M(I)} \\ &\leq C\||u|^{\frac{8}{n-4}}u\|_{N(I)} \leq C\|u\|_{W(I)}^{\frac{n+4}{n-4}} \leq C\eta^{\frac{n+4}{n-4}}, \end{aligned} \tag{4.8.10}$$

where $C > 0$ depends only on n. Hence, if we define $v(t)$ for $t \in I$ by

$$v(t) = \int_{t_-}^{t_0} e^{i(t-s)(\Delta^2+\varepsilon\Delta)}|u(s)|^{\frac{8}{n-4}}u(s)ds, \tag{4.8.11}$$

we get from (4.8.5) and (4.8.8)–(4.8.11) that, if η is sufficiently small, then

$$\|v\|_{W(I^2)} \geq \frac{1}{4}\eta - \eta^{K_2} - C\eta^{\frac{n+4}{n-4}} \geq \frac{1}{8}\eta. \tag{4.8.12}$$

For later use, we remark that, by (4.8.11), v satisfies the linear equation

$$i\partial_t v + \Delta^2 v + \varepsilon\Delta v = 0, \tag{4.8.13}$$

and we get that

$$\|v\|_{L^\infty(I,\dot{H}^2)} \le \|v(t_0)\|_{\dot{H}^2} \le \lambda^{-1}\left(\|u(t_0)\|_{\dot{H}^2} + \|u(t_-)\|_{\dot{H}^2}\right) \le 2\lambda^{-1}\mathcal{E}^{\frac{1}{2}}. \quad (4.8.14)$$

Besides, the Strichartz estimates (4.3.19) give that

$$\begin{aligned}&\|u\|_{M(I)} \le C\|u(t_0)\|_{\dot{H}^2} + C\||u|^{\frac{8}{n-4}}u\|_{N(I)} \le C\mathcal{E}^{\frac{1}{2}} + C\eta^{\frac{n+4}{n-4}}, \text{ and}\\ &\|u_-\|_{M(I)} \le C\|u(t_-)\|_{\dot{H}^2} \le C\mathcal{E}^{\frac{1}{2}},\end{aligned} \quad (4.8.15)$$

where $C>0$ does not depend on u and I. Then (4.8.9), (4.8.10), and (4.8.15) give

$$\|v\|_{M(I^2)} \le C\mathcal{E}^{\frac{1}{2}} + C\mathcal{E}^{\frac{1}{2}} + 2C\eta^{\frac{n+4}{n-4}} \le 3C\mathcal{E}^{\frac{1}{2}}. \quad (4.8.16)$$

Independently, integration by parts and boundedness of Riesz transforms give that there exists $C>0$ independent of I such that for any function $g \in M(I)$,

$$\|g\|_{W(I)} \le C\|g\|_{M(I)}^{\frac{1}{2}}\|g\|_{Z(I)}^{\frac{1}{2}}. \quad (4.8.17)$$

Then, (4.8.12), (4.8.16), and (4.8.17) give that

$$\|v\|_{Z(I^2)} \ge C\eta^2\mathcal{E}^{-\frac{1}{2}}, \quad (4.8.18)$$

where $C>0$ depends only on n and λ. Now, we prove that v enjoys additional regularity. Let us define

$$v_{av}(t,x) = \frac{1}{V}\int_{B_0(2)} v(t,x-ry)\chi(y)dy, \quad (4.8.19)$$

where χ is a bump function as in (4.7.1), $r = \eta^{n+5}|I|^{\frac{1}{4}}$, and $V = \int \chi dx$. We claim that

$$\|v - v_{av}\|_{Z(I^2)} \le C\mathcal{E}^{\frac{n+4}{2(n-4)}}\eta^{\frac{2n+10}{n+4}}. \quad (4.8.20)$$

Now we prove (4.8.20). For $k \in \mathbb{R}^n$, we let τ_k be defined on a function g by $\tau_k g(x) = g(x+k)$. We first assume that $5 \le n \le 12$. Then, by (4.3.7) and Hölder's inequality, letting $f(u) = |u|^{\frac{8}{n-4}}u$, we get that

$$\begin{aligned}&\|v - \tau_k v\|_{L^\infty(I^2,L^{\frac{2(n+4)}{n-4}})}\\ &\le \sup_{t\in I^2}\left\|\int_{t_-}^{t_0} e^{i(t-s)(\Delta^2+\varepsilon\Delta)}\left(f(u(s)) - f(\tau_k u(s))\right)ds\right\|_{L^{\frac{2(n+4)}{n-4}}}\\ &\le C\sup_{t\in I^2}\int_{t_-}^{t_0}|t-s|^{-\frac{2n}{n+4}}\|f(u(s)) - f(\tau_k u(s))\|_{L^{\frac{2(n+4)}{n+12}}}ds\\ &\le C|I|^{-\frac{n-4}{n+4}}\left\|\left(|u|^{\frac{8}{n-4}} + |\tau_k u|^{\frac{8}{n-4}}\right)|u-\tau_k u|\right\|_{L^\infty L^{\frac{2(n+4)}{n+12}}}\\ &\le C|I|^{-\frac{n-4}{n+4}}\|u\|_{L^\infty L^{2^\sharp}}^{\frac{8}{n-4}}\|u-\tau_k u\|_{L^\infty L^{2^\sharp}}^{\frac{12-n}{n+4}}\|u-\tau_k u\|_{L^\infty L^{\frac{2n}{n-2}}}^{\frac{2(n-4)}{n+4}}\\ &\le C|I|^{-\frac{n-4}{n+4}}\mathcal{E}^{\frac{-n^2+24n-16}{2(n^2-16)}}\|u-\tau_k u\|_{L^\infty L^{\frac{2n}{n-2}}}^{\frac{2(n-4)}{n+4}}.\end{aligned}$$

By conservation of energy and Sobolev's inequality,

$$\|u - \tau_k u\|_{L^\infty(I,L^{\frac{2n}{n-2}})} \le |k|\|\nabla u\|_{L^\infty(I,L^{\frac{2n}{n-2}})} \le C|k|\mathcal{E}^{\frac{1}{2}}. \quad (4.8.21)$$

Combining (4.8.19), (4.8.21) and the above computation, we get with Hölder's inequality that

$$
\begin{aligned}
\|v-v_{av}\|_{Z(I^2)} &\le \frac{1}{V}\int_{B_0(2)}\chi(y)\|v-\tau_{ry}v\|_{Z(I^2)}dy\\
&\le C|I|^{\frac{n-4}{2(n+4)}}\int_{B_0(2)}\chi(y)\|v-\tau_{ry}v\|_{L^\infty(I^2,L^{\frac{2(n+4)}{n-4}})}dy\\
&\le C\left(r|I|^{-\frac14}\right)^{\frac{2(n-4)}{n+4}}\mathcal{E}^{\frac{n+4}{2(n-4)}}.
\end{aligned} \tag{4.8.22}
$$

Since $\eta<1$, (4.8.22) gives (4.8.20) when $5\le n\le 12$. When $n\ge 13$, we first estimate the gradient of v with (4.3.7). We have that

$$
\begin{aligned}
\|\nabla v\|_{L^\infty(I^2,L^{\frac{2n}{n-6}})} &\le \int_{t_-}^{t_0}\|\nabla e^{i(t-s)(\Delta^2+\varepsilon\Delta)}f(u(s))\|_{L^\infty([t_-,t_0],L^{\frac{2n}{n-6}})}ds\\
&\le C\int_{t_-}^{t_0}|t-s|^{-\frac32}\|\nabla f(u)\|_{L^\infty([t_-,t_0],L^{\frac{2n}{n+6}})}ds\\
&\le C|I|^{-\frac12}\||u|^{\frac{8}{n-4}}\|_{L^\infty([t_-,t_0],L^{\frac n4})}\|\nabla u\|_{L^\infty([t_-,t_0],L^{\frac{2n}{n-2}})}\\
&\le C|I|^{-\frac12}\mathcal{E}^{\frac{n+4}{2(n-4)}}.
\end{aligned} \tag{4.8.23}
$$

Besides, Sobolev's inequality and (4.8.14) imply that $\|\nabla v\|_{L^\infty(I^2,L^{\frac{2n}{n-2}})}\le C\mathcal{E}^{\frac12}$, and by Hölder's inequality, this gives

$$
\|\nabla v\|_{L^\infty(I^2,L^{2^\sharp})}\le C\mathcal{E}^{\frac{n}{2(n-4)}}|I|^{-\frac14}. \tag{4.8.24}
$$

Then, an application of Sobolev's inequality with (4.8.23) and (4.8.24) yields

$$
\begin{aligned}
\|v-\tau_k v\|_{L^\infty(I^2,L^{\frac{2(n+4)}{n-4}})} &\le \|v-\tau_k v\|_{L^\infty(I^2,L^{\frac{2n}{n-8}})}^{\frac{n-4}{n+4}}\|v-\tau_k v\|_{L^\infty(I^2,L^{2^\sharp})}^{\frac{8}{n+4}}\\
&\le C|I|^{-\frac{n-4}{2(n+4)}}\mathcal{E}^{\frac{n^2+8n-16}{2(n^2-16)}}\left(|k||I|^{-\frac14}\right)^{\frac{8}{n+4}}.
\end{aligned}
$$

Then, by proceeding as in (4.8.22), we finally find

$$
\|v-v_{av}\|_{Z(I^2)}\le C\mathcal{E}^{\frac{n^2+8n-16}{2(n^2-16)}}\left(r|I|^{-\frac14}\right)^{\frac{8}{n+4}}. \tag{4.8.25}
$$

Combining (4.8.22) and (4.8.25), we get (4.8.20) for all $n\ge 5$. Now by (4.8.18) and (4.8.20), we get that if η is sufficiently small, namely $\eta<C\mathcal{E}^{-5n}$, then

$$
\|v_{av}\|_{Z(I^2)}\ge C\eta^2\mathcal{E}^{-\frac12}. \tag{4.8.26}
$$

Independently, (4.8.14) and Sobolev's inequality give the bound

$$
\|v_{av}\|_{L^{2^\sharp}(I^2\times\mathbb{R}^n)}\le C|I|^{\frac{n-4}{2n}}\mathcal{E}^{\frac12}. \tag{4.8.27}
$$

An application of Hölder's inequality with (4.8.26) and (4.8.27) then gives

$$
\|v_{av}\|_{L^\infty(I^2\times\mathbb{R}^n)}\ge C\eta^{\frac{n+4}{2}}|I|^{-\frac{n-4}{8}}\mathcal{E}^{-\frac{n+2}{4}}, \tag{4.8.28}
$$

and we obtain with (4.8.28) that there exists a point $(t_0, x_0) \in I^2 \times \mathbb{R}^n$ such that

$$\left| \int_{B_0(2)} \chi(y) v(t_0, x_0 - ry) dy \right| \geq \frac{1}{2} C \eta^{\frac{n+4}{2}} |I|^{-\frac{n-4}{8}} \mathcal{E}^{-\frac{n+2}{4}}. \tag{4.8.29}$$

It follows from (4.8.29) and Hölder's inequality that

$$M\left(v(t_0), B_{x_0}(2r)\right) \geq C \eta^{K_1} |I| \mathcal{E}^{-\frac{n+2}{2}}, \tag{4.8.30}$$

where $K_1 = n^2 + 6n + 4$. Now, since v satisfies (4.8.13), using (4.7.3), we get that, for any $t \in I$,

$$\partial_t \left(M\left((v(t), B_{x_0}(2\eta^{-K_1} |I|^{\frac{1}{4}}) \right) \right)^{\frac{3}{4}} \leq C \mathcal{E}^{\frac{3}{4}} \eta^{K_1} |I|^{-\frac{1}{4}}. \tag{4.8.31}$$

Integrating (4.8.31) over I, using (4.8.30), we get that for any $t \in I$,

$$M\left(v(t), B_{x_0}(2\eta^{-K_1} |I|^{\frac{1}{4}}) \right) \geq C \eta^{K_1} |I| \mathcal{E}^{-\frac{n+2}{2}}. \tag{4.8.32}$$

In particular, (4.8.32) holds true for $t = t_0$. Independently, since I is unexceptional, by (4.8.5) we can find some time $\tau \in I$ such that

$$\|u_-(\tau)\|_{L^{\frac{2(n+4)}{n-4}}(\mathbb{R}^n)} \leq C \eta^{K_2} |I|^{-\frac{n-4}{2(n+4)}}. \tag{4.8.33}$$

Then, (4.8.33) and Hölder's inequality gives

$$M\left(u_-(\tau), B_{x_0}(2\eta^{-K_1} |I|^{\frac{1}{4}}) \right) \leq C \eta^{8K_1} |I| \tag{4.8.34}$$

and, again, since u_- satisfies (4.3.1) with $h = 0$, we get with (4.8.34) that for any $t \in I$

$$M\left(u_-(t), B_{x_0}(2\eta^{-K_1} |I|^{\frac{1}{4}}) \right) \leq C |I| \eta^{\frac{4}{3} K_1} \mathcal{E}. \tag{4.8.35}$$

Now, by (4.8.9), (4.8.11), and estimates (4.8.30) and (4.8.35), we get that, at time t_0,

$$M\left(u(t_0), B_{x_0}(2\eta^{-K_1} |I|^{\frac{1}{4}}) \right) \geq C |I| \eta^{K_1} \mathcal{E}^{-\frac{n+2}{2}}. \tag{4.8.36}$$

A final application of (4.7.3), using (4.8.36), gives (4.8.7). This ends the proof of Step 4.8.2. □

A consequence of Step 4.8.2 is as follows.

Step 4.8.3. *Let $u \in C([t_-, t_+], H^2_{rad})$ be a radially symmetrical solution of (4.5.1) and let I be an unexceptional interval such that $|I| \leq 1$ if $\varepsilon = 1$. Then*

$$\int_I \int_{B_0(2\eta^{-4K_1} |I|^{\frac{1}{4}})} \frac{|u(t,x)|^{\frac{2n}{n-4}}}{|x|} dx \geq C \eta^{13K_1} \mathcal{E}^{-4n} |I|^{\frac{3}{4}}, \tag{4.8.37}$$

where $K_1 = n^2 + 6n + 4$, and $C > 0$ is a constant depending only on n and λ.

Proof of Step 4.8.3. By Hölder's inequality and (4.8.7) the following bound from below holds true. Namely, that for any $t \in I$,

$$\int_{B_{x_0}(2\eta^{-K_1}|I|^{\frac{1}{4}})} |u(t,x)|^{2^\sharp} dx \geq C\eta^{9K_1}\mathcal{E}^{-4n}. \tag{4.8.38}$$

Now we claim that $|x_0| \leq \eta^{-4K_1}|I|^{\frac{1}{4}}$. Indeed, if this is not the case, then there exists at least $\eta^{-3(n-1)K_1}/4^{n-1}$ disjoint balls which can be obtained by rotating $B_{x_0}(2\eta^{-K_1})$. Using the radial symmetry assumption and (4.8.38) we get that, for any $t \in I$,

$$2^\sharp E(u(t)) \geq \|u(t)\|_{L^{2^\sharp}}^{2^\sharp} \geq \frac{1}{4}\eta^{-3(n-1)K_1} C\eta^{9K_1}\mathcal{E}^{-4n} \geq C\eta^{-2K_1}\mathcal{E}^{-4n}, \tag{4.8.39}$$

and (4.8.39) contradicts $E(u(t)) \leq \mathcal{E}$ if η is sufficiently small. Then, by (4.8.7) we get that for any $t \in I$,

$$M\left(u(t), B_0(2\eta^{-4K_1}|I|^{\frac{1}{4}})\right) \geq C\eta^{K_1}\mathcal{E}^{-\frac{n+2}{2}}|I| \tag{4.8.40}$$

provided that $\eta < \mathcal{E}^{-5n}\eta_1$ where η_1 is sufficiently small depending only on n and λ. Using Hölder's inequality and (4.8.40), we obtain (4.8.37). □

The bound from below in Step 4.8.3 can be combined with the bound stemming from the localized Morawetz estimate (4.7.5), and we then get that the following holds true.

Step 4.8.4. *Let $u \in C([t_-, t_+], H^2_{rad})$ be a radially symmetrical solution of (4.5.1) and let $I = \bigcup_{j_1 \leq j \leq j_2} I_j$ be a collection of consecutive unexceptional intervals for u. In case $\varepsilon = 1$, suppose that $|I| \leq \eta^{16K_1}/256$. Then there exists $j_1 \leq j_0 \leq j_2$ such that*

$$|I_{j_0}| \geq K|I|, \tag{4.8.41}$$

where $K = C\eta^{100K_1}\mathcal{E}^{-20n}$, and $C > 0$ is a constant depending only on n and λ.

Proof of Step 4.8.4. Estimates (4.8.37) and (4.7.5) give that

$$\begin{aligned} C\eta^{13K_1}\mathcal{E}^{-4n} \sum_{j_1 \leq j \leq j_2} |I_j|^{\frac{3}{4}} &\leq \sum_j \int_{I_j}\int_{B_0(2\eta^{-4K_1}|I_j|^{\frac{1}{4}})} \frac{|u|^{2^\sharp}}{|x|} dx \\ &\leq \int_I \int_{B_0(2\eta^{-4K_1}|I|^{\frac{1}{4}})} \frac{|u|^{2^\sharp}}{|x|} dx \\ &\leq C\eta^{-12K_1}\left(\mathcal{E} + \mathcal{E}^{\frac{n}{n-4}}\right)|I|^{\frac{3}{4}}. \end{aligned} \tag{4.8.42}$$

Let $\tilde{K} = C\mathcal{E}^{5n}\eta^{-25K_1}$. We get from (4.8.42) that

$$\left(\max_j |I_j|\right)^{-\frac{1}{4}} \sum_j |I_j| \leq \sum_j |I_j|^{\frac{3}{4}} \leq \tilde{K}|I|^{\frac{3}{4}} \leq \tilde{K}\left(\sum_j |I_j|\right)^{\frac{3}{4}}, \tag{4.8.43}$$

and Step 4.8.4 easily follows from (4.8.43). □

At this point we need a combinatorial argument. Such a result goes back to Bourgain [2] and Tao [26]. In the form we use it here, the proposition is due to Killip, Visan, and Zhang [18].

Proposition 4.8.2. *Let I be an interval which is tiled by finitely many intervals $I_1, \ldots, I_N$. Suppose that for any contiguous family $\{I_j : j \in \mathcal{J}\}$ there exists $j_* \in \mathcal{J}$ so that*

$$|I_{j_*}| \geq K| \cup_{j\in\mathcal{J}} I_j|$$

for some $K > 0$. Then there exists $M \geq \ln(N)/\ln(2K^{-1})$, and distinct indices $j_1, \ldots, j_M$, such that

$$\begin{aligned} &|I_{j_1}| \geq 2|I_{j_2}| \geq \cdots \geq 2^{M-1}|I_{j_M}| \text{ , and} \\ &dist(I_{j_l}, I_{j_k}) \leq (2K)^{-1}|I_{j_l}| \end{aligned} \tag{4.8.44}$$

for all $l < k$.

At last we need the following step.

Step 4.8.5. *Let $u \in C([t_-, t_+], H^2_{rad})$ be a radially symmetrical solution of (4.5.1) and let $I_{j_1}, \ldots, I_{j_M}$ be a disjoint family of unexceptional intervals for u obeying (4.8.44) with $K = C\eta^{100K_1}\mathcal{E}^{-20n}$. In case $\varepsilon = 1$, suppose also that $|I_{j_1}| \leq 1$. Then $M \leq C\mathcal{E}\eta^{-5000n^2}\ln(1/\eta)$, where C depends only on n and λ.*

Proof of Proposition 4.8.5. We let $t_* \in I_{j_M}$. We can combine (4.8.40) with (4.7.3) and (4.8.44) to get that for any $1 \leq k \leq M$, the following mass concentration estimate holds true. Namely that

$$M\left(u(t_*), B_0(2\eta^{-101K_1}|I_{j_k}|^{\frac{1}{4}})\right) \geq C\eta^{K_1}\mathcal{E}^{-\frac{n+2}{2}}|I_{j_k}| \tag{4.8.45}$$

provided that $\eta < C\mathcal{E}^{-21n}$. Besides, (4.7.2) also gives that

$$M\left(u(t_*), B_0(R)\right) \leq C\mathcal{E}R^4. \tag{4.8.46}$$

Let us consider the family of annuli

$$A_k = \{x : \eta^{K_1}|I_{j_k}|^{\frac{1}{4}} \leq |x| \leq 2\eta^{-101K_1}|I_{j_k}|^{\frac{1}{4}}\}.$$

Then (4.8.45) and (4.8.46) give the following bound from below for the mass in A_k. Namely that

$$\int_{A_k} |u(t_*, x)|^2 dx \geq C\eta^{K_1}\mathcal{E}^{-\frac{n+2}{2}}|I_{j_k}|, \tag{4.8.47}$$

and with Hölder's inequality we deduce from (4.8.47) that

$$\int_{A_k} |u(t_*, x)|^{2^\sharp} dx \geq C\eta^{K_3}, \tag{4.8.48}$$

where $K_3 = 5(10^3)n^2$. Now, it only remains to observe that, thanks to (4.8.44), if $l = 405K_1\ln\left(\frac{1}{\eta}\right)/\ln 2$, then the annuli $A_{j_{1+kl}}$ are disjoint for all k. By conservation of energy and (4.8.48) it follows that there are at most $C\mathcal{E}\eta^{-K_3}$ such annuli, and this proves Step 4.8.5. □

Thanks to Steps 4.8.1 to 4.8.5 we are now in position to prove Proposition 4.8.1.

Proof of Proposition 4.8.1. Suppose first that $\varepsilon \le 0$. As mentioned before, see (4.8.6), it is easy to bound the number N_e of exceptional intervals. Now, consider a gap $J = \cup_{j\in\mathcal{J}} I_j$ between two exceptional intervals. The gap J is made exclusively of unexceptional intervals I_j and then, applying (4.8.41), (4.8.44), and Step 4.8.5, we see that

$$|\mathcal{J}| \le C\eta^{-C\eta^{-5000n^2}}$$

if $\eta \le \eta_1 \mathcal{E}^{-5n}$ for η_1 sufficiently small depending only on n and λ. Since there are at most N_e such gaps, we get the desired conclusion. In case $\varepsilon = 1$, we first split $[t_-, t_+]$ into subintervals $\tilde{I}_{k\in\mathcal{K}}$ such that $|\tilde{I}_k| \le 1/256\eta^{48n^2}$. Then, for any k, we can apply on I_k the same strategy as in the case $\varepsilon \le 0$, and we see that the $Z(\tilde{I}_k)$-norm of u is bounded by a constant $C(\mathcal{E}, \eta, n)$. Then, since there are at most $C\eta^{-48n^2}|t_+ - t_-| + 1$ such intervals, we also get Proposition 4.8.1 when $\varepsilon = 1$. □

4.9 Scattering for the critical equation

We briefly discuss scattering in this section and prove that, by standard procedures, an estimate like (4.8.1) implies scattering when $\varepsilon \le 0$. By scaling invariance we may assume $\varepsilon = -1$ or $\varepsilon = 0$. In proposition 4.9.1 we prove that solutions of (4.1.2) with $p = 2^\sharp - 1$ and $\lambda > 0$ converge to a scattering state. We construct the scattering operator in Proposition 4.9.2.

Proposition 4.9.1. *Let $n \ge 5$. Given any $u \in C(\mathbb{R}, H^2_{rad})$ a radially symmetrical solution of* (4.1.2) *with $p = 2^\sharp - 1$, $\lambda > 0$, and $\varepsilon \le 0$, there exists $u^\pm \in H^2_{rad}$ such that*

$$\|u(t) - e^{it(\Delta^2+\varepsilon\Delta)}u^\pm\|_{H^2} \to 0 \tag{4.9.1}$$

as $t \to \pm\infty$. The functions $u^\pm$ are unique, they are determined by (4.9.1)*, and we have that*

$$\begin{aligned} M(u_0) &= M(u^\pm)\ , \text{ and} \\ 2E(u_0) &= \|u^\pm\|^2_{\dot{H}^2} - \varepsilon\|u^\pm\|^2_{\dot{H}^1}. \end{aligned} \tag{4.9.2}$$

This defines two mappings $S_\pm : u(0) \mapsto u^\pm$ from H^2_{rad} into H^2_{rad}, and S_+ and S_- are continuous in H^2_{rad}.

Proof. By time reversal symmetry it suffices to prove (4.9.1) for u^+. From Proposition 4.8.1, we see that u has bounded Z-norm over $\mathbb{R}_+$, and from Proposition 4.5.2, this provides an a priori bound for the $W(\mathbb{R}_+)$-norm of u. Independently, Since $e^{it(\Delta^2+\varepsilon\Delta)}$ is an isometry on H^2, (4.9.1) is equivalent to proving that there exists $u^+ \in H^2$ such that

$$\|e^{-it(\Delta^2+\varepsilon\Delta)}u(t) - u^+\|_{H^2} \to 0 \tag{4.9.3}$$

as $t \to +\infty$. Now we prove that $e^{-it(\Delta^2+\varepsilon\Delta)}u(t)$ satisfies a Cauchy criterion. Duhamel's formula gives that

$$e^{-it_1(\Delta^2+\varepsilon\Delta)}u(t_1) - e^{-it_0(\Delta^2+\varepsilon\Delta)}u(t_0) = i\lambda \int_{t_0}^{t_1} e^{-is(\Delta^2+\varepsilon\Delta)}|u(s)|^{\frac{8}{n-4}}u(s)ds. \tag{4.9.4}$$

By (4.3.20) with $(a,b)=(2,2n/(n-2))$, we see from the finiteness of the $W(\mathbb{R}_+)$-norm of u that the righthand side in (4.9.4) is like $o(1)$ in $\dot{H}^2$ as $t_0,t_1\to+\infty$. In particular, $e^{-it(\Delta^2+\varepsilon\Delta)}u(t)$ satisfies a Cauchy criterion, and there exists $u^+\in H^2$ such that (4.9.3) holds true. We also get that

$$u^+=u_0+i\lambda\int_0^\infty e^{-is(\Delta^2+\varepsilon\Delta)}|u(s)|^{\frac{8}{n-4}}u(s)ds, \tag{4.9.5}$$

and u^+ is unique. Let us now prove that the convergence holds true in the L^2 sense. From (4.5.7) in Proposition 4.5.2 and (4.8.1) in Proposition 4.8.1, we get that $u\in L^{\frac{2(n+4)}{n}}(\mathbb{R}_+\times\mathbb{R}^n)$. An application of the Strichartz estimates (4.3.10) shows that the right hand side in (4.9.4) is like $o(1)$ in L^2 when t_0 and t_1 tend to infinity. In particular the convergence holds true in the L^2 sense. Now we prove (4.9.2). The first equation in (4.9.2) is a direct consequence of conservation of mass and of the convergence in L^2. Concerning the second equation, since $u\in Z(\mathbb{R}_+)$, we can find a sequence $t_k\to+\infty$ such that the $L^{\frac{2(n+4)}{n-4}}$-norm of $u(t_k)$ tends to zero. Combining this with conservation of mass, we get that the $L^{2^\sharp}$-norm of $u(t_k)$ tends to zero as k tends to infinity. Let $\omega(t)=e^{it(\Delta^2+\varepsilon\Delta)}u^+$. Then, we have

$$\begin{aligned}2E(u_0)&=2E(u(t_k))\\&=\|u(t_k)\|^2_{\dot{H}^2}-\varepsilon\|u(t_k)\|^2_{\dot{H}^1}+o(1)\\&=\|\omega(t_k)\|^2_{\dot{H}^2}-\varepsilon\|\omega(t_k)\|^2_{\dot{H}^1}+o(1)\\&=\|u^+\|^2_{\dot{H}^2}-\varepsilon\|u^+\|^2_{\dot{H}^1}+o(1),\end{aligned}$$

and letting $k\to+\infty$ we get that the second equation in (4.9.2) holds true. The continuity in $\dot{H}^2$ of the mapping $u_0\mapsto u^+$ follows from estimate (4.3.20), Proposition 4.6.1, and equation (4.9.5). The continuity in L^2 follows from the Strichartz estimates (4.3.10) and the a priori bound on the Z-norm in Proposition 4.8.1. We may proceed as when proving the control of the $L^{\frac{2(n+4)}{n}}(L^{\frac{2(n+4)}{n}})$-norm in Proposition 4.5.2. □

Conversely to Proposition 4.9.1, it is easy to show that the operators $S_\pm$ are surjective.

Proposition 4.9.2. *Let $n\geq 5$. For any $u^+\in H^2_{rad}$, respectively $u^-\in H^2_{rad}$, there exists a unique $u\in C(\mathbb{R},H^2_{rad})$, solution of (4.1.2) with $\lambda>0$ and $p=2^\sharp-1$, such that (4.9.1) holds true. In particular, $S_\pm$ are homeomorphisms from H^2_{rad} onto H^2_{rad}.*

Proof. Again, by time reversal symmetry, we need only prove Proposition 4.9.2 for u^+. Let $\omega(t)=e^{it(\Delta^2+\varepsilon\Delta)}u^+$. Then by the Strichartz estimates (4.3.10), $\omega\in W(\mathbb{R})$ and, given $\delta>0$, there exists T_δ such that the $W([T_\delta,+\infty))$-norm of ω is less than δ. For $u\in W([T_\delta,+\infty))$, we define

$$\Phi(u)(t)=\omega(t)-i\lambda\int_t^\infty e^{i(t-s)(\Delta^2+\varepsilon\Delta)}|u(s)|^{\frac{8}{n-4}}u(s)ds. \tag{4.9.6}$$

It is easily seen that for δ sufficiently small, Φ defines a contraction mapping on the set

$$\begin{aligned}X_{T_\delta}=&\{u\in W([T_\delta,+\infty))\cap L^{\frac{2(n+4)}{n}}([T_\delta,+\infty),L^{\frac{2(n+4)}{n}});\|u\|_{W([T_\delta,\infty))}\leq C\delta,\\&\|u\|_{L^{\frac{2(n+4)}{n}}([T_\delta,+\infty),L^{\frac{2(n+4)}{n}})}\leq C\|u^+\|_{L^2}\}\ ,\end{aligned}$$

equipped with the $L^{\frac{2(n+4)}{n}}([T_\delta,+\infty),L^{\frac{2(n+4)}{n}})$-norm. Thus Φ admits a unique fixed point u. It follows from the Strichartz estimates (4.3.19) and (4.3.20) that $u\in C([T_\delta,+\infty),H^2)\cap M([T_\delta,+\infty))$. Besides, we can observe that

$$u(T_\delta+t)=e^{it(\Delta^2+\varepsilon\Delta)}u(T_\delta)+i\lambda\int_{T_\delta}^{T_\delta+t}e^{i(t-s)(\Delta^2+\varepsilon\Delta)}|u(s)|^{\frac{8}{n-4}}u(s)ds.$$

Then, u solves (4.1.2) with $p=2^\sharp-1$ on $[T_\delta,+\infty)$. Hence, using the radial symmetry assumption, we see from Theorem 12 that u can be extended on $\mathbb{R}$. Now, (4.9.1) follows from (4.9.6) and the boundedness of u in W and $L^{\frac{2(n+4)}{n}}(L^{\frac{2(n+4)}{n}})$-norms. Uniqueness follows from the fact that any radially symmetrical solution of (4.1.2) with $p=2^\sharp-1$ and $\lambda>0$ has a restriction in X_T for some $T\geq T_\delta$, and uniqueness of the fixed point of Φ in such spaces. The continuity statements can be proved with similar arguments to those used in the proof of Proposition 4.9.1. □

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Chapter 5

Asymptotic analysis for L^2-critical fourth-order Schrödinger equations

Abstract

We prove a structure theorem for sequences of solutions to the L^2-critical fourth-order Schrödinger equation, and isolate some special solutions at the threshold for global well-posedness.

Preprint submitted under the name "Asymptotic analysis for L^2-critical fourth-order Schrödinger equations".

5.1 Introduction and statement of the results

Fourth-order Schrödinger equations have been introduced by Karpman [7] and Karpman and Shagalov [8] to take into account the role of small fourth-order dispersion terms in the propagation of intense laser beams in a bulk medium with Kerr nonlinearity. The L^2-critical homogeneous case of these equations is written as

$$i\partial_t u + \Delta^2 u + \lambda |u|^{\frac{8}{n}} u = 0, \tag{5.1.1}$$

where $\lambda \in \mathbb{R}$, and $u : I \times \mathbb{R}^n \to \mathbb{C}$ is a complex-valued function. Equation (5.1.1) has been recently investigated in Fibich, Ilan, and Papanicolaou [5]. In case the power $8/n$ is replaced by a smaller power, any initial data in L^2 gives rise to a global solution. In the L^2-critical case we discuss here, much less is known. It is suspected, see the numerical work in Fibich, Ilan and Papanicolaou [5], that in case $\lambda < 0$ there exist smooth initial data which do not lead to a global solution. On the other hand, it is natural to conjecture that global existence holds true in case $\lambda > 0$. By standard developments, sufficiently small initial data in L^2 lead to global solutions. Our goal here is to develop the asymptotic analysis of sequences of solutions of (5.1.1) in a very general setting. Such analysis is known to hold true in the case of second order equations and to be, in this case, of fundamental interest for proving global existence and scattering conjectures. As a remark, an advantage of carrying over the analysis at the L^2-level we discuss here is that it can then be used for any higher regularity level $\dot{H}^s$ that possesses good stability properties with only minor additional work. The $\dot{H}^2$-critical case provides an interesting case of $\dot{H}^s$ regularity having good stability properties, see Pausader [13].

Before stating our results we need to introduce some notations. For a general function defined on space time $I \times \mathbb{R}^n$, we define the scattering norm $\|\cdot\|_Z$ by

$$\|u\|_{Z(I)} = \|u\|_{L^{\frac{2(n+4)}{n}}(I, L^{\frac{2(n+4)}{n}})}.$$

and the stronger norm $\|\cdot\|_{\hat{Z}}$ by

$$\|u\|_{\hat{Z}(I)} = \|u\|_{L^\infty(I,L^2)} + \|u\|_{Z(I)}.$$

We omit I in the notation of the norm when $I = \mathbb{R}$. In what follows, we define $\Lambda(N)$ to be the supremum of the $\|u\|_Z$ over all maximal lifespan solutions u of (5.1.1) of L^2-norm less than N. As is easily checked, Λ is increasing in N and $\Lambda(x) \le Cx < +\infty$ in a neighborhood of 0. We let N_{max} be the supremum of the $N > 0$ such that $\Lambda(N) < +\infty$.

For any $(h_0, t_0, x_0) \in \mathbb{R}_+^* \times \mathbb{R} \times \mathbb{R}^n$, we let $\tau_{(h_0,t_0,x_0)}$ be the linear transformation acting on functions defined on $\mathbb{R} \times \mathbb{R}^n$ given by

$$\tau_{(h_0,t_0,x_0)} u(t,x) = h_0^{\frac{n}{2}} u(h_0^4(t-t_0), h_0(x-x_0)).$$

For any choice of (h_0, t_0, x_0), $\tau_{(h_0,t_0,x_0)}$ preserves the $\hat{Z}$-norm. For $(h_0, x_0) \in \mathbb{R}_+^* \times \mathbb{R}^n$, we let $g_{(h_0,x_0)}$ be the isometry of L^2 given by

$$g_{(h_0,x_0)} f(x) = h_0^{\frac{n}{2}} f(h_0(x-x_0)).$$

As a general remark on τ and g, it can be noted that if u is a solution of (5.1.1), then $\tau_{(h_0,t_0,x_0)}u$ is also a solution of (5.1.1) having the same $L^\infty L^2$ and Z-norms, but with initial data $g_{(h_0,x_0)}u(-t_0h_0^4)$. We call scale-core any sequence $(h_k,t_k,x_k)_{k\geq 0}$ in $\mathbb{R}_+^\star \times \mathbb{R}\times\mathbb{R}^n$. We say that two scale-cores $(h_k,t_k,x_k)_{k\geq 0}$ and $(h'_k,t'_k,x'_k)_{k\geq 0}$ are orthogonal if

$$\log\left|\frac{h_k}{h'_k}\right| + h_k^4\,|t_k - t'_k| + h_k\,|x_k - x'_k| \to +\infty \tag{5.1.2}$$

as $k \to +\infty$. A decomposition $u_k = \sum_i v_k^i$ is said to be asymptotically L^2-orthogonal if $\|u_k\|_{L^2}^2 = \sum_i \|v_k^i\|_{L^2}^2 + o(1)$ as $k\to+\infty$. When two variables A and k are in a formula, we let $\lim_{A,k} = \lim_{A\to+\infty}\limsup_{k\to+\infty}$. The first theorem we prove is stated as follows

Theorem 13. *Let $(u_k)_k$ be a sequence of strong solutions of* (5.1.1) *such that $\sup_k \|u_k(0)\|_{L^2} < N_{max}$. Then there exists a sequence $(U^\alpha)_\alpha$ of global strong solutions of* (5.1.1)*, and a sequence $(h^\alpha,t^\alpha,x^\alpha)_\alpha$ of orthogonal scale-cores such that, up to a subsequence,*

$$u_k = \sum_{\alpha=1}^{A} \tau_{(h_k^\alpha,t_k^\alpha,x_k^\alpha)}U^\alpha + e^{it\Delta^2}w_k^A + r_k^A \tag{5.1.3}$$

for all k and A, where $w_k^A \in L^2$ and $r_k^A \in L^\infty L^2 \cap L^{\frac{2(n+4)}{n}}L^{\frac{2(n+4)}{n}}$ for all k and A, $\lim_{A,k}\|e^{it\Delta^2}w_k^A\|_Z = 0$, and $\lim_{A,k}\|r_k^A\|_{\hat{Z}} = 0$. Moreover, for any A, and t, (5.1.3) *is asymptotically L^2-orthogonal and*

$$\|u_k\|_Z^{\frac{2(n+4)}{n}} = \sum_{\alpha=1}^{+\infty}\|U^\alpha\|_Z^{\frac{2(n+4)}{n}} + o(1), \tag{5.1.4}$$

where $o(1)\to 0$ as $k\to+\infty$.

In complement to Theorem 13 we also prove in Theorem 14 below that there are solutions u of (5.1.1) with an L^2-norm exactly like N_{max}. We also provide a threefold scenario for u. Theorem 14 is stated as follows.

Theorem 14. *Suppose that $N_{max} < +\infty$. Then there exists $u \in C(I,L^2)$ a maximal-lifespan solution of L^2-norm exactly N_{max} and locally finite Z-norm, such that $\|u\|_{Z(I')} = +\infty$ for $I' = (T_*,0)$ and $I' = (0,T^*)$, where $I = (T_*,T^*)$. Besides, there exist two smooth functions $h: I\to\mathbb{R}_+^\star$ and $x: I\to\mathbb{R}^n$ such that $K = \{g_{(h(t),x(t))}u(t) : t\in I\}$ is precompact in L^2 and one of the following three scenarii holds true: (soliton-like solution) there holds $I=\mathbb{R}$ and $h(t)=1$ for all t; (double high-to-low cascade) there holds $I=\mathbb{R}$, $\limsup_{t\to\pm\infty} h(t) = +\infty$, and $h(t)\geq 1$ for all t; (self-similar solution) there holds $I=(0,+\infty)$ and $h(t) = t^{\frac{1}{4}}$ for all t.*

Following standard notations, we use the notation $X \lesssim Y$ whenever there exists some constant C, possibly depending on the dimension n so that $X \leq CY$. Similarly we write $X \simeq Y$ when $X\lesssim Y \lesssim X$.

Theorem 13 in the second order case goes back to Bahouri and Gerard [1], Begout and Vargas [2], Bourgain [3], Carles and Keraani [4], Keraani [9] and

Merle and Vega [12]. The involved L^2-critical case was addressed in Carles and Keraani [4] when $n = 1$, in Bourgain [3] and Merle and Vega [12] when $n = 2$, and in Begout and Vargas [2] when $n \geq 3$. Theorem 14 in the second order case was obtained only very recently in Kenig and Merle [10], and Killip, Tao and Visan [11]. An important specific feature of the fourth order case we discuss here is the regularising effect associated with the linear propagator.

5.2 Proof of Theorem 13

An important step in the proof of Theorem 13 is given by the Strichartz estimates obtained in Pausader [13], and the precised Sobolev inequality in Gerard, Meyer and Oru [6]. Let $q = \frac{2(n+4)}{n}$, $r = \frac{2(n+4)}{n+2}$, and $\sigma = \frac{n}{n+4}$. As shown in Pausader [13], for any $u_0 \in L^2$,

$$\|e^{it\Delta^2}u_0\|_{L^q(\dot{H}^{\sigma,r})} \lesssim \|u_0\|_{L^2}. \tag{5.2.1}$$

Independently, it follows from Gerard, Meyer and Oru [6] that, for any $u \in \dot{B}^\sigma_{r,2}(\mathbb{R}^n)$,

$$\|u\|_{L^q(\mathbb{R}^n)} \lesssim \|u\|^{\frac{2}{n+2}}_{\dot{B}^\sigma_{r,\infty}} \|u\|^{\frac{n}{n+2}}_{\dot{B}^\sigma_{r,2}}, \tag{5.2.2}$$

where $\dot{B}^\sigma_{r,2}$ and $\dot{B}^\sigma_{r,\infty}$ are standard homogeneous Besov spaces. Since the Littlewood-Paley projectors P_M commute with the linear propagator of (5.1.1), using the Littlewood Paley theorem, (5.2.1) and (5.2.2) we get that

$$\begin{aligned}\|e^{it\Delta^2}u_0\|_Z &\lesssim \left(\sup_M \|P_M e^{it\Delta^2}u_0\|_{L^q\dot{H}^{\sigma,r}}\right)^\alpha \left(\sum_M \|P_M e^{it\Delta^2}u_0\|^2_{L^q\dot{H}^{\sigma,r}}\right)^{\frac{\beta}{2}} \\ &\lesssim \left(\sup_M \|P_M u_0\|_{L^2}\right)^\alpha \left(\sum_M \|P_M u_0\|^2_{L^2}\right)^{\frac{\beta}{2}} \\ &\lesssim \|u_0\|^\alpha_{\dot{B}^0_{2,\infty}} \|u_0\|^\beta_{L^2},\end{aligned} \tag{5.2.3}$$

where $\alpha = 8/(n+2)(n+4)$ and $\beta = n(n+6)/(n+2)(n+4)$, and the sum and the supremum are taken over all dyadic integers. Now, thanks to (5.2.3), we may follow the analysis in Bahouri and Gerard [1]. In particular, mimicking the proof in Bahouri and Gerard [1] we obtain that for $(v_k)_k$ a bounded sequence in L^2 , there exists a sequence $(V^\alpha)_\alpha$ in L^2, and pairwise orthogonal scale-cores $((h^\alpha_k)_k, (t^\alpha_k)_k, (x^\alpha_k)_k)_\alpha$ such that, up to a subsequence, for any $A \geq 1$,

$$v_k = \sum_{\alpha=1}^{A} g_{(h^\alpha_k, x^\alpha_k)}\left(e^{-i(h^\alpha_k)^4 t^\alpha_k \Delta^2} V^\alpha\right) + w^\alpha_k \tag{5.2.4}$$

for all k, where $w^A_k \in L^2$ for all k and A, and $\lim_{A,k} \|e^{it\Delta^2}w^A_k\|_Z = 0$. Moreover, (5.2.4) is asymptotically L^2-orthogonal for all A, and

$$\|e^{it\Delta^2}u_k\|^{\frac{2(n+4)}{n}}_Z = \sum_{\alpha=1}^{+\infty} \|e^{it\Delta^2}V^\alpha\|^{\frac{2(n+4)}{n}}_Z + o(1) \tag{5.2.5}$$

for all k, where $o(1) \to 0$ as $k \to +\infty$. Moreover, see Pausader [13] for the $\dot{H}^2$-critical case, the following stability property follows from the Strichartz estimates (5.2.1). Namely that for any $B > 0$ and $\varepsilon > 0$, there exists $\delta > 0$ such that if v is an approximate solution of (5.1.1) in the sense that

$$i\partial_t v + \Delta^2 v - \lambda |v|^{\frac{8}{n}} v = e \tag{5.2.6}$$

in some interval I with $0 \in I$, and $\|v\|_{Z(I)} \le B$, $\|e\|_{L^{\frac{2(n+4)}{n+8}}(I, L^{\frac{2(n+4)}{n+8}})} \le \delta$, then for any $u_0 \in L^2$ satisfying $\|e^{it\Delta^2}(v(0) - u_0)\|_{Z(I)} \le \delta$, there exists a unique strong solution $u \in L^{\frac{2(n+4)}{n}}(I \times \mathbb{R}^n)$ of (5.1.1) which satisfies

$$\|u - v\|_{Z(I)} \le \varepsilon, \text{ and } \|u - v\|_{L^\infty(I, L^2)} \lesssim \|v(0) - u_0\|_{L^2} + \varepsilon. \tag{5.2.7}$$

Let $(V, \mathbf{h}, \mathbf{z})$ be such that $V \in L^2$, with $\|V\|_{L^2} < N_{max}$, and $(\mathbf{h}, \mathbf{z})$ be a scale-core such that $h_k^4 t_k$ has a limit $l \in [-\infty, +\infty]$. We say that U is the nonlinear profile associated to $(V, \mathbf{h}, \mathbf{z})$ if U is a solution of (5.1.1), and

$$\|U(-h_k^4 t_k) - e^{-ih_k^4 t_k \Delta^2} V\|_{L^2} \to 0$$

as $k \to +\infty$. In what follows, we use repeatedly that if $(\mathbf{h}, \mathbf{t}, \mathbf{x})$ and $(\mathbf{h}', \mathbf{t}', \mathbf{x}')$ are orthogonal scale cores, then, for any u, v with finite Z-norm,

$$\|\tau_{(h_k, t_k, x_k)} u |\tau_{(h'_k, t'_k, x'_k)} v|^{\frac{n+8}{n}}\|_{L^1(\mathbb{R}^{n+1})} \to 0 \tag{5.2.8}$$

as $k \to +\infty$. Let $N_\infty = \sup_k \|u_k(0)\|_{L^2} < N_{max}$. We let $(\mathbf{h}^\alpha, \mathbf{z}^\alpha)$, V^α, and $\mathbf{w}^A$ be given by (5.2.4) applied to the sequence $(v_k = u_k(0))_k$. Passing to subsequences, and using a diagonal extraction argument, we can assume that $h_k^4 t_k$ has a limit in $[-\infty, \infty]$. We let U^α be the nonlinear profile associated to $(V^\alpha, \mathbf{h}^\alpha, \mathbf{z}^\alpha)$. Since $\|U^\alpha(0)\|_{L^2} = \|V^\alpha\|_{L^2} < N_{max}$, we get that all the nonlinear profiles are globally defined. Letting $W_k^A(t) = e^{it\Delta^2} w_k^A$, we remark that

$$p_k^A = \sum_{\alpha=1}^{A} \tau_{(h_k^\alpha, z_k^\alpha)} U^\alpha + W_k^A$$

satisfy (5.2.6) with

$$e = e_k^A = \sum_{\alpha=1}^{A} f(\tau_{(h_k^\alpha, z_k^\alpha)} U^\alpha) - f(\sum_{\alpha=1}^{A} \tau_{(h_k^\alpha, z_k^\alpha)} U^\alpha + W_k^A) - f(W_k^A)$$

and initial data $p_k^A(0) = u_k(0) + o_A(1)$, where $f(x) = |x|^{\frac{8}{n}} x$. First, we claim that

$$\limsup_k \|\sum_{\alpha=1}^{A} \tau_{(h_k^\alpha, z_k^\alpha)} U^\alpha\|_Z \lesssim_{N_\infty} 1 \tag{5.2.9}$$

independently of A. Indeed, since Λ is sublinear around 0, and bounded on $[0, N_\infty]$, using (5.2.8), we get that

$$\begin{aligned}
\limsup_k \| \sum_{\alpha=1}^{A} \tau_{(h_k^\alpha, z_k^\alpha)} U^\alpha \|_Z^{\frac{2(n+4)}{n}} &\leq \sum_{\alpha=1}^{A} \| U^\alpha \|_Z^{\frac{2(n+4)}{n}} \\
&\lesssim_{N_\infty} \sum_{\alpha=1}^{A} \Lambda(\| U^\alpha \|_{L^2})^{\frac{2(n+4)}{n}} \\
&\lesssim_{N_\infty} \sum_{\alpha=1}^{A} \| U^\alpha \|_{L^2}^2 \\
&\lesssim_{N_\infty} N_\infty^2
\end{aligned} \tag{5.2.10}$$

and (5.2.9) is proved. Now, using (5.2.8), we get that

$$\| f(\sum_{\alpha=1}^{A} \tau_{(h_k^\alpha, z_k^\alpha)} U^\alpha) - \sum_{\alpha=1}^{A} f(\tau_{(h_k^\alpha, z_k^\alpha)} U^\alpha) \|_{L^{\frac{2(n+4)}{n+8}}(\mathbb{R}^{n+1})} = o_A(1) \tag{5.2.11}$$

as $k \to +\infty$. Besides,

$$\begin{aligned}
&\| f(\sum_{\alpha=1}^{A} \tau_{(h_k^\alpha, z_k^\alpha)} U^\alpha + W_k^A) - f(\sum_{\alpha=1}^{A} \tau_{(h_k^\alpha, z_k^\alpha)} U^\alpha) \|_{L^{\frac{2(n+4)}{n+8}}(\mathbb{R}^{n+1})} \\
&\lesssim \| W_k^A \|_Z \left(\| W_k^A \|_Z^{\frac{8}{n}} + \| \sum_{\alpha=1}^{A} \tau_{(h_k^\alpha, z_k^\alpha)} U^\alpha \|_Z^{\frac{8}{n}} \right) \\
&\lesssim_{N_\infty} \| W_k^A \|_Z \left(\| W_k^A \|_Z^{\frac{8}{n}} + 1 \right) \\
&\lesssim_{N_\infty} \| W_k^A \|_Z
\end{aligned} \tag{5.2.12}$$

Independently,

$$\begin{aligned}
\| p_k^A \|_Z &\leq \| \sum_{\alpha=1}^{A} \tau_{(h_k^\alpha, z_k^\alpha)} U^\alpha \|_Z + \| W_k^A \|_Z \\
&\lesssim_{N_\infty} 1.
\end{aligned} \tag{5.2.13}$$

Now, by (5.2.11)–(5.2.13), we get that $\lim_{A,k} \| e_k^A \|_{L^{\frac{2(n+4)}{n+8}}(\mathbb{R}^{n+1})} = 0$ and using (5.2.7), we get, since $p_k^A(0) = u_k(0) + o_A(1)$, that $\lim_{A,k} \| r_k^A \|_{\hat{Z}} = 0$. This finishes the proof of Theorem 13.

5.3 Proof of Theorem 14

Given $u \in C(I, L^2)$ a strong solution of (5.1.1) satisfying the conclusion of Theorem 14, we refer to the function $g(t) = g_{(h(t), x(t))}$ in Theorem 14 as the scaling function of u. We note that if $g_{(h_k, x_k)} u_k$ and $g_{(h_k', x_k')} u_k$ both stay in a L^2-ball of center a and a', and radii r and r' such that $\|a\|_{L^2} > r + r'$, then $g_{(h_k, x_k)} g_{(h_k', x_k')}^{-1}$ cannot converge weakly to 0. In particular $h_k^{-1} h_k'$ and $h_k(x_k - x_k')$ remain in a compact subset. This is the case if $g_{(h_k, x_k)} u_k$ and $g_{(h_k', x_k')} u_k$ both converge in L^2 to nonzero functions. A direct consequence of this remark is that if $\{g(t)u(t) : t \in I\}$ and $\{g'(t)u(t) : t \in I\}$ are precompact in L^2, then there exists a compact set K such that $g(t)g'(t)^{-1} \in K$ for all $t \in I$. In particular the rescaling function of u is well defined, up to multiplication by a function $\bar{g}$ of compact image. As a preliminary step in the proof we claim that for any sequence of nonlinear solutions $(u_k)_k$ defined on a maximal interval $(-T_k, T^k)$, and any sequence of time $(t_k)_k$ such that $\|u_k\|_{L^2} \to N_{max}$ and

$$\lim_k \|u_k\|_{Z(-T_k,t_k)} = \lim_k \|u_k\|_{Z(t_k,T^k)} = +\infty,$$

there exists two sequences (h_k) and (y_k), and a function $w \in L^2$ of L^2-norm N_{max} such that, up to a subsequence, $g_{(h_k,y_k)}(u_k(t_k)) \to w$ in L^2. This can be proved with similar arguments to the ones developed in Tao, Visan and Zhang [14], using the linear decomposition (5.2.4) and the stability theory as in (5.2.7).

Now, with this claim, we are in position to prove the first part of Theorem 14. A consequence of the stability property (5.2.7) is that the set $\{N : \Lambda(N) < +\infty\}$ is open. Hence, we can always find a sequence of nonlinear solutions u_k such that $\|u_k\|_{L^2} < N_{max}$ and $\|u_k\|_{L^2}$ converges to N_{max}, while $\|u_k\|_Z$ converges to $+\infty$. By time translation invariance, we can suppose that $\|u_k\|_{Z((-\infty,0])} = \|u_k\|_{Z([0,+\infty)} = \frac{1}{2}\|u_k\|_Z$. Applying the above claim and passing to a subsequence, we get that there exists (h_k, x_k) such that $g_{(h_k,x_k)}u_k(0) \to V$ in L^2. Let U be the maximal nonlinear solution of (5.1.1) with initial data V, defined on $I = (-T_*, T^*)$. Suppose, for example that $T^* = +\infty$, and that $\|U\|_{Z(\mathbb{R}_+)} < +\infty$. Then, using (5.2.7) on $\mathbb{R}_+$ with $v = U$, and $u = \tau_{(h_k,0,x_k)}u_k$, we see that $\|u_k\|_{Z(\mathbb{R}_+)}$ is bounded uniformly in k, which is a contradiction with the hypothesis on u_k. Now, we prove the compactness property of U. Using the above preliminary step, it is easily proved by contradiction that for any $\epsilon > 0$, there exist $t_1, \ldots, t_j$, $j = j(\epsilon)$, such that for any time $t \in (-T_*, T^*)$, there exist $i = i(t)$, and $g(t) = g_{(h(t),y(t))}$ with the property that $\|u(t_i) - g(t)u(t)\|_{L^2} \leq \epsilon$. Let us apply this with $\epsilon = N_{max}/4$. We get a function $g(t) = g_{(h(t),y(t))}$, and a finite set of times $t_1, \ldots, t_j$ such that for any t, there exists i satisfying $\|u(t_i) - g(t)u(t)\|_{L^2} \leq N_{max}/4$. We claim that $\{g(t)u(t)\}$ is precompact in L^2. Suppose by contradiction that this is not true. Then, there exist $\epsilon > 0$, and a sequence s_k such that for any k and p,

$$\|g(s_k)u(s_k) - g(s_{k+p})u(s_{k+p})\|_{L^2} > \epsilon. \tag{5.3.1}$$

According to what we said above, and passing to a subsequence, we can assume that there exist two times $\bar{t}, \bar{t}'$, and a sequence $g'_k = g_{(h'_k,y'_k)}$ such that, for any k,

$$\begin{aligned} \|u(\bar{t}) - g(s_k)u(s_k)\|_{L^2} &< N_{max}/4, \text{ and} \\ \|u(\bar{t}') - g'_k u(s_k)\|_{L^2} &< \frac{\epsilon}{4}. \end{aligned} \tag{5.3.2}$$

Passing to a subsequence, and using the remark we made at the begining of this section, it is easily seen that $h(s_k)^{-1}h'_k$ and $(h'_k)^{-1}h(s_k)y(s_k) - y'_k$ remain in a compact subset. Hence, up to considering a subsequence, we can find g_∞ such that $g(s_k)\left(g'_k\right)^{-1} \to g_\infty$ strongly. Now, using (5.3.2), we get that

$$\begin{aligned}&\|g(s_k)u(s_k) - g(s_{k+1})u(s_{k+1})\|_{L^2}\\&\le \|g(s_k)u(s_k) - g_\infty u(\bar{t}')\|_{L^2} + \|g_\infty u(\bar{t}') - g(s_{k+1})u(s_{k+1})\|_{L^2}\\&\le \|g'_k u(s_k) - g'_k g(s_k)^{-1} g_\infty u(\bar{t}')\|_{L^2} + \|g'_{k+1}u(s_{k+1}) - g'_{k+1}g(s_{k+1})^{-1}g_\infty u(\bar{t}')\|_{L^2}\\&\le \frac{\epsilon}{2} + o(1).\end{aligned}$$

Clearly, this contradicts (5.3.1) and finishes the proof of the first part of Theorem 14. Now we discuss the proof of the second part of Theorem 14. For $\delta > 0$, and t, we define the interval $J_\delta(t)$ by

$$J_\delta(t) = (t - \delta h(t)^4, t + \delta h(t)^4). \tag{5.3.3}$$

We start by collecting a few results on the scaling function. First, we claim that if $u \in C(I, L^2)$ is a maximal lifespan solution of (5.1.1) of locally finite Z-norm, and $g(t)$ is such that $K = \{g(t)u(t) : t \in I\}$ is precompact, then there exists $\delta > 0$ such that for any time $t_0 \in I$, the following holds true: for $J_\delta(t_0)$ as in (5.3.3), $J_\delta(t_0) \subset I$, $h(t) \simeq_u h(t_0)$ for any $t \in J_\delta(t_0)$, and

$$\|u\|_{Z(J_\delta(t_0))} \simeq 1. \tag{5.3.4}$$

In particular, if h is bounded from below, then $I = \mathbb{R}$, and if u blows up at time T, then $h(t) \le (T-t)^{\frac{1}{4}}$. Indeed, since K is precompact, using Strichartz estimates and local well-posedness theory as developed in Pausader [13], we see that there exists δ such that for any $w \in K$, the maximal nonlinear solution W of (5.1.1) with initial data w is defined on $(-\delta, \delta)$ and satisfies $\|W\|_{Z(-\delta,\delta)} \lesssim 1$. By rescaling, this gives the first claim concerning $J_\delta(t_0)$, and one inequality in (5.3.4). On the other hand, compactness of K implies that there exists η such that for any t,

$$\eta \le \|u\|_{Z(J_\delta(t))}. \tag{5.3.5}$$

This gives the second inequality in (5.3.4). Now, let us prove that in $J_\delta(t_0)$ we have $h(t) \simeq h(t_0)$. To achieve this let us consider

$$\mathcal{J} = \{\kappa : \exists t_0 \in I, \exists t \in J_\delta(t_0) : \|u\|_{Z(J_\kappa(t,t_0))} \ge \eta\},$$

where $J_\kappa(t,t_0) = (t - \kappa\delta h(t_0)^4, t + \kappa\delta h(t_0)^4)$. Note that, if $t \in J_\delta(t_0)$, and $A > 1$, then $h(t) = Ah(t_0)$ implies that $A^{-4} \in \mathcal{J}$, while $h(t) < ah(t_0)$ implies $a^4 \in \mathcal{J}$. Let us suppose that $\inf \mathcal{J} = 0$. Using (5.3.5), this implies that there exists a sequence of times $t_k \in I$, a sequence $t'_k \in J(t_k)$, and a sequence $\kappa_k \to 0$ such that

$$\|u_k\|_{Z(J_{\kappa_k}(t'_k,t_k))} \ge \eta. \tag{5.3.6}$$

Extracting a subsequence and rescaling, we obtain in this way a sequence of solutions $u_k = \tau_{(h(t_k), -h(t_k)^{-4}t_k, x(t_k))}u$ such that $\|u_k\|_{Z(-\delta,\delta)} \lesssim 1$, $u_k(0) \to w$ in L^2, and $\|u_k\|_{Z(s_k-\kappa_k, s_k+\kappa_k)} \ge \eta$ for $s_k \to t_*$. Let W be the nonlinear solution of (5.1.1) with initial data $W(0) = w$. Using (5.2.7), we see that $\|W\|_{Z(-\delta,\delta)} \lesssim 1$, hence, there exists $D > \delta$ such that the $Z((-D,D))$-norm of W is finite. Besides, for any $\kappa > 0$, $\|W\|_{Z(t_*-\kappa, t_*+\kappa)} \ge \eta$. Noting that this contradicts the dominated convergence theorem, we get that $\inf \mathcal{J} > 0$, and this finishes the proof of the above claim. As a remark, since for any t, we have $\|u\|_{Z(J_\delta(t))}^{\frac{2(n+4)}{n}} \simeq_u 1$, and $\int_{J_\delta(t)} h(s)^{-4} ds \simeq_u 1$, we have that

$$\int_J h(s)^{-4} ds \lesssim 1 + \|u\|_{Z(J)}^{\frac{2(n+4)}{n}} \lesssim 1 + \int_J h(s)^{-4} ds, \tag{5.3.7}$$

for any interval $J \subset I$. Now, we claim that one can always suppose that g as given by Theorem 14 is smooth. To prove this, choose t_k to be a sequence in I such that $I \subset \cup_k J_\delta(t_k)$, and choose $\tilde{g}(t) = g_{(\tilde{h}(t),\tilde{x}(t))}$ to satisfy $\tilde{g}(t_k) = g(t_k)$ and $\tilde{h}$, $\tilde{x}$ are piecewise affine. Then $\tilde{g}(t)g(t)^{-1}$ remains in a compact set, and one can modify $\tilde{g}$ in a small neighborhood of t_k so that it becomes smooth and $\tilde{g}(t)g(t)^{-1}$ remains in a compact set. The claim follows. Suppose now that u_k is a sequence of maximal lifespan solutions defined on I_k satisfying that

$$K = \{g_k(t)u(t) : t \in I_k, k \geq 0\}$$

is precompact for some rescaling functions g_k, and suppose that

$$u_k \to W \text{ in } C_{loc}(I, L^2) \cap L_{loc}^{\frac{2(n+4)}{n}}(I, L^{\frac{2(n+4)}{n}}),$$

where $W \in C(I, L^2)$ is a maximal lifespan solution to (5.1.1) with rescaling function $G(t) = g_{(H(t),X(t))}$. Then, for any $t \in I$, we have that

$$0 < \liminf_k h_k(t) \simeq_u H(t) \simeq_u \limsup_k h_k(t) < +\infty. \tag{5.3.8}$$

Let us prove (5.3.8). For this purpose, we let u_k be a sequence of maximal lifespan solutions as above converging to $W \in C(I, L^2)$ with rescaling function G. Suppose that there exists a sequence of time $s_p \in I$, and, for each p, a subsequence k' such that

$$H(s_p)h_{k'}^{-1}(s_p) < 1/p. \tag{5.3.9}$$

By assumption, $u_k \to W$ in $C(J, L^2)$ for any compact interval $s_p \in J \subset I$. In particular, we can find an element $k \geq p$ such that $\|u_k(s_p) - W(s_p)\|_{L^2} \leq N_{max}/4$, and (5.3.9) holds true with k instead of k'. This defines a sequence $k(p)$ such that for any p,

$$u_{k(p)}(s_p) = W(s_p) + r_p$$

with $\|r_p\|_{L^2} \leq N_{max}/4$. Considering maybe a subsequence, we can assume that $G(s_p)W(s_p) \to \bar{w}$ in L^2. In particular $\|\bar{w}\|_{L^2} = N_{max}$. However

$$G(s_p)W(s_p) = G(s_p)g_{k(p)}(s_p)^{-1}g_{k(p)}(s_p)u_{k(p)}(s_p) - G(s_p)r_p.$$

Now by assumption, $g_{k(p)}(s_p)u_{k(p)}(s_p)$ remains in a compact subset, and we have that $G(s_p)g_{k(p)}(s_p)^{-1}$ converges weakly to 0. As a consequence any weak limit w' of $G(s_p)W(s_p)$ satisfies $\|w'\|_{L^2} \leq \|r_p\|_{L^2} \leq N_{max}/4$. This contradicts $\|\bar{w}\|_{L^2} = N_{max}$. This finishes the proof of (5.3.8). Now we prove the second part of Theorem 14. We only briefly sketch the proof, and refer to Killip, Tao and Visan [11] for a similar proof in the context of the L^2-critical Schrödinger equation. Thanks to all what we proved above only slight modifications with respect to the arguments developped in Killip, Tao and Visan [11] are needed. First, we define the oscillation function

$$Osc(\kappa) = \inf_{t_0 \in I} \frac{\sup_{t \in I \cap J_\kappa(t_0)} h(t)}{\inf_{t \in I \cap J_\kappa(t_0)} h(t)}.$$

This is an increasing function of κ. Let us assume that it is bounded. Then we can find arbitrarily long intervals on which the solution does not oscillate, and using (5.2.7), (5.3.7) and (5.3.8), mimicking the proof in Killip, Tao and Visan [11], we find a solution that behaves as in the first scenario in Theorem 14, namely the soliton-like solution scenario. Now, we assume that $\lim_{\kappa \to \infty} Osc(\kappa) = +\infty$, and we introduce the quantity

$$a(t_0) = \frac{\inf_{t \in I, t \leq t_0} h(t)^{-1} + \inf_{t \in I, t \geq t_0} h(t)^{-1}}{h(t_0)^{-1}}$$

for every $t_0 \in J$. We suppose that $\inf_{t_0 \in I} a(t_0) = 0$. Then, we can find intervals on which the solution present arbitrarily large relative peak. Using (5.3.7) and (5.3.8), mimicking the proof in Killip, Tao and Visan [11], we find a solution satisfying the second scenario in Theorem 14, namely the high-to-low cascade scenario. Finally, in case $a = \inf_{t_0 \in I} a(t_0) > 0$, the solution has arbitrarily large oscillations, but no relative peak. Using (5.3.7) and (5.3.8), mimicking the proof in Killip Tao and Visan [11], we find a solution behaving as in the last scenario in Theorem 14, namely the self-similar solution scenario. This finishes the proof of the second part of Theorem 14.

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Chapter 6

The cubic fourth-order Schrödinger equation

Abstract

We investigate the cubic fourth-order defocusing Schrödinger equation in energy space in arbitrary space dimension. We prove global well-posedness when $n \leq 8$, scattering when $5 \leq n \leq 8$, and ill-posedness when $n \geq 9$. The cubic fourth-order Schrödinger equation is energy-critical precisely when $n = 8$.

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6.1 Introduction

Fourth-order Schrödinger equations have been introduced by Karpman [13] and Karpman and Shagalov [14] to take into account the role of small fourth-order dispersion terms in the propagation of intense laser beams in a bulk medium with Kerr nonlinearity. Such fourth-order Schrödinger equations have been studied from the mathematical viewpoint in Fibich, Ilan and Papanicolaou [6] who describe various properties of the equation in the subcritical regime, with part of their analysis relying on very interesting numerical developments. Related references are by Ben-Artzi, Koch, and Saut [2] who gave sharp dispersive estimates for the biharmonic Schrödinger operator, Guo and Wang [9] who proved global well-posedness and scattering in H^s for small data, Hao, Hsiao and Wang [10, 11] who discussed the Cauchy problem in a high-regularity setting, and Segata [27] who proved scattering in the case the space dimension is one. We refer also to Pausader [21] where the energy critical case for radially symmetrical initial data is discussed.

We focus in this paper on the study of the initial value problem for the cubic fourth-order defocusing equation in arbitrary space dimension $\mathbb{R}^n$, $n \geq 1$. The equation is written as

$$i\partial_t u + \Delta^2 u + |u|^2 u = 0 \ , \tag{6.1.1}$$

where $u = I \times \mathbb{R}^n \to \mathbb{C}$ is a complex valued function, and $u_{|t=0} = u_0$ is in H^2. The equation is critical when $n = 8$. The fourth-order dispersion scaling property leads to the heuristic that smooth solutions of the free homogeneous equation have their L^∞ norm which decays like $t^{-\frac{n}{4}}$. However, the situation is not so transparent and all frequency parts of the function have their L^∞-norm that decays much faster, like $t^{-\frac{n}{2}}$, but at a rate which depends on the frequency. Uniformly, the rate of decay $t^{-\frac{n}{4}}$ is the best possible, but it is not optimal when the solution is localized in frequency. This subtelty leads to various differences between the dispersion behaviors of second-order Schrödinger equations and of (6.1.1). Our first theorem provides a complete picture of global well-posedness for (6.1.1). It is stated as follows. We let H^2 be the the space of L^2 functions whose first and second derivatives are in L^2, and $\mathcal{S}$ be the space of Schwartz functions.

Theorem 15. *The following two assertions hold true.* (*i*) *Assume* $1 \leq n \leq 8$. *Then any* $u_0 \in H^2$ *leads to a global solution* $u \in C(\mathbb{R}, H^2)$ *of* (6.1.1) *with initial data* $u(0) = u_0$. *Moreover, for any* $t \in \mathbb{R}$, *the mapping* $u(0) \mapsto u(t)$ *is analytic from* H^2 *into itself.* (*ii*) *Assume* $n \geq 9$. *Then the Cauchy problem for* (6.1.1) *is ill-posed in* H^2 *in the sense that for any* $t \neq 0$, *the mapping* $u(0) \mapsto u(t)$, *if it exists, cannot be continuous at* 0. *More precisely, for any* $n \geq 9$, *and any* $\varepsilon > 0$, *there exist* $u_0 \in \mathcal{S}$, $t_\varepsilon \in (0, \varepsilon)$, *and* $u \in C\left([0, \varepsilon], H^2\right)$ *a solution of* (6.1.1) *with initial data* u_0 *such that* $\|u_0\|_{H^2} < \varepsilon$ *while* $\|u(t_\varepsilon)\|_{H^2} > \varepsilon^{-1}$.

When long time existence and global well-posedness hold true the natural question to ask is whether or not scattering holds true as well. It is well known for Schrödinger equations, and more generally for dispersive equations, that there are two regimes for scattering corresponding to a splitting between high and low dimensions. In our case the splitting is between $n \leq 4$ and $n \geq 5$. Our second theorem settles the scattering question when $n \geq 5$, and thus also $n \leq 8$

in order to get long time existence and global well-posedness from Theorem 15. It is stated as follows.

Theorem 16. *Assume* $5 \le n \le 8$. *Then there is scattering in* H^2 *for* (6.1.1). *Moreover the scattering operator is analytic.*

As a remark on Theorem 15, when $1 \le n \le 8$ it can be proved that for any $s \ge 2$, and any initial data $u_0 \in H^s$, the associated u actually belongs to $C(\mathbb{R}, H^s)$. Moreover, the mapping $u(0) \mapsto u(t)$ is analytic from H^s into itself. When $n \le 7$, the cubic Schrödinger equation (6.1.1) is subcritical. For such dimensions Theorems 15 and 16 are easy to obtain and have been already stated in Fibich, Ilan, and Papanicolaou [6] and Pausader [21]. The difficult case in Theorem 15, part (i), and in Theorem 16 is when $n = 8$. In this dimension the equation is energy-critical, namely critical with respect to its natural energy space $\dot{H}^2$. In particular, when $n = 8$, there is a rescaling invariance rule for (6.1.1) given by

$$u \mapsto \tau_{(h,t_0,x_0)}u = h^2 u(h^4(t-t_0), h(x-x_0)) \tag{6.1.2}$$

which sends a solution of (6.1.1) with initial data $u(0) = u_0$ to another solution with data at time $t = t_0$ given by

$$g_{(h,x_0)}u_0 = h^2 u_0(h(x-x_0)), \tag{6.1.3}$$

and which leaves the energy and $\dot{H}^2$-norm unchanged. The associated loss of compactness makes that (6.1.1) is particularly difficult to handle in the critical dimension $n = 8$. In the radially symmetrical case the difficulty was overcome in Pausader [21]. We prove here that we can get rid of the radially symmetrical assumption. As a remark, with the arguments we develop here and adaptations of the analysis in Visan [32], global well-posedness and scattering in Theorems 15 and 16 continue to hold true when $n \ge 8$ and the cubic nonlinearity is replaced by the n-dimensional energy-critical nonlinearity.

Our paper is organised as follows. We fix notations in Section 6.2 and recall important preliminary results from Pausader [21] in Section 6.3. In Section 6.4, we prove that the Cauchy problem is ill-posed when $n \ge 9$. In order to do so we use a low-dispersion regime argument which was essentially given in Christ, Colliander and Tao [4]. We also refer to Lebeau [19] and Thomann [30, 31] for results in different settings. Starting from Section 6.5 we focus on the energy-critical case, and so on the $n = 8$ part of our theorems. We prove in Section 6.5, using ideas of concentration compactness developed in Kenig and Merle [16] and Killip, Tao and Visan [18], that any failure of global wellposedness implies the existence of some special solutions satisfying three possible scenarii. The remaining part of the analysis consists in excluding these hypothetical special solutions working at the level of $\dot{H}^2$-solutions. The first scenario is that there is a self-similar-like solution. It is not consistent with conservation of energy, conservation of local mass and compactness up to rescaling. We exclude this scenario in Section 6.6. The two other scenarii are that there is a soliton-like solution or that there is a low-to-high cascade-like solution. In these two scenarii the solution is away from the L^2-like region, namely we have that $h \le 1$ with respect to the notation of Theorem 18. We use this to prove an interaction Morawetz estimate in Sections 6.7 and 6.8, following previous analysis from

Colliander, Keel, Staffilani, Takaoka and Tao [5], Ryckman and Visan [24] and Visan [32]. The estimate we prove is not an a priori estimate. A major difficulty is that the estimate scales like the $\dot{H}^{\frac{1}{4}}$-norm and thus creates a 7/4-difference in scaling with the $\dot{H}^2$-norm control we have. In Section 6.9, we exclude soliton-like solution by proving that it is not consistent with the frequency-localized interaction Morawetz estimates and compactness up to rescaling. The last scenario is excluded in Section 6.10 by proving that any low-to-high-like solution has an unexpected L^2-regularity. Then, conservation of L^2-norm, frequency-localized interaction Morawetz estimates and conservation of energy allows us to exclude this existence of low-to-high cascade-like cascade solutions. Finally, in Section 6.11, we prove Theorem 16.

6.2 Notations

We fix notations we use throughout the paper. In what follows, we write $A \lesssim B$ to signify that there exists a constant C depending only on n such that $A \leq CB$. When the constant C depends on other parameters, we indicate this by a subscript, for exemple, $A \lesssim_u B$ means that the constant may depend on u. Similar notations hold for $\gtrsim$. Similarly we write $A \simeq B$ when $A \lesssim B \lesssim A$.

We let $L^q = L^q(\mathbb{R}^n)$ be the usual Lebesgue spaces, and $L^r(I, L^q)$ be the space of measurable functions from an interval $I \subset \mathbb{R}$ to L^q whose $L^r(I, L^q)$ norm is finite, where

$$\|u\|_{L^r(I,L^q)} = \left(\int_I \|u(t)\|_{L^q}^r dt \right)^{\frac{1}{r}}.$$

When there is no risk of confusion we may write $L^q L^r$ instead of $L^q(I, L^r)$. Two important conserved quantities of equation (6.1.1) are the mass and the energy. The mass is defined by

$$M(u) = \int_{\mathbb{R}^n} |u(x)|^2 dx \tag{6.2.1}$$

and the energy is defined by

$$E(u) = \int_{\mathbb{R}^n} \left(\frac{|\Delta u(x)|^2}{2} + \frac{|u(x)|^4}{4} \right) dx. \tag{6.2.2}$$

In what follows we let $\mathcal{F}f = \hat{f}$ be the Fourier transform of f given by

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} f(y) e^{i\langle y,\xi\rangle} dy$$

for all $\xi \in \mathbb{R}^n$. The biharmonic Schrödinger semigroup is defined for any tempered distribution g by

$$e^{it\Delta^2} g = \mathcal{F}^{-1} e^{it|\xi|^4} \mathcal{F} g. \tag{6.2.3}$$

Let $\psi \in C_c^\infty(\mathbb{R}^n)$ be supported in the ball $B(0,2)$, and such that $\psi = 1$ in $B(0,1)$. For any dyadic number $N = 2^k, k \in \mathbb{Z}$, we define the following

Littlewood-Paley operators:

$$\begin{aligned}\widehat{P_{\leq N}f}(\xi) &= \psi(\xi/N)\hat{f}(\xi),\\ \widehat{P_{>N}f}(\xi) &= (1-\psi(\xi/N))\hat{f}(\xi),\\ \widehat{P_N}f(\xi) &= (\psi(\xi/N)-\psi(2\xi/N))\,\hat{f}(\xi).\end{aligned} \tag{6.2.4}$$

Similarly we define $P_{<N}$ and $P_{\geq N}$ by the equations $P_{<N} = P_{\leq N} - P_N$ and $P_{\geq N} = P_{>N} + P_N$. These operators commute one with another. They also commute with derivative operators and with the semigroup $e^{it\Delta^2}$. In addition they are self-adjoint and bounded on L^p for all $1 \leq p \leq \infty$. Moreover, they enjoy the following Bernstein property:

$$\begin{aligned}&\|P_{\geq N}f\|_{L^p} \lesssim_s N^{-s}\||\nabla|^s P_{\geq N}f\|_{L^p} \lesssim_s N^{-s}\||\nabla|^s f\|_{L^p}\\ &\||\nabla|^s P_{\leq N}f\|_{L^p} \lesssim_s N^s\|P_{\leq N}f\|_{L^p} \lesssim_s N^s\|f\|_{L^p}\\ &\||\nabla|^{\pm s} P_N f\|_{L^p} \lesssim_s N^{\pm s}\|P_N f\|_{L^p} \lesssim_s N^{\pm s}\|f\|_{L^p}\end{aligned} \tag{6.2.5}$$

for all $s \geq 0$, and all $1 \leq p \leq \infty$, independently of f, N, and p, where $|\nabla|^s$ is the classical fractional differentiation operator. We refer to Tao [28] for more details. Given $a \geq 1$, we let a' be the conjugate of a, so that $\frac{1}{a} + \frac{1}{a'} = 1$.

Several norms have to be considered in the analysis of the critical case of (6.1.1). For $I \subset \mathbb{R}$ an interval, they are defined as

$$\begin{aligned}\|u\|_{M(I)} &= \|\Delta u\|_{L^{\frac{2(n+4)}{n-4}}(I,L^{\frac{2n(n+4)}{n^2+16}})},\\ \|u\|_{W(I)} &= \|\nabla u\|_{L^{\frac{2(n+4)}{n-4}}(I,L^{\frac{2n(n+4)}{n^2-2n+8}})},\\ \|u\|_{Z(I)} &= \|u\|_{L^{\frac{2(n+4)}{n-4}}(I,L^{\frac{2(n+4)}{n-4}})}, \text{ and}\\ \|u\|_{N(I)} &= \|\nabla u\|_{L^2(I,L^{\frac{2n}{n+2}})}.\end{aligned} \tag{6.2.6}$$

Accordingly, we let $M(\mathbb{R})$ be the completion of $\mathcal{S}(\mathbb{R}^{n+1})$ with the norm $\|\cdot\|_{M(\mathbb{R})}$, and $M(I)$ be the set consisting of the restrictions to I of functions in $M(\mathbb{R})$. We adopt similar definitions for W, Z, and N. We also need the following stronger norms in order to fully exploit the Strichartz estimates in Section 6.3. Following standard notations, we say that a pair (q, r) is Schrödinger-admissible, for short S-admissible, if $2 \leq q, r \leq \infty$, $(q, r, n) \neq (2, \infty, 2)$, and

$$\frac{2}{q} + \frac{n}{r} = \frac{n}{2}. \tag{6.2.7}$$

We define the full Strichartz norm of regularity s by

$$\|u\|_{\dot{S}^s(I)} = \sup_{(a,b)} \left(\sum_N N^{2s+\frac{4}{a}}\|P_N u\|^2_{L^a(I,L^b)}\right)^{\frac{1}{2}}, \tag{6.2.8}$$

where the supremum is taken over all S-admissible pairs (a, b) as in (6.2.7), $s \in \mathbb{R}$ and $I \subset \mathbb{R}$ is an interval. We also define the dual norm,

$$\|h\|_{\dot{\bar{S}}^s(I)} = \inf_{(a,b)} \left(\sum_N N^{2s-\frac{4}{a}}\|P_N h\|^2_{L^{a'}(I,L^{b'})}\right)^{\frac{1}{2}} \tag{6.2.9}$$

where again, the infimum is taken over all S-admissible pairs (a,b) as in (6.2.7), $s\in\mathbb{R}$, and I is an interval. We let $\dot{S}^s(I)$ be the set of tempered distributions of finite $\dot{S}^s(I)$-norm. Finally, for a product $\pi=\Pi_i a_i$, we use the notation $\mathcal{O}(\pi)$ to denote an expression which is schematically like π, i.e. that is a finite combination of products $\pi'=\Pi_i b_i$ where in each π', each b_i stands for a_i or for $\bar{a}_i$.

6.3 Preliminary results

We recall results from Pausader [21]. We refer to Pausader [21] for their proof. A first result from Pausader [21] is that the following fundamental Strichartz-type estimates hold true. Note that these estimates, because of the gain of derivatives, contradict the Galilean invariance one could have expected for the fourth order Schrödinger equation.

Proposition 6.3.1. *Let $u\in C(I,H^{-4})$ be a solution of*

$$i\partial_t u+\Delta^2 u+h=0, \tag{6.3.1}$$

and $u(0)=u_0$. Then, for any S-admissible pairs (q,r) and (a,b) as in (6.2.7)*, and any $s\in\mathbb{R}$,*

$$\||\nabla|^s u\|_{L^q(I,L^r)}\lesssim\left(\||\nabla|^{s-\frac{2}{q}}u_0\|_{L^2}+\||\nabla|^{s-\frac{2}{q}-\frac{2}{a}}h\|_{L^{a'}(I,L^{b'})}\right) \tag{6.3.2}$$

whenever the right hand side in (6.3.2) *is finite.*

A consequence of the Strichartz estimates (6.3.2) and of the commutation properties of the linear propagator $e^{it\Delta^2}$ is the following estimate, for any solution u as above:

$$\begin{aligned}\|u\|_{\dot{S}^s(I)}&\lesssim\|u_0\|_{\dot{H}^s}+\|h\|_{\dot{\bar{S}}^s(I)}\\&\lesssim\|u_0\|_{\dot{H}^s}+\||\nabla|^{s-\frac{2}{a}}h\|_{L^{a'}(I,L^{b'})},\end{aligned} \tag{6.3.3}$$

where (a,b) is an S-admissible pair as in (6.2.7), and the norms are defined in (6.2.8) and (6.2.9) above. Let $u\in C(I,\dot{H}^2)$ be defined on some interval I such that $0\in I$ and such that $u\in L^3_{loc}(I\times\mathbb{R}^n)$. We say that u is a solution of (6.1.1) provided that the following equality holds in the sense of tempered distributions for all times:

$$u(t)=e^{it\Delta^2}u_0+i\int_0^t e^{i(t-s)\Delta^2}\left(|u|^2u\right)(s)ds. \tag{6.3.4}$$

Note that, by Strichartz estimates, if $u_0\in L^2$ and $|u|^2u\in L^1_{loc}(I,L^2)$, then (6.3.4) is equivalent to the fact that u solves (6.1.1) in H^{-4} with $u(0)=u_0$.

The following Propositions 6.3.2 and 6.3.3, still from Pausader [21], are important for the energy-critical case $n=8$. Proposition 6.3.2 settles the question of local well-posedness. Proposition 6.3.3 settles the question of stability.

Proposition 6.3.2. *Let $n=8$. There exists $\delta>0$ such that for any initial data $u_0\in\dot{H}^2$, and any interval $I=[0,T]$, if*

$$\|e^{it\Delta^2}u_0\|_{W(I)}<\delta, \tag{6.3.5}$$

then there exists a unique solution $u \in C(I, \dot{H}^2)$ *of* (6.1.1) *with initial data* u_0. *This solution has conserved energy, and satisfies* $u \in \dot{S}^2(I)$. *Moreover,*

$$\|u\|_{\dot{S}^2(I)} \lesssim \|u_0\|_{\dot{H}^2} + \delta^3, \tag{6.3.6}$$

and if $u_0 \in H^2$, *then* $u \in \dot{S}^0(I) \cap \dot{S}^2(I)$,

$$\|u\|_{\dot{S}^0(I)} \lesssim \|u_0\|_{L^2},$$

and u *has conserved mass. Besides, in this case, the solution depends continuously on the initial data in the sense that there exists* δ_0, *depending on* δ, *such that, for any* $\delta_1 \in (0, \delta_0)$, *if* $\|v_0 - u_0\|_{H^2} \leq \delta_1$, *and if we let* v *be the local solution of* (6.1.1) *with initial data* v_0, *then* v *is defined on* I *and* $\|u - v\|_{\dot{S}^0(I)} \lesssim \delta_1$.

In addition to Proposition 6.3.2 we also have Proposition 6.3.3.

Proposition 6.3.3. *Let* $n = 8$, $I \subset \mathbb{R}$ *be a compact time interval such that* $0 \in I$, *and* $\tilde{u}$ *be an approximate solution of* (6.1.1) *in the sense that*

$$i\partial_t \tilde{u} + \Delta^2 \tilde{u} + |\tilde{u}|^2 \tilde{u} = e \tag{6.3.7}$$

for some $e \in N(I)$. *Assume that* $\|\tilde{u}\|_{Z(I)} < +\infty$ *and* $\|\tilde{u}\|_{L^\infty(I, \dot{H}^2)} < +\infty$. *There exists* $\delta_0 > 0$, $\delta_0 = \delta_0(\Lambda, \|\tilde{u}\|_{Z(I)}, \|\tilde{u}\|_{L^\infty(I,\dot{H}^2)})$, *such that if* $\|e\|_{N(I)} \leq \delta$, *and* $u_0 \in \dot{H}^2$ *satisfies*

$$\|\tilde{u}(0) - u_0\|_{\dot{H}^2} \leq \Lambda \;\; \textit{and} \;\; \|e^{it\Delta^2}\left(\tilde{u}(0) - u_0\right)\|_{W(I)} \leq \delta \tag{6.3.8}$$

for some $\delta \in (0, \delta_0]$, *then there exists* $u \in C(I, \dot{H}^2)$ *a solution of* (6.1.1) *such that* $u(0) = u_0$. *Moreover,* u *satisfies*

$$\begin{aligned} &\|u - \tilde{u}\|_{W(I)} \leq C\delta \;, \\ &\|u - \tilde{u}\|_{\dot{S}^2} \leq C\left(\Lambda + \delta\right) \;, \; \textit{and} \\ &\|u\|_{\dot{S}^2} \leq C, \end{aligned} \tag{6.3.9}$$

where $C = C(\Lambda, \|\tilde{u}\|_{Z(I)}, \|\tilde{u}\|_{L^\infty(I,\dot{H}^2)})$ *is a nondecreasing function of its arguments.*

In our analysis, we need to consider $\dot{H}^2$-solutions. These solutions do not satisfy conservation of mass. However the next proposition shows that there is still something remaining from that conservation law for these solutions. Proposition 6.3.4 shows that the local mass of a solution of (6.1.1) varies slowly in time provided that the radius R is sufficiently large. We define the local mass $M\left(u, B(x_0, R)\right)$ over the ball $B(x_0, R)$ of a function $u \in L^2_{loc}$ by

$$M\left(u, B(x_0, R)\right) = \int_{\mathbb{R}^n} |u(x)|^2 \psi^4\left((x - x_0)/R\right) dx, \tag{6.3.10}$$

where, ψ is as in (6.2.4). Proposition 6.3.4 from Pausader [21], states as follows.

Proposition 6.3.4. *Let* $n \geq 5$, *and* $u \in C(I, \dot{H}^2)$ *be a solution of* (6.1.1). *Then we have that*

$$|\partial_t M\left(u(t), B(x_0, R)\right)| \lesssim \frac{E(u)^{\frac{3}{4}}}{R} M\left(u(t), B(x_0, R)\right)^{\frac{1}{4}} \tag{6.3.11}$$

for all $t \in I$.

We refer to Pausader [21] for a proof of the above propositions.

6.4 Ill-posedness results

In this section we use a quantitative analysis of the small dispersion regime to prove ill-posedness results for the cubic equation when $n > 8$. The idea is that now the equation is supercritical with repect to the regularity-setting in which we work, namely H^2. Hence one can always use rescaling arguments to make any "separation-mechanism" between two different solutions happen sooner and sooner while making the H^2-norm smaller and smaller. It remains then to find two solutions whose distance goes to ∞ as time evolves. To achieve this, we follow the proof in Christ, Colliander and Tao [4] by considering the small dispersion regime. See also Lebeau [19] for previous results, and Thomann [30, 31] for instability results in different contexts.

Before we prove our main theorem, we need the following lemma concerning the small dispersion regime.

Lemma 6.4.1. *Let $k > n/2$. Then, for any $\phi \in \mathcal{S}$, there exists $c > 0$ such that for any $\nu \in (0,1)$, there exists a unique solution $w^\nu \in C([-T,T], H^k)$ of the problem*

$$i\partial_t w + \nu^4\Delta^2 w + |w|^2 w = 0 \tag{6.4.1}$$

with initial data $w^\nu(0) = \phi$, where $T = c|\log\nu|^c$. Besides, the solution satisfies $w^\nu \in C([-T,T], H^p)$ for any p, and

$$\|w^\nu - w^0\|_{L^\infty([-T,T],H^k)} \lesssim_{\phi,k} \nu^3, \tag{6.4.2}$$

where

$$w^0(t,x) = \phi(x)\exp\left(i|\phi(x)|^2 t\right) \tag{6.4.3}$$

is a solution of the ODE formally obtained by setting $\nu = 0$ in (6.4.1).

Proof. Letting $u = w^\nu - w^0$, we see that u solves the Cauchy problem

$$i\partial_t u + \nu^4\Delta^2 u = \nu^4\Delta^2 w^0 + |w^0|^2 w^0 - |w^0 + u|^2(w^0 + u) \tag{6.4.4}$$

with $u(0) = 0$. Let $k > n/2$ be given. Since $w^0 \in C^\infty(\mathcal{S})$, standard developments ensure that there exists a unique solution $u \in C([-t,t], H^k)$ to (6.4.4), and that u can be continued as long as $\|u\|_{H^k}$ remains bounded. Besides, $u \in C([-t,t], H^p)$ for any $p \geq 0$ (in the sense that t does not depend on p). Consequently, it suffices to prove that there exists $c > 0$ such that for any $s < c|\log\nu|^c$, we have that $\|u(s)\|_{H^k} \leq \nu^3$. Now, taking derivatives ∂^α of equation (6.4.4), multiplying by $\partial^\alpha\bar{u}$, taking the imaginary part and integrating, for all α such that $|\alpha| \leq k$, we get that

$$\begin{aligned}&\partial_s\|u(s)\|_{H^k}^2\\ &\lesssim \|u\|_{H^k}\left(\nu^4\|\Delta^2 w^0(s)\|_{H^k} + \||w^0+u|^2(w^0+u) - |w^0|^2 w^0\|_{H^k}\right).\end{aligned} \tag{6.4.5}$$

By (6.4.3) we see that, for $p \geq 0$,

$$\|w^0\|_{H^p} \lesssim_{\phi,p} t^p. \tag{6.4.6}$$

Independently, since H^k is an algebra, we get that

$$\begin{aligned}\||w^0+u|^2(w^0+u) - |w^0|^2 w^0\|_{H^k} &\lesssim \sum_{j=0}^{2}\|\mathcal{O}\left(\left(w^0\right)^j u^{3-j}\right)\|_{H^k}\\ &\lesssim \|u\|_{H^k}\left(1 + \|w^0\|_{H^k} + \|u\|_{H^k}\right)^2.\end{aligned} \tag{6.4.7}$$

Now, using (6.4.5)–(6.4.7), we see that, in the sense of distributions,

$$\partial_s \|u(s)\|_{H^k} \lesssim_{\phi,k} \nu^4 \left(1+|s|^{k+4}\right) + \|u(s)\|_{H^k} \left(1+|s|^k + \|u(s)\|_{H^k}\right)^2. \quad (6.4.8)$$

An application of Gromwall's lemma gives the bound

$$\|u(s)\| \lesssim_{k,\phi} \nu^4 \exp\left(C\left(1+|s|^C\right)\right) \quad (6.4.9)$$

for all s such that $\|u(s)\|_{H^k} \leq 1$. By (6.4.9) we see that $\|u(s)\|_{H^k} \leq 1$ holds for all times $|s| \leq c|\log \nu|^c$, $c > 0$ sufficiently small. This gives (6.4.2) and finishes the proof of Lemma 6.4.1. □

Now, we are in position to prove the main theorem of this section which states that the flow map $u_0 \mapsto u(t)$, from H^2 into H^2 which maps the initial data to the associated solution fails to be continuous at 0. As a remark, note that (6.4.10) is false when $n \leq 8$ since the H^2-norm controls the energy.

Theorem 17. *Let $n > 8$. Given $\varepsilon > 0$, there exists a solution $u \in C([0,\varepsilon], H^2)$ such that*

$$\|u(0)\|_{H^2} < \varepsilon \quad \text{and} \quad \|u(t_\varepsilon)\|_{H^2} > \varepsilon^{-1}, \quad (6.4.10)$$

for some $t_\varepsilon \in (0,\varepsilon)$. Besides, we can choose u such that $u(0) \in \mathcal{S}$ and $u \in C([0,\varepsilon], H^k)$ for any $k > 0$.

Proof of Theorem 17. For $\phi \in \mathcal{S}$ and $\nu \in (0,1]$, we let w^ν be the solution of equation (6.4.1) with initial data $w^\nu(0) = \phi$. By Lemma 6.4.1, we see that for $|s| \leq c|\log \nu|^c$, (6.4.2) holds true for w^0 as in (6.4.3). Now, for $\lambda \in (0,\infty)$, we let

$$u^{(\nu,\lambda)}(t,x) = \lambda^2 w^\nu(\lambda^4 t, \lambda \nu x). \quad (6.4.11)$$

Then $u^{(\nu,\lambda)}$ solves (6.1.1) with initial data $u^{(\nu,\lambda)}(0,x) = \lambda^2\phi(\lambda\nu x)$. A simple calculation gives

$$\begin{aligned} \|u^{(\nu,\lambda)}(0)\|_{H^2}^2 &= \frac{\lambda^4}{(2\pi)^n} (\lambda\nu)^{-2n} \int_{\mathbb{R}^n} |\hat{\phi}(\xi/(\lambda\nu))|^2 (1+|\xi|^2)^2 d\xi \\ &\lesssim \lambda^4 (\lambda\nu)^{-n} \Big(\int_{\mathbb{R}^n} |\hat{\phi}(\eta)|^2 |\lambda\nu\eta|^4 d\eta + \int_{\mathbb{R}^n} |\hat{\phi}(\eta)|^2 d\eta \Big) \\ &\lesssim_\phi \lambda^4 (\lambda\nu)^{4-n}, \end{aligned} \quad (6.4.12)$$

provided that $\lambda\nu \geq 1$. Now, given $\varepsilon > 0$, and $\nu > 0$, we fix

$$\lambda = \lambda_{\nu,\varepsilon} = \left(\varepsilon^2 \nu^{n-4}\right)^{-\frac{1}{n-8}} \quad (6.4.13)$$

such that $\lambda^4 (\lambda\nu)^{4-n} = \varepsilon^2$, and $\lambda\nu = \left(\varepsilon\nu^2\right)^{-\frac{2}{n-8}} > 1$. Independently, by (6.4.3), we see that

$$\|w^0(t)\|_{\dot{H}^2} \gtrsim_\phi t^2 + O(t),$$

and, consequently, using (6.4.2), we get that for $|s| \leq c|\log \nu|^c$ sufficiently large independently of ν, there holds that

$$\|w^\nu(s)\|_{\dot{H}^2} \gtrsim_\phi s^2. \quad (6.4.14)$$

Consequently, using (6.4.11), (6.4.13) and (6.4.14) we get that

$$\begin{aligned}\|u^{(\nu,\lambda)}(\lambda^{-4}t)\|_{H^2}^2 &\geq \|u^{(\nu,\lambda)}(\lambda^{-4}t)\|_{\dot{H}^2}^2 \\ &\geq \lambda^4 (\lambda\nu)^{4-n} \|w^\nu(t)\|_{\dot{H}^2}^2 \\ &\gtrsim_\phi \varepsilon^2 t^4\end{aligned} \tag{6.4.15}$$

for t sufficiently large. Now, given ε, we let $\nu > 0$ be sufficiently small such that

$$\begin{aligned}&\varepsilon^2 t_\nu^4 > \varepsilon^{-2}\ , \text{ for } t_\nu = c|\log\nu|^c\ , \text{ and} \\ &\varepsilon^{\frac{16-n}{n-8}} \nu^{\frac{4(n-4)}{n-8}} < \varepsilon.\end{aligned} \tag{6.4.16}$$

We choose $\lambda = \lambda_{\nu,\varepsilon}$ as in (6.4.13). Using (6.4.16), we get that $t_\varepsilon = \lambda^{-4}t_\nu < \varepsilon$, and then (6.4.12) and (6.4.15) give (6.4.10). This finishes the proof. □

6.5 Reduction to three scenarii

¿From now on we start with the analysis of the energy-critical case $n = 8$. In this section we prove that the analysis can be reduced to the study of some very special solutions. In order to do so, we borrow ideas from previous works developed in the context of Schrödinger and wave equations by Bahouri and Gerard [1], Kenig and Merle [16], Keraani [17], Killip, Tao and Visan [18], and Tao, Visan and Zhang [29]. We refer also to Pausader [22] for a similar result developed in the context of the L^2-critical fourth-order Schrödinger equation. For any $E > 0$, we let

$$\Lambda(E) = \sup\{\|u\|_{Z(I)}^6 : E(u) \leq E\}, \tag{6.5.1}$$

where the supremum is taken over all maximal-lifespan solutions $u \in C(I, \dot{H}^2)$ of (6.1.1) satisfying $E(u) \leq E$. In light of Proposition 6.3.2 and of the Strichartz estimates (6.3.2), we know that there exists $\delta > 0$ such that, for any $E \leq \delta$, $\Lambda(E) \lesssim_\delta E < +\infty$. Besides, Λ is clearly an increasing function of E. Hence, we can define

$$E_{max} = \sup\{E > 0 : \Lambda(E) < \infty\}. \tag{6.5.2}$$

The goal in Sections 6.5–6.10 is to prove that $E_{max} = +\infty$. Theorem 18 below is a first step in this direction.

Theorem 18. *Suppose that $E_{max} < +\infty$. There exists $u \in C(I, \dot{H}^2)$ a maximal-lifespan solution of energy exactly E_{max} such that the $Z(I')$-norm of u is infinite for $I' = (T_*, 0)$ and $I' = (0, T^*)$, where $I = (T_*, T^*)$. Besides, there exist two smooth functions $h : I \to \mathbb{R}_+^*$ and $x : I \to \mathbb{R}^n$ such that*

$$K = \{g_{(h(t),x(t))}u(t) : t \in I\} \tag{6.5.3}$$

is precompact in $\dot{H}^2$, where the transformation $g(t) = g_{(h(t),x(t))}$ is as in (6.1.3). Furthermore, one can assume that one of the following three scenarii holds true: (soliton-like solution) there holds $I = \mathbb{R}$ and $h(t) = 1$ for all t; (double low-to-high cascade) there holds $\liminf_{t\to\bar{T}} h(t) = 0$ for $\bar{T} = T_*, T^*$, and $h(t) \leq 1$ for all t; (self-similar solution) there holds $I = (0, +\infty)$ and $h(t) = t^{\frac{1}{4}}$ for all t.

As a remark, since $E(u) = E_{max}$, the solution u in Theorem 18 is such that $u \neq 0$. Assuming Propositions 6.6.1, 6.9.1 and 6.10.1 which exclude the three scenarii in Theorem 18, the following corollary holds true.

Corollary 6.5.1. *For any $E > 0$, there exists $C = C(E)$ such that, for any $u_0 \in \dot{H}^2$ satisfying $E(u_0) \leq E$, if $u \in C(I, \dot{H}^2)$ is the maximal solution of* (6.1.1) *with initial data $u(0) = u_0$, then $I = \mathbb{R}$ and $\|u\|_{\dot{S}^2(\mathbb{R})} \leq C$.*

Proof of Corollary 6.5.1. First, using [21, Proposition 2.6.], we see that a bound on the Z-norm of u implies a bound on the $\dot{S}^2$-norm of u. Hence if Corollary 6.5.1 is false, then $E_{max} < +\infty$. Applying Theorem 18, we find a maximal solution satisfying one of the three scenarii in Theorem 18. Then, using Propositions 6.6.1, 6.9.1 and 6.10.1, we get a contradiction. Hence $E_{max} = +\infty$. ☐

Now we prove Theorem 18.

Proof of Theorem 18. In several ways the proof is similar to the one developed in the L^2-critical case in Pausader [22]. We prove the more general statement that Theorem 18 holds true in any dimension $n \geq 5$ when (6.1.1) is replaced by the $\dot{H}^2$-critical equation. In particular, this is the case when $n = 8$. Therefore, in this proof, (6.1.1) always refers to the energy-critical equation in dimension n, and the energy E and Λ must be replaced by

$$\begin{aligned} E(u) &= \int_{\mathbb{R}^n} \left(\frac{1}{2}|\Delta u(x)|^2 + \frac{n-4}{2n}|u(x)|^{\frac{2n}{n-4}} \right) dx \text{ and} \\ \Lambda(E) &= \sup\{\|u\|_Z^{\frac{2(n+4)}{n-4}} : E(u) \leq E\}, \end{aligned}$$

where the supremum is taken over all maximal solutions of the energy-critical equation of energy less or equal to E. Besides, the definition of τ and g as in (6.1.2) and (6.1.3) and Propositions 6.3.2 and 6.3.3 refer to their n-dimensional energy-critical counterparts. A consequence of the precised Sobolev's inequality in Gerard, Meyer and Oru [8] and of the Strichartz estimates (6.3.2) is that, for any $u_0 \in \dot{H}^2$,

$$\begin{aligned} \|e^{it\Delta^2}u_0\|_{Z(\mathbb{R})} &\lesssim \|e^{it\Delta^2}|\nabla|u_0\|^{\frac{n-4}{n-2}}_{L^{\frac{2(n+4)}{n-2}}L^{\frac{2(n+4)}{n-2}}} \|e^{it\Delta^2}|\nabla|u_0\|^{\frac{2}{n-2}}_{L^\infty L^{\frac{2n}{n-2}}} \\ &\lesssim \|u_0\|^{\frac{n-4}{n-2}}_{\dot{H}^2} \|e^{it\Delta^2}|\nabla|u_0\|^{\frac{2}{n}}_{L^\infty \dot{H}^1} \|e^{it\Delta^2}|\nabla|u_0\|^{\frac{4}{n(n-2)}}_{L^\infty \dot{B}^1_{2,\infty}} \qquad (6.5.4) \\ &\lesssim \|u_0\|^{\frac{n^2-2n-4}{n(n-2)}}_{\dot{H}^2} \|u_0\|^{\frac{4}{n(n-2)}}_{\dot{B}^2_{2,\infty}}, \end{aligned}$$

where for $s = 1, 2$, $\dot{B}^s_{2,\infty}$ is a standard homogeneous Besov space. Now, thanks to (6.5.4), we may follow the analysis in Bahouri and Gerard [1] and Keraani [17]. In the following, we call scale-core a sequence (h_k, t_k, x_k) such that for every k, $h_k > 0$, $t_k \in \mathbb{R}$ and $x_k \in \mathbb{R}^n$. Mimicking the proof in Keraani [17] we obtain that for $(v_k)_k$ a bounded sequence in $\dot{H}^2$, there exists a sequence $(V^\alpha)_\alpha$ in $\dot{H}^2$, and scale-cores $(h_k^\alpha, t_k^\alpha, x_k^\alpha)$ such that for any $\alpha \neq \beta$,

$$\left| \log \frac{h_k^\alpha}{h_k^\beta} \right| + (h_k^\alpha)^4 \left| t_k^\alpha - t_k^\beta \right| + h_k^\alpha \left| x_k^\alpha - x_k^\beta \right| \to +\infty \qquad (6.5.5)$$

as $k \to +\infty$, with the property that, up to a subsequence, for any $A \geq 1$,

$$v_k = \sum_{\alpha=1}^{A} g_{(h_k^\alpha, x_k^\alpha)} \left(e^{-i(h_k^\alpha)^4 t_k^\alpha \Delta^2} V^\alpha \right) + w_k^A \tag{6.5.6}$$

for all k, where $w_k^A \in \dot{H}^2$ for all k and A, and

$$\lim_{A \to +\infty} \limsup_{k \to +\infty} \| e^{it\Delta^2} w_k^A \|_Z = 0. \tag{6.5.7}$$

Moreover, we have the following estimates:

$$\begin{aligned} \| e^{it\Delta^2} v_k \|_Z^{\frac{2(n+4)}{n-4}} &= \sum_{\alpha=1}^{+\infty} \| e^{it\Delta^2} V^\alpha \|_Z^{\frac{2(n+4)}{n-4}} + o(1) \text{ and,} \\ E(v_k) &= \sum_{\alpha=1}^{A} E(e^{-i(h_k^\alpha)^4 t_k \Delta^2} V^\alpha) + \| w_k^A \|_{\dot{H}^2}^2 + o(1) \end{aligned} \tag{6.5.8}$$

for all k, where $o(1) \to 0$ as $k \to +\infty$. Let $(V, (h_k)_k, (t_k)_k, (x_k)_k)$ be such that $V \in \dot{H}^2$, and $(h_k, t_k, x_k) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^n$ is a scale-core such that $h_k^4 t_k$ has a limit $l \in [-\infty, +\infty]$ as $k \to +\infty$. We say that U is the nonlinear profile associated to $(V, (h_k)_k, (t_k)_k, (x_k)_k)$ if U is a solution of (6.1.1) defined on a neighborhood of $-l$, and

$$\| U(-h_k^4 t_k) - e^{-ih_k^4 t_k \Delta^2} V \|_{\dot{H}^2} \to 0$$

as $k \to +\infty$. Using the analysis in Pausader [21], it is easily seen that a nonlinear profile always exists and is unique. Besides if

$$E(U) = \lim_k E(e^{-ih_k^4 t_k \Delta^2} V) \tag{6.5.9}$$

is such that $E(U) < E_{max}$, then the associated nonlinear profile U is globally defined, and

$$\| U \|_{\dot{S}^2(\mathbb{R})} \lesssim_{E(U)} 1.$$

Now, we enter more specifically into the proof of Theorem 18. A consequence of Proposition 6.3.3 is that there exists a sequence of nonlinear solutions u_k such that $E(u_k) < E_{max}$, $E(u_k) \to E_{max}$, and

$$\| u_k \|_{Z(-\infty,0)} \ , \ \| u_k \|_{Z(0,+\infty)} \to +\infty. \tag{6.5.10}$$

We let $((h_k^\alpha)_k, (t_k^\alpha)_k, (x_k^\alpha)_k) = (\mathbf{h}^\alpha, \mathbf{z}^\alpha)$, V^α, and $\mathbf{w}^A$ be given by (6.5.6) applied to the sequence $(v_k = u_k(0))_k$. Passing to subsequences, and using a diagonal extraction argument, we can assume that, for all α, $(h_k^\alpha)^4 t_k^\alpha$ has a limit in $[-\infty, \infty]$. We let U^α be the nonlinear profile associated to $(V^\alpha, \mathbf{h}^\alpha, \mathbf{z}^\alpha)$. Suppose first that there exists α such that $0 < E(U^\alpha) < E_{max}$. Then, applying (6.5.8) and (6.5.9), we see that there exists $\varepsilon > 0$ such that for any β, $E\left(U^\beta\right) < E_{max} - \varepsilon$, and we get that all the nonlinear profiles are globally defined. Letting $W_k^A(t) = e^{it\Delta^2} w_k^A$, we remark that

$$p_k^A = \sum_{\alpha=1}^{A} \tau_{(h_k^\alpha, z_k^\alpha)} U^\alpha + W_k^A$$

satisfies (6.3.7) with

$$e = e_k^A = f(\sum_{\alpha=1}^{A} \tau_{(h_k^\alpha, z_k^\alpha)} U^\alpha + W_k^A) - \sum_{\alpha=1}^{A} f(\tau_{(h_k^\alpha, z_k^\alpha)} U^\alpha)$$

and initial data $p_k^A(0) = u_k(0) + o_A(1)$, where $f(x) = |x|^{\frac{8}{n-4}} x$. First, we claim that

$$\limsup_k \| \sum_{\alpha=1}^{A} \tau_{(h_k^\alpha, z_k^\alpha)} U^\alpha \|_Z \lesssim_{E_{max},\varepsilon} 1 \tag{6.5.11}$$

independently of A. Indeed, we remark that when $(h_k^\alpha, t_k^\alpha, x_k^\alpha)$ and $(h_k^\beta, t_k^\beta, x_k^\beta)$ satisfy (6.5.5), then for any u, v with finite Z-norm, there holds that

$$\| |\tau_{(h_k^\beta, t_k^\beta, x_k^\beta)} v|^{\frac{n+12}{n-4}} \tau_{(h_k^\alpha, t_k^\alpha, x_k^\alpha)} u \|_{L^1(\mathbb{R}, L^1)} \to 0 \tag{6.5.12}$$

as $k \to +\infty$, where $\tau_{(h_k, t_k, x_k)}$ is as in (6.1.2). Now, since Λ is sublinear around 0, and bounded on $[0, E_{max} - \varepsilon]$, using (6.5.8) and (6.5.12), we get that

$$\begin{aligned}
\limsup_k \| \sum_{\alpha=1}^{A} \tau_{(h_k^\alpha, z_k^\alpha)} U^\alpha \|_Z &= \left(\sum_{\alpha=1}^{A} \| U^\alpha \|_Z^{\frac{2(n+4)}{n-4}} \right)^{\frac{n-4}{2(n+4)}} \\
&\lesssim \left(\sum_{\alpha=1}^{A} \Lambda(E(U^\alpha)) \right)^{\frac{n-4}{2(n+4)}} \\
&\lesssim_{E_{max},\varepsilon} \left(\sum_{\alpha=1}^{A} E\left(U^\alpha\right) \right)^{\frac{n-4}{2(n+4)}} \\
&\lesssim_{E_{max},\varepsilon} 1.
\end{aligned}$$

Using again (6.5.12), we get that

$$\| f(\sum_{\alpha=1}^{A} \tau_{(h_k^\alpha, z_k^\alpha)} U^\alpha) - \sum_{\alpha=1}^{A} f(\tau_{(h_k^\alpha, z_k^\alpha)} U^\alpha) \|_{L^2(\mathbb{R}, L^2)} = o_A(1) \tag{6.5.13}$$

as $k \to +\infty$. On the other hand, using the blow-up criterion in Pausader [21, Proposition 2.6.], and the bound $\|U^\alpha\|_Z \leq \Lambda\left(E(U^\alpha)\right) \leq \Lambda\left(E_{max} - \varepsilon\right)$, we get that, for any α,

$$\|U^\alpha\|_M \lesssim_{E_{max},\varepsilon} 1.$$

Using the Leibnitz and chain rules for fractional derivative in Kato [15] and Visan [32, Appendix A], we obtain that

$$\| f(\sum_{\alpha=1}^{A} \tau_{(h_k^\alpha, z_k^\alpha)} U^\alpha) - \sum_{\alpha=1}^{A} f(\tau_{(h_k^\alpha, z_k^\alpha)} U^\alpha) \|_{L^2(\mathbb{R}, \dot{H}^{\frac{n+8}{n+4}, \frac{2n(n+4)}{n^2+6n+16}})} \lesssim_{A, E_{max},\varepsilon} 1. \tag{6.5.14}$$

Interpolating between (6.5.13) and (6.5.14), we get that

$$\| f(\sum_{\alpha=1}^{A} \tau_{(h_k^\alpha, z_k^\alpha)} U^\alpha) - \sum_{\alpha=1}^{A} f(\tau_{(h_k^\alpha, z_k^\alpha)} U^\alpha) \|_N = o_A(1). \tag{6.5.15}$$

Now, we claim that, letting $s_k^A = \sum_{\alpha=1}^{A} \tau_{(h_k^\alpha, z_k^\alpha)} U^\alpha$, there holds that

$$\limsup_k \|s_k^A\|_M \lesssim_{E_{max},\varepsilon} 1, \tag{6.5.16}$$

independently of A. Indeed, s_k^A satisfies the equation

$$i\partial_t s_k^A + \Delta^2 s_k^A + \sum_{\alpha=1}^{A} f(\tau_{(h_k^\alpha, z_k^\alpha)} U^\alpha) = 0,$$

with initial data

$$s_k^A(0) = \sum_{\alpha=1}^{A} \tau_{(h_k^\alpha, z_k^\alpha)} U^\alpha(0) = \sum_{\alpha=1}^{A} g_{(h_k^\alpha, x_k^\alpha)} e^{-i(h_k^\alpha)^4 t_k^\alpha \Delta^2} V^\alpha + o_A(1),$$

and consequently (6.5.8) and (6.5.9) give that

$$\|s_k^A(0)\|_{\dot{H}^2}^2 \leq 2E\left(s_k^A(0)\right) \lesssim_{E_{max}} 1 + o_A(1).$$

Using the Strichartz estimates (6.3.2), (6.5.11) and (6.5.15), we get that

$$\begin{aligned}
\|s_k^A\|_M &\lesssim \|s_k^A(0)\|_{\dot{H}^2} + \|\sum_{\alpha=1}^{A} f(\tau_{(h_k^\alpha, z_k^\alpha)} U^\alpha)\|_N \\
&\lesssim E\left(s_k^A(0)\right)^{\frac{1}{2}} + o_A(1) + \|f(\sum_{\alpha=1}^{A} \tau_{(h_k^\alpha, z_k^\alpha)} U^\alpha)\|_N \\
&\lesssim_{E_{max}} 1 + o_A(1) + \|s_k^A\|_Z^{\frac{8}{n-4}} \|s_k^A\|_W \\
&\lesssim_{E_{max}} 1 + o_A(1) + \|s_k^A\|_Z^{\frac{8}{n-4}} \|s_k^A\|_Z^{\frac{1}{2}} \|s_k^A\|_M^{\frac{1}{2}} \\
&\lesssim_{E_{max},\varepsilon} 1 + o_A(1) + \|s_k^A\|_M^{\frac{1}{2}} \\
&\lesssim_{E_{max},\varepsilon} 1 + o_A(1)
\end{aligned} \tag{6.5.17}$$

and (6.5.17) proves (6.5.16). Independently,

$$\begin{aligned}
&\|f(\sum_{\alpha=1}^{A} \tau_{(h_k^\alpha, z_k^\alpha)} U^\alpha + W_k^A) - f(\sum_{\alpha=1}^{A} \tau_{(h_k^\alpha, z_k^\alpha)} U^\alpha)\|_{L^2(\mathbb{R},L^2)} \\
&\lesssim \|W_k^A\|_Z \left(\|W_k^A\|_Z^{\frac{8}{n-4}} + \|\sum_{\alpha=1}^{A} \tau_{(h_k^\alpha, z_k^\alpha)} U^\alpha\|_Z^{\frac{8}{n-4}} \right) \\
&\lesssim_{E_{max},\varepsilon} \|W_k^A\|_Z \left(\|W_k^A\|_Z^{\frac{8}{n-4}} + 1 \right) \\
&\lesssim_{E_{max},\varepsilon} \|W_k^A\|_Z
\end{aligned} \tag{6.5.18}$$

and again, using (6.5.16) and the product and Leibnitz rules for fractional derivatives, we get that

$$\|f(\sum_{\alpha=1}^{A} \tau_{(h_k^\alpha, z_k^\alpha)} U^\alpha + W_k^A) - f(\sum_{\alpha=1}^{A} \tau_{(h_k^\alpha, z_k^\alpha)} U^\alpha)\|_{L^2(\mathbb{R},\dot{H}^{\frac{n+8}{n+4}, \frac{2n(n+4)}{n^2+6n+1}})} \lesssim_{E_{max},\varepsilon} 1. \tag{6.5.19}$$

Interpolating between (6.5.18) and (6.5.19), we obtain that

$$\|f(\sum_{\alpha=1}^{A}\tau_{(h_k^\alpha,z_k^\alpha)}U^\alpha + W_k^A) - f(\sum_{\alpha=1}^{A}\tau_{(h_k^\alpha,z_k^\alpha)}U^\alpha)\|_N \lesssim_{E_{max},\varepsilon} \|W_k^A\|_Z^{\frac{4}{n+8}} \quad (6.5.20)$$

and (6.5.7), (6.5.15) and (6.5.20) show that

$$\limsup_k \|e_k^A\|_N = o(1) \quad (6.5.21)$$

as $A \to +\infty$. Independently,

$$\begin{aligned}\|p_k^A\|_W &\leq \|\sum_{\alpha=1}^{A}\tau_{(h_k^\alpha,z_k^\alpha)}U^\alpha\|_W + \|W_k^A\|_W \\ &\lesssim_{E_{max},\varepsilon} 1 + o_A(1).\end{aligned} \quad (6.5.22)$$

Now using Proposition 6.3.3, (6.5.21) and (6.5.22), since $p_k^A(0) = u_k(0) + o_A(1)$, we get that

$$\begin{aligned}\limsup_k \|u_k\|_Z^{\frac{2(n+4)}{n-4}} &\lesssim \lim_{A\to+\infty}\limsup_k \|p_k^A\|_Z^{\frac{2(n+4)}{n-4}} \\ &\lesssim \sum_\alpha \|U^\alpha\|_Z^{\frac{2(n+4)}{n-4}} \lesssim_{E_{max},\varepsilon} \sum_\alpha E\left(U^\alpha\right) \lesssim_{E_{max},\varepsilon} 1\end{aligned}$$

and this contradicts (6.5.10). Now, suppose that for all α, we have that $V^\alpha = 0$. Then Strichartz estimates (6.3.2) and (6.5.8) give that

$$\begin{aligned}\|e^{it\Delta^2}u_k(0)\|_W &\leq \|e^{it\Delta^2}u_k(0)\|_M^{\frac{1}{2}}\|e^{it\Delta^2}u_k(0)\|_Z^{\frac{1}{2}} \\ &\lesssim E_{max}^{\frac{1}{2}}\|e^{it\Delta^2}u_k(0)\|_Z^{\frac{1}{2}} \to 0\end{aligned}$$

as $k \to +\infty$, and Proposition 6.3.2 gives that $\|u_k\|_Z \to 0$, which contradicts (6.5.10). Consequently, we know that there exists a scale core (h_k, t_k, y_k), and $V \in \dot{H}^2$ such that

$$u_k(0) = g_{(h_k,y_k)}e^{-it_kh_k^4\Delta^2}V + w_k,$$

where $E(w_k) \to 0$. Now, up to passing to a subsequence, we can assume that $t_kh_k^4 \to l \in [-\infty, +\infty]$. If $l \in \mathbb{R}$, then, replacing V by $e^{-il\Delta^2}V$, we can assume that $l = 0$, and changing slightly w_k, we can assume that for any k, $t_k = 0$. We then get that $u_k(0) = g_{(h_k,y_k)}V + o(1)$ in $\dot{H}^2$, and in particular $E(V) = E_{max}$. Otherwise, by time reversal symmetry, we can assume that $l = -\infty$, and then, we find that

$$\begin{aligned}\|e^{it\Delta^2}u_k(0)\|_{Z([0,+\infty))} &\leq \|\tau_{(h_k,t_k,y_k)}\left(e^{it\Delta^2}V\right)\|_{Z([0,+\infty))} + \|w_k\|_{Z([0,+\infty))} \\ &\leq \|e^{it\Delta^2}V\|_{Z([-h_k^4t_k,+\infty))} + o(1) \\ &= o(1),\end{aligned}$$

and by standard developements, we get that, for k sufficiently large, $\|u_k\|_{Z(\mathbb{R}_+)}$ remains bounded. Once again, this contradicts (6.5.10). Let U be the maximal nonlinear solution of (6.1.1) with initial data V, defined on $I = (-T_*, T^*)$.

Suppose, for example that $T^* = +\infty$, and that $\|U\|_{Z(\mathbb{R}_+)} < +\infty$. Then, using Proposition 6.3.3 on $\mathbb{R}_+$ with $v = U$, and $u = \tau_{(h_k^{-1},0,-y_k)} u_k$, we see that $\|u_k\|_{Z(\mathbb{R}_+)}$ is bounded uniformly in k, which is a contradiction with (6.5.10). Consequently, we have that

$$\|U\|_{Z(0,T^*)} = \|U\|_{Z(-T_*,0)} = +\infty$$

and $E(U) = E_{max}$. Now, we prove the compactness property of U. In the sequel, we let $N_{min} > 0$ be sufficiently small so that $\|u\|_{\dot{H}^2} \leq N_{min}$ implies $E(u) < E_{max}/4$. Proceeding as above, it is easily proved by contradiction that for any $\varepsilon > 0$, there exist $t_1, \ldots, t_j$, $j = j(\varepsilon)$, such that for any time $t \in (-T_*, T^*)$, there exist $i = i(t)$, and $g(t) = g_{(h(t),y(t))}$ with the property that $\|u(t_i) - g(t)u(t)\|_{\dot{H}^2} \leq \varepsilon$. Let us apply this with $\varepsilon = N_{min}$. We get a function $g(t) = g_{(h(t),y(t))}$, and a finite set of times $t_1, \ldots, t_j$ such that for any t, there exists i satisfying

$$\|u(t_i) - g(t)u(t)\|_{\dot{H}^2} \leq N_{min}.$$

We claim that $K = \{g(t)u(t) : t \in (-T_*, T^*)\}$ is precompact in $\dot{H}^2$. Suppose by contradiction that this is not true. Then, there exist $\varepsilon > 0$, and a sequence s_k such that for any k and p,

$$\|g(s_k)u(s_k) - g(s_p)u(s_p)\|_{\dot{H}^2} > \varepsilon. \tag{6.5.23}$$

According to what we said above, and passing to a subsequence, we can assume that there exist two times $\bar{t}, \bar{t}'$, and a sequence $g'_k = g_{(h'_k,y'_k)}$ such that, for any k,

$$\begin{aligned} &\|u(\bar{t}) - g(s_k)u(s_k)\|_{\dot{H}^2} < N_{min}, \text{ and} \\ &\|u(\bar{t}') - g'_k u(s_k)\|_{\dot{H}^2} < \frac{\varepsilon}{4}. \end{aligned} \tag{6.5.24}$$

Passing to a subsequence, it is easily seen that that $(h'_k)^{-1} h(s_k)$ remains in a compact subset of $(0,\infty)$ and that and $y(s_k) - h(s_k)^{-1}h'_k y'_k$ remains in a compact subset of $\mathbb{R}^n$. Hence, up to considering a subsequence, we can find g_∞ such that $g(s_k)\,(g'_k)^{-1} \to g_\infty$ strongly. Now, using (6.5.24) and the fact that $g_{(h,y)}$ is an isometry on $\dot{H}^2$ for all (h,y), we get that

$$\begin{aligned} &\|g(s_k)u(s_k) - g(s_{k+1})u(s_{k+1})\|_{\dot{H}^2} \\ &\leq \|g(s_k)u(s_k) - g_\infty u(\bar{t}')\|_{\dot{H}^2} + \|g_\infty u(\bar{t}') - g(s_{k+1})u(s_{k+1})\|_{\dot{H}^2} \\ &\leq \|g'_k u(s_k) - g'_k g(s_k)^{-1} g_\infty u(\bar{t}')\|_{\dot{H}^2} + \|g'_{k+1}u(s_{k+1}) - g'_{k+1}g(s_{k+1})^{-1}g_\infty u(\bar{t}')\|_{\dot{H}^2} \\ &\leq \frac{\varepsilon}{2} + o(1). \end{aligned}$$

Clearly, this contradicts (6.5.23) and proves the compactness property of K. The remaining part follows the line of the work in Tao, Visan and Zhang [29] and Killip, Tao and Visan [18]. However, in order to obtain a low-to-high cascade (instead of a high-to-low cascade), we make the following slight modification. We use the notations in Killip, Tao and Visan [18], except for $h(t) = N(t)^{-1}$. In case $Osc(\kappa)$ is unbounded, instead of a, we introduce the quantity

$$b(t_0) = \inf\left(\frac{h(t_0)}{\inf_{t\geq t_0} h(t)}, \frac{h(t_0)}{\inf_{t\leq t_0} h(t)}\right).$$

Then, if $\sup_{t_0 \in J} b(t_0) = +\infty$, we can find intervals on which the solution presents arbitrarily large relative peak. In particular it becomes possible to find a solution satisfying the low-to-high cascade scenario. Finally, in case $\sup_{t_0 \in J} b(t_0) < +\infty$, the solution has arbitrarily large oscillation, but no relative peak. Mimicking the proof in Killip, Tao and Visan [18], but changing future (resp past)-focusing time into future (resp past)-defocusing time, one can find a solution behaving as in the self-similar case scenario. Theorem 18 follows. □

6.6 The self-similar case

In this section, we deal with the easiest case in Theorem 18, namely, the self-similar-like solution. We prove that it is not consistent with conservation of the energy, compactness up to rescaling, and almost conservation of the local L^2-norm as expressed in (6.3.11). More precisely, we prove that the following proposition holds true.

Proposition 6.6.1. *Let $u \in C(I, \dot{H}^2)$ be a maximal-lifespan solution such that $K = \{g(t)u(t) : t \in I\}$ is precompact in $\dot{H}^2$ for some function g as in* (6.1.3). *If $n = 8$, and $I \neq \mathbb{R}$, then $u = 0$. In particular, the self-similar scenario in Theorem 18 does not hold true.*

Proof. Let $u \in C(I, \dot{H}^2)$ be a solution as above, with $I \neq \mathbb{R}$, and let $v(t) = g(t)u(t)$. Without loss of generality, we can assume that $\inf I = 0$ and that $(0, 2) \subset I$. Fix $0 < t < 1$. First, using Hölder's inequality, we get that, for any $\delta > 0$,

$$\int_{B(-h(t)x(t),\delta)} |u(t,x)|^2 dx \lesssim_{E_{max}} \delta^4. \tag{6.6.1}$$

Independently, let $x_0 \in \mathbb{R}^n$, $R > \delta > 0$, $D = B(x_0, R) \setminus B(-h(t)x(t), \delta)$, and $D' = B(x(t) + x_0/h(t), R/h(t)) \setminus B(0, \delta/h(t))$. Using Hölder's inequality once again, we get that

$$\begin{aligned}
\int_D |u(t,x)|^2 dx &= h(t)^4 \int_{D'} |v(t,x)|^2 dx \\
&\leq h(t)^4 \left(\int_{|x| \geq \frac{\delta}{h(t)}} |v(t,x)|^4 dx \right)^{\frac{1}{2}} \left(\int_{B(\frac{x_0}{h(t)} + x(t), \frac{R}{h(t)})} dx \right)^{\frac{1}{2}} \\
&\lesssim \epsilon(\delta/h(t))^{\frac{1}{2}} R^4,
\end{aligned} \tag{6.6.2}$$

where ϵ is given by

$$\epsilon(R) = \sup_{t \in I} \int_{|x| \geq R} |v(t,x)|^4 dx.$$

A consequence of the compactness of K as in Theorem 18 is that

$$\epsilon(R) \to 0 \ , \ \text{as } R \to +\infty. \tag{6.6.3}$$

Combining (6.6.1) and (6.6.2), we get that for any ball B_R of radius $R > \delta$,

$$\int_{B_R} |u(t,x)|^2 dx \lesssim_{E_{max}} \delta^4 + R^4 \epsilon(\delta/h(t))^{\frac{1}{2}}. \tag{6.6.4}$$

Using almost conservation of local mass, as expressed in (6.3.11), and (6.6.4), we get, for any $x_0 \in \mathbb{R}^8$ and any $R > 4$, that the following bound at time 1 holds true

$$\begin{aligned} M\left(u(1), B(x_0, R)\right)^{\frac{3}{4}} &\lesssim_{E_{max}} \frac{1}{R} + M\left(u(t), B(x_0, 2R)\right)^{\frac{3}{4}} \\ &\lesssim_{E_{max}} \frac{1}{R} + \left(\delta^4 + R^4 \epsilon(\delta/h(t))^{\frac{1}{2}}\right)^{\frac{3}{4}}, \end{aligned} \tag{6.6.5}$$

where the local mass is as in (6.3.10). Letting $t \to 0$ and using (6.6.3), and then letting $\delta \to 0$, we get with (6.6.5) that

$$M\left(u(1), B(x_0, R)\right) \lesssim_{E_{max}} R^{-\frac{4}{3}}. \tag{6.6.6}$$

Letting $R \to \infty$ in (6.6.6), we obtain

$$\|u(1)\|_{L^2} = 0. \tag{6.6.7}$$

Clearly (6.6.7) contradicts $u \neq 0$. This proves Proposition 6.6.1. □

6.7 An interaction Morawetz estimate

To deal with the remaining two scenarii in Theorem 18, in which there is no prescribed finite-time blow-up, we need a new ingredient that bounds the amount of nonlinear presence of the solution at a given scale. Natural candidates to achieve this are Morawetz estimates and in our case, interaction Morawetz estimates. In light of Theorem 18, we need to work exclusively with $\dot{H}^2$-solutions. Interaction Morawetz estimates scale like the $\dot{H}^{\frac{1}{4}}$-norm. Because of this 7/4-differrence in scaling, following Colliander, Keel, Staffilani, Takaoka and Tao [5], Ryckman and Visan [24] and Visan [32], we seek for frequency-localized interaction Morawetz estimates. This is the purpose of Sections 6.7 and 6.8. In Section 6.7 we derive an a priori interaction estimate that applies to all solutions $u \in C(H^2)$, and in Section 6.8 we use it to obtain a frequency-localized version of these estimates. The frequency localized version applies only to the special $\dot{H}^2$-solutions given by Theorem 18. We prove here that the following proposition holds true.

Proposition 6.7.1. *Let $n \geq 7$ and let $u \in C([T_1, T_2], H^2)$ be a solution of (6.3.1), with forcing term $h \in \dot{\bar{S}}^2([T_1, T_2]) + \dot{\bar{S}}^0([T_1, T_2])$. Then the following estimate holds true:*

$$\begin{aligned} &\sum_{j=1}^{n} \int_{T_1}^{T_2} \int_{\mathbb{R}^{2n}} \{h, u\}_m(t, y) \frac{(x-y)_j}{|x-y|} \{\partial_j u, u\}_m(t, x) dx dy dt \\ &+ \sum_{j=1}^{n} \int_{T_1}^{T_2} \int_{\mathbb{R}^{2n}} |u(t, y)|^2 \frac{(x-y)_j}{|x-y|} \{h, u\}_p^j(t, x) dx dy dt \\ &+ \int_{T_1}^{T_2} \int_{\mathbb{R}^{2n}} \frac{|u(t,x)|^2 |u(t,y)|^2}{|x-y|^5} dx dy dt \lesssim \sup_{t = T_1, T_2} \|u(t)\|_{L^2}^2 \|u(t)\|_{\dot{H}^{\frac{1}{2}}}^2, \end{aligned} \tag{6.7.1}$$

where $\{,\}_m$ and $\{,\}_p$, the mass and momentum bracket are defined by

$$\{f, g\}_m = Im(f\bar{g}) \text{ , } and, \{f, g\}_p = Re(f\nabla\bar{g} - g\nabla\bar{f}). \tag{6.7.2}$$

In addition to Proposition 6.7.1, in order to exploit the bound given in (6.7.1), we also prove that the following lemma holds true.

Lemma 6.7.1. *Assume $n \geq 6$. Then*

$$\| |\nabla|^{-\frac{n-5}{4}} u \|_{L^4} \simeq \| (\sum_N N^{-\frac{n-5}{2}} |P_N u|^2)^{\frac{1}{2}} \|_{L^4} \lesssim \| |\nabla|^{-\frac{n-5}{2}} |u|^2 \|_{L^2}^{\frac{1}{2}}, \quad (6.7.3)$$

for all $u \in \dot{H}^2$ such that $|\nabla|^{-\frac{3}{2}} |u|^2 \in L^2$, where the summation is over all dyadic numbers.

Proof. The equivalence of norms is classical. We first claim that for any $g \in \mathcal{S}$, and any $n \geq 6$,

$$\| |\nabla|^{-\frac{n-5}{4}} g \|_{L^4} \lesssim \| |\nabla|^{-\frac{n-5}{2}} |g|^2 \|_{L^2}^{\frac{1}{2}}. \quad (6.7.4)$$

We prove (6.7.4). Let $\phi(\xi) = |\xi|^{-\frac{n-5}{4}} (\psi(\xi) - \psi(2\xi))$ where ψ is as in (6.2.4). Using the Cauchy-Schwartz inequality we get that for any dyadic N,

$$\begin{aligned} &\left(P_N |\nabla|^{-\frac{n-5}{4}} g \right)(x) \\ &= N^{-\frac{n-5}{4}} \left(g * \mathcal{F}^{-1} (\phi(\xi/N)) \right)(x) \\ &= N^{\frac{3n+5}{4}} \int_{\mathbb{R}^n} g(x-y) \check{\phi}(Ny) dy \\ &\leq N^{\frac{3n+5}{4}} \left(\int_{\mathbb{R}^n} |g(x-y)|^2 |\check{\phi}(Ny)| dy \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} |\check{\phi}(Ny)| dy \right)^{\frac{1}{2}} \\ &\lesssim N^{\frac{n+5}{4}} \left(\int_{\mathbb{R}^n} |g(x-y)|^2 |\check{\phi}(Ny)| dy \right)^{\frac{1}{2}} \end{aligned} \quad (6.7.5)$$

uniformly in N. Since $\phi \in \mathcal{S}$, for any $y \in \mathbb{R}^n$, we get

$$\sum_N (N|y|)^{\frac{n+5}{2}} |\check{\phi}(Ny)| \lesssim \sum_N (N|y|)^{\frac{n+5}{2}} (1 + N|y|)^{-2n} \lesssim 1, \quad (6.7.6)$$

where the summation is over all dyadic numbers N. Consequently, using (6.7.5), (6.7.6) and the fact that $\check{\phi} \in \mathcal{S}$, we get that

$$\begin{aligned} \sum_N |P_N |\nabla|^{-\frac{n-5}{4}} g|^2(x) &\lesssim \sum_N N^{\frac{n+5}{2}} \int_{\mathbb{R}^n} |g(x-y)|^2 |\check{\phi}(Ny)| dy \\ &\lesssim \int_{\mathbb{R}^n} \frac{|g(x-y)|^2}{|y|^{\frac{n+5}{2}}} \left(\sum_N (N|y|)^{\frac{n+5}{2}} |\check{\phi}(Ny)| \right) dy \\ &\lesssim \left(|\nabla|^{-\frac{n-5}{2}} |g|^2 \right)(x), \end{aligned} \quad (6.7.7)$$

and using the Littlewood-Paley theorem, (6.7.7) gives (6.7.4) for g smooth. Density arguments then give (6.7.3). This ends the proof of Lemma 6.7.1. □

Proof of Proposition 6.7.1. Since the estimate we want to prove is linear, we can assume that u is smooth and use density arguments to recover the general case. We adopt the convention that repeated indices are summed. Given some real function a, we define the Morawetz action centered at 0 by

$$M_a^0(t) = 2 \int_{\mathbb{R}^n} \partial_j a(x) \mathrm{Im}(\bar{u}(t,x) \partial_j u(t,x)) dx. \quad (6.7.8)$$

Following the computation in Pausader [21], we get that

$$\partial_t M_a^0(t) = 2\int_{\mathbb{R}^n} \Big(2\partial_j u \partial_k \bar{u} \partial_{jk}\Delta a - \frac{1}{2}\left(\Delta^3 a\right)|u|^2 - 4\partial_{jk}a\partial_{ik}u\partial_{ij}\bar{u} + \Delta^2 a|\nabla u|^2 + \partial_j a\{u,h\}_p^j\Big)dx. \tag{6.7.9}$$

Similarly, we define the Morawetz action centered at y, $M_a^y(t) = M_{a_y}^0(t)$ for $a_y(x) = |x-y|$. Finally, we define the interaction Morawetz action by the following formula:

$$\begin{aligned} M^i(t) &= \int_{\mathbb{R}^n} |u(t,y)|^2 M_a^y(t)dy \\ &= 2\mathrm{Im}\left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |u(t,y)|^2\frac{x-y}{|x-y|}\nabla u(t,x)\bar{u}(t,x)dxdy\right). \end{aligned} \tag{6.7.10}$$

We can directly estimate

$$|M^i(t)| \le \|u\|_{L^\infty L^2}^2\|u\|_{L^\infty \dot{H}^{\frac{1}{2}}}^2. \tag{6.7.11}$$

Now, we get an estimate on the variation of M^i by writing that

$$\begin{aligned} \partial_t M^i =& 2\int_{\mathbb{R}^n}\{u,h\}_m(y)M_a^y dy + 4\mathrm{Im}\int_{\mathbb{R}^n}\partial_j u(y)\partial_{jk}\bar{u}(y)\partial_k M_a^y dy \\ &+ 2\mathrm{Im}\left(\int_{\mathbb{R}^n}\bar{u}(y)\nabla u(y)\nabla\Delta M_a^y dy\right) + \int_{\mathbb{R}^n}|u(y)|^2\partial_t M_a^y dy. \end{aligned} \tag{6.7.12}$$

This gives that

$$\begin{aligned} &\partial_t M^i = \\ &4\int_{\mathbb{R}^n\times\mathbb{R}^n} \mathrm{Im}\left(\bar{u}(y)\partial_j u(y)\right)\partial_j^y\Delta\left(\partial_k^x a(x-y)\right)\mathrm{Im}\left(\partial_k u(x)\bar{u}(x)\right)dxdy \\ &+ 8\int_{\mathbb{R}^n\times\mathbb{R}^n} \mathrm{Im}\left(\partial_i u(y)\partial_{ij}\bar{u}(y)\right)\partial_j^y\left(\partial_k^x a(x-y)\right)\mathrm{Im}\left(\partial_k u(x)\bar{u}(x)\right)dxdy \\ &+ 4\int_{\mathbb{R}^n\times\mathbb{R}^n} \{u,h\}_m(y)\partial_k^x a(x-y)\mathrm{Im}\left(\partial_k u(x)\bar{u}(x)\right)dxdy \\ &+ 4\int_{\mathbb{R}^n\times\mathbb{R}^n} |u(y)|^2\partial_{jk}^x\left(\Delta a(x-y)\right)\partial_j u(x)\partial_k\bar{u}(x)dxdy \\ &- \int_{\mathbb{R}^n\times\mathbb{R}^n} |u(y)|^2\left(\Delta^3 a(x-y)\right)|u(x)|^2dxdy \\ &- 8\int_{\mathbb{R}^n\times\mathbb{R}^n} |u(y)|^2\left(\partial_{jk}^x a(x-y)\right)\partial_{ik}u(x)\partial_{ij}\bar{u}(x)dxdy \\ &+ 2\int_{\mathbb{R}^n\times\mathbb{R}^n} |u(y)|^2\left(\Delta^2 a(x-y)\right)|\nabla u(x)|^2dxdy \\ &+ 2\int_{\mathbb{R}^n\times\mathbb{R}^n} |u(y)|^2\partial_j^x a(x-y)\{u,h\}_p^j(x)dxdy, \end{aligned} \tag{6.7.13}$$

where ∂_j^x denotes derivation with respect to x_j, and ∂_k^y derivation with respect to y_k. Most of the terms in (6.7.13) have the right sign if we let $a(z) = |z|$. Now we focus on the first two terms in (6.7.13). In the sequel, we let $z = x-y$. Using the fact that $\mathrm{Re}\,(AB) = \mathrm{Re}\,(A)\,\mathrm{Re}\,(B) - \mathrm{Im}\,(A)\,\mathrm{Im}\,(B)$, we get the equality:

$$\begin{aligned}&\int_{\mathbb{R}^{2n}} \mathrm{Im}\,(\bar{u}(y)\partial_j u(y)) \left(\partial_j^y \partial_k^x \Delta a(z)\right) \mathrm{Im}\,(\partial_k u(x)\bar{u}(x))\, dxdy \\ &= -\frac{1}{4}\int_{\mathbb{R}^{2n}} |u(y)|^2 \Delta^3 a(z) |u(x)|^2 dxdy - R((\nabla u \otimes u); (\nabla u \otimes u)),\end{aligned} \tag{6.7.14}$$

where we let R be the bilinear form on $\mathcal{S}(\mathbb{R}^n, \mathbb{R}^n) \otimes \mathcal{S}(\mathbb{R}^n, \mathbb{R})$ defined by

$$R((\vec{\alpha} \otimes \beta); (\vec{\gamma} \otimes \delta)) = \mathrm{Re} \int_{\mathbb{R}^{2n}} \alpha_j(y)\bar{\delta}(y) \left(\partial_{jk}^x \Delta a(z)\right) \bar{\gamma}_k(x)\beta(x) dxdy. \tag{6.7.15}$$

For the second term, we proceed as follows:

$$\begin{aligned}&\int_{\mathbb{R}^{2n}} \mathrm{Im}\,(\partial_{ij}\bar{u}(x)\partial_i u(x)) \left(\partial_j^x \partial_k^y a(z)\right) \mathrm{Im}\,(\partial_k u(y)\bar{u}(y))\, dxdy \\ &= \frac{1}{4}\int_{\mathbb{R}^n} |\nabla u(x)|^2 \Delta^2 a(z) |u(y)|^2 dxdy + Q((\nabla\partial_i u \otimes u); (\nabla u \otimes \partial_i u)),\end{aligned} \tag{6.7.16}$$

where we define the quadratic form Q on $\mathcal{S}(\mathbb{R}^n, \mathbb{R}^n) \otimes \mathcal{S}(\mathbb{R}^n, \mathbb{R})$ by

$$Q\left((\vec{\alpha} \otimes \beta); (\vec{\gamma} \otimes \delta)\right) = \mathrm{Re} \int_{\mathbb{R}^{2n}} \alpha_k(x)\bar{\delta}(x) \frac{1}{|z|} \left(\delta_{jk} - \frac{z_j z_k}{|z|^2}\right) \bar{\gamma}_j(y)\beta(y) dydx. \tag{6.7.17}$$

As one can check by computing the Fourier transform of its kernel, Q is non-negative. Hence, applying the Cauchy-Schwartz inequality, we get

$$\begin{aligned}|Q((\nabla\partial_i u \otimes u); (\nabla u \otimes \partial_i u))| &\leq |Q((\nabla\partial_i u \otimes u)^2)|^{\frac{1}{2}} |Q((\nabla u \otimes \partial_i u)^2)|^{\frac{1}{2}} \\ &\leq \frac{1}{2} Q((\nabla\partial_i u \otimes u)^2) + \frac{1}{2} Q((\nabla u \otimes \partial_i u)^2)\end{aligned} \tag{6.7.18}$$

and if R and Q are as in (6.7.15) and (6.7.17), we observe that

$$\begin{aligned}Q((\nabla u \otimes \partial_i u)^2) &= Q((\nabla\partial_i u \otimes u)^2) - R\left((\nabla u \otimes u)^2\right) \\ &+ 2\mathrm{Re} \int_{\mathbb{R}^{2n}} \partial_k u(x)\bar{u}(x) \left(\partial_{ijk}^x a(z)\right) \partial_{ij}\bar{u}(y) u(y) dxdy \\ &= Q((\nabla\partial_i u \otimes u)^2) + R((\nabla u \otimes u)^2) \\ &+ \mathrm{Re} \int_{\mathbb{R}^{2n}} |u(x)|^2 \left(\partial_{ij}^x \Delta a(z)\right) \partial_i \bar{u}(y) \partial_j u(y) dxdy.\end{aligned} \tag{6.7.19}$$

Consequently, applying (6.7.14), (6.7.16), (6.7.18) and (6.7.19), we get that

$$\begin{aligned}&4\int_{\mathbb{R}^n \times \mathbb{R}^n} \mathrm{Im}\,(\bar{u}(y)\partial_j u(y))\, \partial_j^y \Delta\left(\partial_k^x a(x-y)\right) \mathrm{Im}\,(\partial_k u(x)\bar{u}(x))\, dxdy \\ &+ 8\int_{\mathbb{R}^n \times \mathbb{R}^n} \mathrm{Im}\,(\partial_i u(y)\partial_{ij}\bar{u}(y))\, \partial_j^y \left(\partial_k^x a(x-y)\right) \mathrm{Im}\,(\partial_k u(x)\bar{u}(x))\, dxdy \\ &\leq -\int_{\mathbb{R}^{2n}} |u(y)|^2 \left(\Delta^3 a(z)\right) |u(x)|^2 dxdy + 8Q\left((\nabla\partial_i u \otimes u)^2\right) \\ &+ 2\int_{\mathbb{R}^{2n}} |u(y)|^2 \left(\Delta^2 a(z)\right) |\nabla u(x)|^2 dxdy \\ &+ 4\mathrm{Re} \int_{\mathbb{R}^{2n}} |u(x)|^2 \left(\partial_{ij}^x \Delta a(z)\right) \partial_i \bar{u}(y) \partial_j u(y) dxdy.\end{aligned} \tag{6.7.20}$$

Now, for $e \in \mathbb{R}^n$ a vector, and u a function, we define

$$\nabla_e u = (e \cdot \nabla u)\,\frac{e}{|e|^2}\ ,\ \text{and}, \nabla_e^{\perp} u = \nabla u - \nabla_e u.$$

Then, applying the Cauchy-Schwartz inequality, we get that

$$\begin{aligned}
&Q((\nabla\partial_i u, u)^2)\\
&= \int_{\mathbb{R}^{2n}} \partial_{ij}\bar{u}(x)u(x)\frac{1}{|x-y|}\left(\delta_{jk} - \frac{(x-y)_j(x-y)_k}{|x-y|^2}\right)\partial_{ik}u(y)\bar{u}(y)dxdy\\
&= \int_{\mathbb{R}^{2n}} \frac{\left[u(x)\nabla^{\perp}_{x-y}\partial_i u(y)\right]\cdot\left[\nabla^{\perp}_{x-y}\partial_i\bar{u}(x)\bar{u}(y)\right]}{|x-y|}dxdy\\
&\leq \int_{\mathbb{R}^{2n}} |u(x)|^2\frac{1}{|x-y|}|\nabla^{\perp}_{x-y}\partial_i u(y)|^2 dxdy\\
&\leq \int_{\mathbb{R}^{2n}} |u(x)|^2\frac{1}{|x-y|}\left(\delta_{jk} - \frac{(x-y)_j(x-y)_j}{|x-y|^2}\right)\partial_{ik}\bar{u}(y)\partial_{ij}u(y)dxdy.
\end{aligned} \tag{6.7.21}$$

Finally, (6.7.13), (6.7.20), and (6.7.21) give

$$\begin{aligned}
\partial_t M^i \leq &-2\int_{\mathbb{R}^n\times\mathbb{R}^n} |u(y)|^2\left(\Delta^3 a(x-y)\right)|u(x)|^2 dxdy\\
&+4\int_{\mathbb{R}^n\times\mathbb{R}^n} \{u,h\}_m(y)\partial^x_k a(x-y)\mathrm{Im}\left(\partial_k u(x)\bar{u}(x)\right)dxdy\\
&+2\int_{\mathbb{R}^n\times\mathbb{R}^n} |u(y)|^2\partial^x_j a(x-y)\{u,h\}^j_p(x)dxdy\\
&+8\int_{\mathbb{R}^n\times\mathbb{R}^n} |u(y)|^2\left(\partial^x_{jk}\Delta a(x-y)\right)\partial_j u(x)\partial_k\bar{u}(x)dxdy\\
&+4\int_{\mathbb{R}^n\times\mathbb{R}^n} |u(y)|^2\left(\Delta^2 a(x-y)\right)|\nabla u(x)|^2 dxdy
\end{aligned} \tag{6.7.22}$$

and the sum of the last two terms equals

$$-4(n-1)\int_{\mathbb{R}^{2n}} \frac{|u(y)|^2}{|x-y|^3}\left((n-1)\delta_{jk} - 6\frac{(x-y)_j(x-y)_k}{|x-y|^2}\right)\partial_j u(x)\partial_k\bar{u}(x)dxdy$$

which is nonpositive when $n \geq 7$. Finally, (6.7.22) and this remark give (6.7.1). □

6.8 A frequency-localized interaction Morawetz estimate

The preceding interaction Morawetz estimate is ill-suited for $\dot{H}^2$-solutions. In order to exploit such an estimate in the context of $\dot{H}^2$-solutions, we need to localize it at high frequencies. The difficulty then is to deal with an inequality that scales like the $\dot{H}^{\frac{1}{4}}$-norm, while using only bounds that scale like the $\dot{H}^2$-norm. To overcome this difference of 7/4 derivatives, we split the solution into high and low frequencies and develop an intricate bootstrap argument to get the inequality. This is made possible because we restrict ourselves to the case of the special solutions obtained in Theorem 18. More precisely, we prove that the following proposition holds true.

Proposition 6.8.1. *Let $n = 8$. Let $u \in C(I, \dot{H}^2)$ be a maximal lifespan-solution of* (6.1.1) *such that $K = \{g(t)u(t) : t \in I\}$ is precompact in $\dot{H}^2$ and such that $\forall t \in I,\ h(t) \leq h(0) = 1$. Then, for any sufficiently small $\varepsilon > 0$,*

$$\begin{aligned} &\| |\nabla|^{-\frac{3}{2}} |P_{\geq 1} u|^2 \|_{L^2(I,L^2)} \lesssim \varepsilon \ , \\ &\|P_{\geq 1} u\|_{\dot{S}^{-\frac{3}{2}}(I)} \lesssim \varepsilon \ , \ \textit{and} \ \|P_{\leq 1} u\|_{\dot{S}^2(I)} \lesssim \varepsilon \end{aligned} \tag{6.8.1}$$

up to replacing u by $g_{(N,0)}u$ for some N.

Proof. We fix $\varepsilon > 0$ sufficiently small to be chosen later on. We remark that for N a dyadic number and for all time,

$$\|P_{\leq N} g^{-1}(t) \left(g(t)u(t)\right) \|_{\dot{H}^2} = \|P_{\leq N h(t)} \left(g(t)u(t)\right) \|_{\dot{H}^2}. \tag{6.8.2}$$

Hence, by compactness of K, and since $h \leq 1$, we have that $\|P_{\leq N} u\|_{L^\infty \dot{H}^2} \to 0$ as $N \to 0$. Let N be such that

$$\|P_{\leq \varepsilon^{-4} N} u\|_{L^\infty \dot{H}^2} \leq \frac{\varepsilon}{2}.$$

Replacing K by $Kg_{(\varepsilon^4 N^{-1},0)}$, and modifying slighly h, one can assume that

$$\begin{aligned} &\|P_{\leq 1} u\|_{L^\infty(I,\dot{H}^2)} \leq \varepsilon \ , \ \text{and} \\ &\|P_{\geq 1} u\|_{L^\infty(I,\dot{H}^s)} \leq \|P_{1 \leq \cdot < \varepsilon^{-4}} u\|_{L^\infty(I,\dot{H}^2)} + \varepsilon^{4(2-s)} \|P_{\geq \varepsilon^{-4}} u\|_{L^\infty(I,\dot{H}^2)} \\ &\qquad \leq \varepsilon, \end{aligned} \tag{6.8.3}$$

for $s \leq 7/4$. We let

$$J\left(C\right) = \{t \geq 0 : \| |\nabla|^{-\frac{3}{2}} |P_{\geq 1} u|^2 \|_{L^2([0,t],L^2)} \leq C\eta\}. \tag{6.8.4}$$

The first step in the proof is to obtain good Strichartz controls on the high and low-frequency parts of u. In the sequel, we let $u_l = P_{<1}u$, and $u_h = P_{\geq 1}u$. Besides the summations are always over all dyadic numbers, unless otherwise specified. We claim that for $J = J(2)$, we have that

$$\begin{aligned} &\| |\nabla|^{-\frac{3}{2}} |P_{\geq 1} u|^2 \|_{L^2(J,L^2)} \leq 2\eta, \\ &\|P_{\leq 1} u\|_{\dot{S}^2(J)} \lesssim \varepsilon \ , \ \text{and} \\ &\|P_{\geq 1} u\|_{\dot{S}^{-\frac{3}{2}}(J)} \lesssim \eta, \end{aligned} \tag{6.8.5}$$

provided that $\varepsilon > 0$ is sufficiently small, and that $\varepsilon < \eta$. In the following, all space-time norms are taken on the interval J. Applying the Strichartz estimates (6.3.3), we get that

$$\begin{aligned} \|P_{\leq 1} u\|_{\dot{S}^2} &\lesssim \|P_{\leq 1} u(0)\|_{\dot{H}^2} + \| |\nabla| P_{\leq 1} \left(|u_l|^2 u_l\right) \|_{L^2(J,L^{\frac{8}{5}})} \\ &\quad + \sum_{j=0}^{2} \| |\nabla| P_{\leq 1} \mathcal{O}\left(u_l^j u_h^{3-j}\right) \|_{L^2(J,L^{\frac{8}{5}})} \\ &\lesssim \varepsilon + \|u_l\|_{\dot{S}^2}^3 + \sum_{j=0}^{2} \| |\nabla| P_{\leq 1} \mathcal{O}\left(u_l^j u_h^{3-j}\right) \|_{L^2(J,L^{\frac{8}{5}})}. \end{aligned} \tag{6.8.6}$$

Now, we estimate the terms in the sum. First, using the Bernstein's properties (6.2.5) and (6.8.3), we get that

$$\begin{aligned} \left\| |\nabla| P_{\leq 1} \mathcal{O}\left(u_l^2 u_h\right) \right\|_{L^2 L^{\frac{8}{5}}} &\lesssim \|u_l^2 u_h\|_{L^2 L^{\frac{8}{5}}} \\ &\lesssim \|u_l\|_{L^4 L^8} \|u_l\|_{L^4 L^8} \|u_h\|_{L^\infty L^{\frac{8}{3}}} \\ &\lesssim \varepsilon \|u_l\|_{\dot{S}^2}^2. \end{aligned} \tag{6.8.7}$$

For the next term, we remark that if $N \geq 4M$ and $N \geq 8$, then the Fourier support of $P_N u P_M v$ is supported in $\{|\xi| \geq 2\}$, and $P_{\leq 1}\left(P_N u P_M v\right) = 0$. Using this remark, the Bernstein's properties (6.2.5), (6.8.3) and the Cauchy-Schwartz inequality, we get

$$\begin{aligned} &\left\| |\nabla| P_{\leq 1} \mathcal{O}\left(u_l u_h^2\right) \right\|_{L^2 L^{\frac{8}{5}}} \\ &\lesssim \left\| P_{\leq 1} \mathcal{O}\left(u_l u_h^2\right) \right\|_{L^2 L^{\frac{8}{5}}} \\ &\lesssim \sum_{M \leq 1, N \leq 8} \left\| P_{\leq 1}\left(P_M u P_N \mathcal{O}\left(u_h^2\right)\right) \right\|_{L^2 L^{\frac{8}{5}}} \\ &\lesssim \left(\sum_{M \leq 1} \|P_M u\|_{L^\infty L^8} \right) \left(\sum_{N \leq 8} \left\| P_N \mathcal{O}\left(u_h^2\right) \right\|_{L^2 L^2} \right) \\ &\lesssim \left(\sum_{M \leq 1} M^{-1} \|P_M u\|_{L^\infty L^8}^2 \right)^{\frac{1}{2}} \left(\sum_{N \leq 8} N^{-3} \left\| P_N \mathcal{O}\left(u_h^2\right) \right\|_{L^2 L^2}^2 \right)^{\frac{1}{2}} \\ &\lesssim \|u_l\|_{L^\infty \dot{H}^2} \left\| |\nabla|^{-\frac{3}{2}} |u_h|^2 \right\|_{L^2 L^2} \\ &\lesssim \varepsilon \eta, \end{aligned} \tag{6.8.8}$$

where we have used in the last inequalities that since $|\nabla|^{-\frac{3}{2}}$ has a positive kernel, we have that $\| |\nabla|^{-\frac{3}{2}} \mathcal{O}(u_h^2) \|_{L^2 L^2} \leq \| |\nabla|^{-\frac{3}{2}} |u_h|^2 \|_{L^2 L^2}$. We treat the last term similarly as follows, by writing that

$$\begin{aligned} &\left\| |\nabla| P_{\leq 1} \mathcal{O}\left(u_h^3\right) \right\|_{L^2 L^{\frac{8}{5}}} \\ &\lesssim \left\| P_{\leq 1} \mathcal{O}\left(u_h^3\right) \right\|_{L^2 L^1} \\ &\lesssim \sum_{1 \leq N \leq 8, M \leq 32} \left\| P_{\leq 1}\left(P_N u_h P_M \mathcal{O}\left(u_h^2\right)\right) \right\|_{L^2 L^1} \\ &+ \sum_{N \geq 8, 4N \geq M \geq N/4} \left\| P_{\leq 1}\left(P_N u_h P_M \mathcal{O}\left(u_h^2\right)\right) \right\|_{L^2 L^1} \\ &\lesssim \left(\sum_M M^3 \|P_M u_h\|_{L^\infty L^2}^2 \right)^{\frac{1}{2}} \left(\sum_M M^{-3} \|P_M |u_h|^2\|_{L^2 L^2}^2 \right)^{\frac{1}{2}} \\ &\lesssim \|u_h\|_{L^\infty H^{\frac{7}{4}}} \left\| |\nabla|^{-\frac{3}{2}} |u_h|^2 \right\|_{L^2 L^2} \\ &\lesssim \varepsilon \eta. \end{aligned} \tag{6.8.9}$$

Finally, we get with (6.8.6)–(6.8.9) that

$$\|u_l\|_{\dot{S}^2} \lesssim \|P_{\leq 1} u\|_{\dot{S}^2} \lesssim \varepsilon + \eta\varepsilon + \varepsilon \|u_l\|_{\dot{S}^2}^2 + \|u_l\|_{\dot{S}^2}^3 \tag{6.8.10}$$

and this proves the second inequality in (6.8.5) with u_l instead of $P_{\leq 1}u$ if $\varepsilon > 0$ is sufficiently small. Using again (6.8.10), we get the second inequality in (6.8.5). Now we turn to the control on u_h. Still using the Strichartz estimates (6.3.3) and Sobolev's inequality, we get that

$$\begin{aligned}
\|u_h\|_{\dot{S}^{-\frac{3}{2}}} &\lesssim \|u_h(0)\|_{\dot{H}^{-\frac{3}{2}}} + \sum_{j=0}^{3} \||\nabla|^{-\frac{5}{2}} P_{\geq 1}\mathcal{O}\left(u_h^j u_l^{3-j}\right)\|_{L^2 L^{\frac{8}{5}}} \\
&\lesssim \varepsilon + \sum_{j=2,3} \|P_{\geq 1}\mathcal{O}\left(u_h^j u_l^{3-j}\right)\|_{L^2 L^{\frac{16}{15}}} \\
&\quad + \sum_{j=0,1} \||\nabla|^{-\frac{5}{2}} P_{\geq 1}\mathcal{O}\left(u_h^j u_l^{3-j}\right)\|_{L^2 L^{\frac{8}{5}}}.
\end{aligned} \tag{6.8.11}$$

By convolution estimate, letting $c_N = N^{-\frac{3}{4}}|P_N u_h|$, we get that

$$\begin{aligned}
\left||u_h|^2 u_h\right| &\lesssim |\sum_{M_1 \geq M_2 \geq M_3} \mathcal{O}\left(P_{M_1}u_h P_{M_2}u_h P_{M_3}u_h\right)| \\
&\lesssim |\sum_{M_1 \geq M_2 \geq M_3} c_{M_3}\left(\frac{M_3}{M_2}\right)^{\frac{3}{4}} c_{M_2}\left(\frac{M_2}{M_1}\right)^{\frac{3}{2}} M_1^{\frac{3}{2}} P_{M_1}u_h| \\
&\lesssim \left(\sum_M c_M^2\right)\left(\sup_M M^{\frac{3}{2}}|P_M u_h|\right).
\end{aligned} \tag{6.8.12}$$

Consequently, using the Bernstein's properties (6.2.5), (6.7.3) and (6.8.12), we get that

$$\begin{aligned}
&\|P_{\geq 1}|u_h|^2 u_h\|_{L^2 L^{\frac{16}{15}}} \\
&\lesssim \||u_h|^2 u_h\|_{L^2 L^{\frac{16}{15}}} \\
&\lesssim \|(\sum_M M^{-\frac{3}{2}}|P_M u_h|^2)^{\frac{1}{2}}\|_{L^4 L^4}^2 \left(\|\sup_M |\nabla|^{\frac{3}{2}} P_M u_h\|_{L^\infty L^{\frac{16}{7}}}\right) \\
&\lesssim \||\nabla|^{-\frac{3}{2}}|u_h|^2\|_{L^2 L^2}\|u_h\|_{L^\infty \dot{H}^2} \\
&\lesssim_{E_{max}} \eta.
\end{aligned} \tag{6.8.13}$$

Note that instead of using the pointwise evaluation of $u_h = \sum P_M u_h$, we can replace u_h by an arbitrary Schwartz function, get the bound, and then use density arguments to recover (6.8.13). When $j = 2$, we proceed as follows, using Sobolev's inequality, the Bernstein's properties (6.2.5), (6.8.3) and the estimate for u_l in (6.8.5),

$$\begin{aligned}
\|\mathcal{O}\left(u_h^2 u_l\right)\|_{L^2 L^{\frac{16}{15}}} &\lesssim \|u_l\|_{L^4 L^8}\|u_h\|_{L^4 L^{\frac{16}{7}}}\|u_h\|_{L^\infty L^{\frac{8}{3}}} \\
&\lesssim \|u_l\|_{\dot{S}^2}\||\nabla|^{-\frac{3}{2}}u_h\|_{L^2 L^{\frac{8}{3}}}^{\frac{1}{2}}\||\nabla|^{\frac{3}{2}}u_h\|_{L^\infty L^2}^{\frac{3}{2}} \\
&\lesssim \varepsilon^{\frac{5}{2}}\|u_h\|_{\dot{S}^{-\frac{3}{2}}}^{\frac{1}{2}}.
\end{aligned} \tag{6.8.14}$$

When $j = 1$, we proceed similarly to get

$$\begin{aligned}
\||\nabla|^{-\frac{5}{2}} P_{\geq 1}\mathcal{O}\left(u_l^2 u_h\right)\|_{L^2 L^{\frac{8}{5}}} &\lesssim \|\mathcal{O}\left(u_l^2 u_h\right)\|_{L^2 L^{\frac{8}{5}}} \\
&\lesssim \|u_l\|_{L^4 L^8}^2\|u_h\|_{L^\infty L^{\frac{8}{3}}} \\
&\lesssim \varepsilon^3,
\end{aligned} \tag{6.8.15}$$

and finally,

$$\begin{aligned}\left\||\nabla|^{-\frac{5}{2}}P_{\geq 1}|u_l|^2u_l\right\|_{L^2L^{\frac{8}{5}}} &\lesssim \left\||\nabla||u_l|^2u_l\right\|_{L^2L^{\frac{8}{5}}}\\ &\lesssim \|u_l\|^3_{\dot{S}^2}\\ &\lesssim \varepsilon^3.\end{aligned} \tag{6.8.16}$$

Combining (6.8.11) and (6.8.13)–(6.8.16), we get that

$$\begin{aligned}\|u_h\|_{\dot{S}^{-\frac{3}{2}}} &\lesssim \varepsilon+\eta+\varepsilon^3+\varepsilon^{\frac{5}{2}}\|u_h\|^{\frac{1}{2}}_{\dot{S}^{-\frac{3}{2}}}\\ &\lesssim \eta.\end{aligned}$$

This ends the proof of of (6.8.5). As a consequence of conservation of energy, (6.8.3), (6.8.5) and Hardy-Littlewood-Sobolev's inequality, we get the following estimates on $J=J(2)$. Namely,

$$\begin{aligned}&\|u_h\|_{L^{\frac{5}{2}}L^{\frac{5}{2}}} \lesssim_{E_{max}} \eta^{\frac{4}{5}}\ ,\ \|u_h\|_{L^3L^{\frac{8}{3}}} \lesssim_{E_{max}} \eta^{\frac{2}{3}}\ ,\ \|u_h\|_{L^{\frac{9}{2}}L^3} \lesssim_{E_{max}} \eta^{\frac{4}{9}}\ ,\ \text{and}\\ &\left\||u_h|^2*|x|^{-1}\right\|_{L^3L^{24}} \lesssim \|u_h\|^2_{L^6L^{\frac{24}{11}}} \lesssim_{E_{max}} \varepsilon^{\frac{4}{3}}\eta^{\frac{2}{3}}\end{aligned} \tag{6.8.17}$$

Now that we have good Strichartz control on the high and low frequencies, we can control the error terms arising in the frequency-localized interaction Morawetz estimates. First, we treat the terms arising from the mass bracket. We claim that on $J=J(2)$, as defined above, we have that

$$\int_J\int_{\mathbb{R}^{2n}}\{P_{\geq 1}\left(|u|^2u\right),u\}_m(t,y)\frac{(x-y)_j}{|x-y|}\{\partial_j u,u\}_m(t,x)dxdy \lesssim \varepsilon^2\eta^2. \tag{6.8.18}$$

Exploiting cancellations, we write

$$\begin{aligned}\{P_{\geq 1}\left(|u|^2u\right),u_h\}_m =&\{P_{\geq 1}\left(|u|^2u-|u_h|^2u_h\right),u_h\}_m\\ &-\{P_{<1}\left(|u_h|^2u_h\right),u_h\}_m+\{|u_h|^2u_h,u_h\}_m.\end{aligned} \tag{6.8.19}$$

The last term in the right-hand side of (6.8.19) vanishes. Using the Bernstein's properties (6.2.5), (6.8.3) and (6.8.17), we get that

$$\begin{aligned}&\left|\int_J\int_{\mathbb{R}^{2n}}\operatorname{Im}\left(\partial_k u_h(x)\bar{u}_h(x)\right)\frac{(x-y)_k}{|x-y|}\{P_{<1}|u_h|^2u_h,u_h\}_m(y)dxdydt\right|\\ &\lesssim \|u_h\|_{L^\infty L^2}\|\nabla u_h\|_{L^\infty L^2}\int_J\left|\left(P_{<1}|u_h|^2u_h\right)u_h\right|dxdt\\ &\lesssim \varepsilon^2\|u_h\|_{L^3L^{\frac{8}{3}}}\|P_{<1}|u_h|^2u_h\|_{L^{\frac{3}{2}}L^{\frac{8}{5}}}\\ &\lesssim \varepsilon^2\|u_h\|_{L^3L^{\frac{8}{3}}}\left\||u_h|^2u_h\right\|_{L^{\frac{3}{2}}L^1}\\ &\lesssim \varepsilon^2\|u_h\|^3_{L^3L^{\frac{8}{3}}}\|u_h\|_{L^\infty L^4}\\ &\lesssim \varepsilon^2\eta^2.\end{aligned} \tag{6.8.20}$$

As for the first term in (6.8.19), using (6.8.3), we get that

$$\begin{aligned}&\left|\int_J\int_{\mathbb{R}^{2n}}\{\partial_k u_h,u_h\}_m(x)\frac{(x-y)_k}{|x-y|}\{P_{\geq 1}\left(|u|^2u-|u_h|^2u_h\right),u_h\}_m(y)dxdydt\right|\\ &\lesssim \sum_{j=0}^{2}\|u_h\|^2_{L^\infty H^1}\int_J\left|\left(P_{\geq 1}\mathcal{O}\left(u_h^ju_l^{3-j}\right)\right)u_h\right|dxdt\end{aligned} \tag{6.8.21}$$

and, using the Bernstein's properties (6.2.5), (6.8.5), and (6.8.17), we obtain

$$\begin{aligned}\int_J \left|P_{\geq 1}\mathcal{O}\left(u_h^2 u_l\right) u_h\right| dxdt &\lesssim \|u_h\|_{L^{\frac{9}{2}}L^3}\|u_h^2 u_l\|_{L^{\frac{9}{7}}L^{\frac{3}{2}}} \\ &\lesssim \|u_h\|^3_{L^{\frac{9}{2}}L^3}\|u_l\|_{L^3L^\infty} \\ &\lesssim \eta^{\frac{4}{3}}\varepsilon.\end{aligned} \tag{6.8.22}$$

Similarly,

$$\begin{aligned}\int_J \left|P_{\geq 1}\mathcal{O}\left(u_h u_l^2\right) u_h\right| dxdt &\lesssim \|u_h\|^2_{L^3L^{\frac{8}{3}}}\|u_l\|^2_{L^6L^8} \\ &\lesssim \eta^{\frac{4}{3}}\varepsilon^2.\end{aligned} \tag{6.8.23}$$

In order to treat the last term, we remark that, in view of the Fourier support, if $M_1, M_2, M_3 \leq 1/8$, then $P_{\geq 1}\left(P_{M_1}uP_{M_2}uP_{M_3}u\right) = 0$. Consequently, letting $c_M = M^2\|P_M u\|_{L^2L^4}$ and $d_M = M^2\|P_M u\|_{L^\infty L^2}$, we get, using again the Bernstein's properties (6.2.5), (6.8.3) and (6.8.5), that

$$\begin{aligned}&\int_J \left|\left(P_{\geq 1}|u_l|^2 u_l\right) u_h\right| dxdt \\ &\lesssim \|u_h\|_{L^\infty L^2}\|P_{\geq 1}\sum_{1\geq M_1\geq M_2\geq M_3} P_{M_1}uP_{M_2}uP_{M_3}u\|_{L^1L^2} \\ &\lesssim \|u_h\|_{L^\infty L^2}\sum_{1\geq M_1\geq 1/8, M_1\geq M_2\geq M_3}\|P_{M_1}uP_{M_2}uP_{M_3}u\|_{L^1L^2} \\ &\lesssim \|u_h\|_{L^\infty L^2}\sum_{1\geq M_1\geq 1/8, M_1\geq M_2\geq M_3}\|P_{M_1}\|_{L^2L^4}\|P_{M_2}u\|_{L^2L^4}\|P_{M_3}u\|_{L^\infty L^\infty} \\ &\lesssim \|u_h\|_{L^\infty L^2}\left(\sum_{1\geq M\geq 1/8}\|P_M u\|_{L^2L^4}\right)\sum_{1\geq M_2\geq M_3} c_{M_2}d_{M_3}\left(\frac{M_3}{M_2}\right)^2 \\ &\lesssim \|u_h\|_{L^\infty L^2}\|u_l\|^3_{\dot{S}^2} \\ &\lesssim \varepsilon^4.\end{aligned} \tag{6.8.24}$$

Combining (6.8.19)–(6.8.24), we see that (6.8.18) holds true. Now, we turn to the last error term, which arises from the momentum bracket. We claim that on $J = J(2)$, we have that

$$\begin{aligned}&\Big|\int_J\int_{\mathbb{R}^{2n}} |u_h(s,y)|^2\frac{(x-y)_j}{|x-y|}\{P_{\geq 1}|u|^2u, u_h\}^j_p(s,x)dxdyds \\ &-\frac{1}{2}\int_J\int_{\mathbb{R}^{2n}}\frac{|u_h(s,y)|^2|u_h(s,x)|^4}{|x-y|}dxdyds\Big| \\ &\lesssim \eta^2\left(\varepsilon^{\frac{7}{3}}\eta^{-\frac{2}{3}} + \varepsilon^2 + \varepsilon^{\frac{16}{3}}\eta^{-\frac{4}{3}}\right).\end{aligned} \tag{6.8.25}$$

In order to prove (6.8.25), we decompose

$$\begin{aligned}\{P_{\geq 1}|u|^2u, u_h\}_p &= \{|u|^2u, u\}_p - \{|u_l|^2u_l, u_l\}_p - \{\left(|u|^2u - |u_l|^2u_l\right), u_l\}_p \\ &- \{P_{<1}|u|^2u, u_h\}_p \\ &= -\frac{1}{2}\nabla\left(|u|^4 - |u_l|^4\right) - \{\left(|u|^2u - |u_l|^2u_l\right), u_l\}_p \\ &- \{P_{<1}|u|^2u, u_h\}_p.\end{aligned} \tag{6.8.26}$$

Besides, we remark that

$$\{f, g\}_p = \nabla\mathcal{O}\left(fg\right) - \mathcal{O}\left(f\nabla g\right). \tag{6.8.27}$$

Now, we estimate

$$\mathcal{R} = \sum_{k=0}^{2}\int_J\int_{\mathbb{R}^{2n}}|u_h(s,y)|^2\frac{(x-y)_j}{|x-y|}\{\mathcal{O}\left(u_l^k u_h^{3-k}\right), u_l\}_p^j(s,x)dxdyds. \tag{6.8.28}$$

The case $k=2$ is treated as follows, using (6.8.27). First

$$\begin{aligned}&\left|\int_J\int_{\mathbb{R}^{2n}}|u_h(y)|^2\frac{(x-y)_j}{|x-y|}\mathcal{O}\left(u_h u_l^2\right)\partial_j u_l(x)dsdxdy\right| \\ &\lesssim \left|\int_J\int_{\mathbb{R}^n}|u_h(y)|^2\mathcal{O}\int_{\mathbb{R}^n}\left(|\nabla|^{-1}u_h\right)\left(|\nabla|\left(\frac{(x-y)_j}{|x-y|}\left(u_l^2\partial_j u_l\right)(x)\right)\right)dxdsdy\right|.\end{aligned} \tag{6.8.29}$$

Now, using the boundedness of the Riesz transform and the Bernstein's properties (6.2.5), we estimate for any $y \in \mathbb{R}^n$,

$$\begin{aligned}&\||\nabla|\left(\frac{(x-y)_j}{|x-y|}\left(u_l^2\partial_j u_l\right)(x)\right)\|_{L^{\frac{8}{5}}} \\ &\lesssim \|\nabla\left(\frac{(x-y)_j}{|x-y|}\left(u_l^2\partial_j u_l\right)(x)\right)\|_{L^{\frac{8}{5}}} \\ &\lesssim \||x-y|^{-1}\mathbb{1}_{\{|x-y|\leq 1\}}\|_{L^2}\|u_l^2\|_{L^\infty}\|\nabla u_l\|_{L^8} \\ &+ \||x-y|^{-1}\mathbb{1}_{\{|x-y|\geq 1\}}\|_{L^\infty}\|u_l^2\|_{L^4}\|\nabla u_l\|_{L^{\frac{8}{3}}} \\ &+ \|\nabla u_l\|_{L^8}\|\partial_j u_l\|_{L^{\frac{8}{3}}}\|u_l\|_{L^8} + \|u_l^2\|_{L^4}\|\nabla^2 u_l\|_{L^{\frac{8}{3}}} \\ &\lesssim \|u_l\|_{L^8}^2\|\nabla u_l\|_{L^{\frac{8}{3}}},\end{aligned} \tag{6.8.30}$$

where $\mathbb{1}_E$ is the characteristic function of the set E. Consequently, using the Bernstein's properties (6.2.5), (6.8.3), (6.8.5) and (6.8.30), we get that

$$\begin{aligned}&\left|\int_J\int_{\mathbb{R}^n}|u_h(y)|^2\mathcal{O}\int_{\mathbb{R}^n}\left(|\nabla|^{-1}u_h\right)\left(|\nabla|\left(\frac{(x-y)_j}{|x-y|}\left(u_l^2\partial_j u_l\right)(x)\right)\right)dxdsdy\right| \\ &\lesssim \int_J\int_{\mathbb{R}^n}|u_h(y)|^2\||\nabla|^{-1}u_h\|_{L^{\frac{8}{5}}}\||\nabla|\left(\frac{(x-y)_j}{|x-y|}u_l^2\partial_j u_l\right)\|_{L^{\frac{8}{5}}} \\ &\lesssim \|u_h\|_{L^\infty L^2}^2\||\nabla|^{-\frac{1}{2}}u_h\|_{L^2L^{\frac{8}{3}}}\|u_l\|_{L^4L^8}^2\|\nabla u_l\|_{L^\infty L^{\frac{8}{3}}} \\ &\lesssim \eta\varepsilon^5.\end{aligned} \tag{6.8.31}$$

Besides, integrating by parts and using (6.8.5) and (6.8.17), we finish the analysis of the case $k = 2$ as follows:

$$\begin{aligned}
&\left| \int_J \int_{\mathbb{R}^{2n}} |u_h(y)|^2 |x-y|^{-1} |\mathcal{O}\left(u_l^3 u_h\right)(x)| dx dy ds \right| \\
&\lesssim \| |u_h|^2 * |x|^{-1} \|_{L^3 L^{24}} \|u_h\|_{L^{\frac{5}{2}} L^{\frac{5}{2}}} \|u_l\|^3_{L^{\frac{45}{4}} L^{\frac{360}{67}}} \\
&\lesssim \|u_h\|^2_{L^6 L^{\frac{24}{11}}} \|u_h\|_{L^{\frac{5}{2}} L^{\frac{5}{2}}} \|u_l\|^3_{\dot{S}^2} \\
&\lesssim \eta \varepsilon^4.
\end{aligned} \tag{6.8.32}$$

The case $k = 1$ is similar. First, with the Bernstein's properties (6.2.5), (6.8.3), (6.8.5) and (6.8.17), we obtain that

$$\begin{aligned}
&\left| \int_J \int_{\mathbb{R}^{2n}} |u_h(y)|^2 \frac{(x-y)_j}{|x-y|} \mathcal{O}\left(u_h^2 u_l\right)(x) \partial_j u_l(x) ds dx dy \right| \\
&\lesssim \|u_h\|^2_{L^\infty L^2} \|u_h^2\|_{L^{\frac{3}{2}} L^{\frac{4}{3}}} \|\nabla u_l\|_{L^3 L^{\frac{24}{5}}} \|u_l\|_{L^\infty L^{24}} \\
&\lesssim \|u_h\|^2_{L^\infty L^2} \|u_l\|_{L^\infty L^4} \|u_l\|_{\dot{S}^2} \|u_h\|^2_{L^3 L^{\frac{8}{3}}} \\
&\lesssim \varepsilon^4 \eta^{\frac{4}{3}}
\end{aligned} \tag{6.8.33}$$

and then,

$$\begin{aligned}
&\left| \int_J \int_{\mathbb{R}^{2n}} |u_h(y)|^2 |x-y|^{-1} \mathcal{O}\left(u_h^2 u_l^2\right)(x) ds dx dy \right| \\
&\lesssim \| |u_h|^2 * |x|^{-1} \|_{L^3 L^{24}} \|u_h\|_{L^\infty L^2} \|u_h\|_{L^\infty L^{\frac{8}{3}}} \|u_l\|^2_{L^3 L^{24}} \\
&\lesssim \|u_h\|^2_{L^6 L^{\frac{24}{11}}} \|u_h\|^2_{L^\infty H^1} \|u_l\|^2_{L^3 L^{12}} \\
&\lesssim \varepsilon^{\frac{16}{3}} \eta^{\frac{2}{3}}.
\end{aligned} \tag{6.8.34}$$

Finally for the case $k = 0$, using the Bernstein's properties (6.2.5), (6.8.3), (6.8.5) and (6.8.17), we write that

$$\begin{aligned}
&\left| \int_J \int_{\mathbb{R}^{2n}} |u_h(y)|^2 \frac{(x-y)_j}{|x-y|} \mathcal{O}\left(u_h^3\right)(x) \partial_j u_l(x) ds dx dy \right| \\
&\lesssim \|u_h\|^2_{L^\infty L^2} \|u_h^3\|_{L^{\frac{3}{2}} L^1} \|\nabla u_l\|_{L^3 L^\infty} \\
&\lesssim \eta^{\frac{4}{3}} \varepsilon^3,
\end{aligned} \tag{6.8.35}$$

and that

$$\begin{aligned}
&\left| \int_J \int_{\mathbb{R}^{2n}} |u_h(y)|^2 |x-y|^{-1} \mathcal{O}\left(u_h^3 u_l\right)(x) ds dx dy \right| \\
&\| |u_h|^2 * |x|^{-1} \|_{L^3 L^{24}} \|u_h\|_{L^3 L^{\frac{8}{3}}} \|u_h\|^2_{L^\infty L^4} \|u_l\|_{L^3 L^{12}} \\
&\lesssim \varepsilon^{\frac{7}{3}} \eta^{\frac{4}{3}}.
\end{aligned} \tag{6.8.36}$$

This finishes the analysis of the second error term in the momentum bracket (6.8.26), namely $\mathcal{R}$. Now we turn to the third error term arising from (6.8.26), i.e.

$$\tilde{\mathcal{R}} = \sum_{k=0}^{3} \int_J \int_{\mathbb{R}^{2n}} |u_h(s,y)|^2 \frac{(x-y)_j}{|x-y|} \{P_{<1} \mathcal{O}\left(u_h^k u_l^{3-k}\right), u_h\}_p^j(s,x) dx dy ds.$$

We treat the term $k=0$ using (6.8.27) as follows. First, we get that

$$\begin{aligned}
&\left|\int_J\int_{\mathbb{R}^n}|u_h(y)|^2\int_{\mathbb{R}^n}u_h\frac{(x-y)_j}{|x-y|}\partial_j\left(P_{<1}\mathcal{O}\left(u_l^3\right)\right)(x)dxdyds\right| \\
&=\left|\int_J\int_{\mathbb{R}^n}|u_h(y)|^2\int_{\mathbb{R}^n}\left(|\nabla|^{-1}u_h\right)\left(|\nabla|\left(\frac{(x-y)_j}{|x-y|}\partial_jP_{<1}\mathcal{O}\left(u_l^3\right)\right)\right)dxdyds\right| \\
&\leq\int_J\int_{\mathbb{R}^n}|u_h(y)|^2\||\nabla|^{-1}u_h\|_{L^{\frac{8}{3}}}\||\nabla|\left(\frac{(x-y)_j}{|x-y|}\partial_jP_{<1}\mathcal{O}\left(u_l^3\right)\right)\|_{L^{\frac{8}{5}}}dsdy.
\end{aligned} \tag{6.8.37}$$

Using the boundedness of the Riesz transform, we see that

$$\begin{aligned}
&\||\nabla|\left(\frac{(x-y)_j}{|x-y|}\partial_jP_{<1}\mathcal{O}\left(u_l^3\right)\right)\|_{L^{\frac{8}{5}}} \\
&\lesssim\|\nabla\left(\frac{(x-y)_j}{|x-y|}\partial_jP_{<1}\mathcal{O}\left(u_l^3\right)\right)\|_{L^{\frac{8}{5}}} \\
&\lesssim\|\mathbb{1}_{\{|x-y|\leq1\}}|x-y|^{-1}\|_{L^2}\|\partial_ju_l\|_{L^8}\|u_l\|_{L^\infty}^2 \\
&+\|\mathbb{1}_{\{|x-y|\geq1\}}|x-y|^{-1}\|_{L^\infty}\|\partial_ju_l\|_{L^8}\|u_l\|_{L^4}^2 \\
&+\|\nabla\partial_ju_l\|_{L^8}\|u_l\|_{L^4}^2+\|\nabla u_l\|_{L^8}\|\nabla u_l\|_{L^4}\|u_l\|_{L^4} \\
&\lesssim\|u_l\|_{L^4}^2\|\nabla u_l\|_{L^8},
\end{aligned} \tag{6.8.38}$$

and, consequently, using the Bernstein's properties (6.2.5), (6.8.3), (6.8.5) and (6.8.38) above, we obtain that

$$\begin{aligned}
&\left|\int_J\int_{\mathbb{R}^n}|u_h(y)|^2\left(\int_{\mathbb{R}^n}u_h(x)\frac{(x-y)_j}{|x-y|}\partial_j\left(P_{<1}|u_l|^2u_l\right)(x)dx\right)dyds\right| \\
&\lesssim\int_J\int_{\mathbb{R}^n}|u_h(y)|^2\||\nabla|^{-1}u_h\|_{L^{\frac{8}{3}}}\|\nabla u_l\|_{L^8}\|u_l\|_{L^4}^2dyds \\
&\lesssim\|u_h\|_{L^\infty L^2}^2\||\nabla|^{-\frac{1}{2}}u_h\|_{L^2L^{\frac{8}{3}}}\|\nabla u_l\|_{L^2L^8}\|u_l\|_{L^\infty L^4}^2 \\
&\lesssim\eta\varepsilon^5.
\end{aligned} \tag{6.8.39}$$

As for the other part, using (6.8.5) and (6.8.17), we get that

$$\begin{aligned}
&\left|\int_J\int_{\mathbb{R}^{2n}}\frac{|u_h(y)|^2}{|x-y|}P_{<1}\mathcal{O}\left(u_l^3\right)(x)u_h(x)dxdyds\right| \\
&\lesssim\||u_h|^2*|x|^{-1}\|_{L^3L^{24}}\|u_h\|_{L^3L^{\frac{8}{3}}}\|u_l\|_{L^3L^{12}}\|u_l\|_{L^\infty L^4}^2 \\
&\lesssim\varepsilon^{\frac{16}{3}}\eta^{\frac{4}{3}}.
\end{aligned} \tag{6.8.40}$$

Now, we treat the case $k=1$ using Bernstein property (6.2.5), (6.8.3), (6.8.5) and (6.8.17) as follows. First we write that

$$\begin{aligned}
&\left|\int_J\int_{\mathbb{R}^{2n}}|u_h(y)|^2\frac{(x-y)_j}{|x-y|}u_h(x)\partial_j\left(P_{<1}\mathcal{O}\left(u_l^2u_h\right)\right)(x)dxdyds\right| \\
&\lesssim\|u_h\|_{L^\infty L^2}^2\|u_h\|_{L^3L^{\frac{8}{3}}}\|u_l^2u_h\|_{L^{\frac{3}{2}}L^{\frac{8}{5}}} \\
&\lesssim\|u_h\|_{L^\infty L^2}^2\|u_h\|_{L^3L^{\frac{8}{3}}}^2\|u_l\|_{L^6L^8}^2 \\
&\lesssim\varepsilon^4\eta^{\frac{4}{3}},
\end{aligned} \tag{6.8.41}$$

and then we write that

$$\begin{aligned}&\left|\int_J\int_{\mathbb{R}^{2n}}\frac{|u_h(y)|^2}{|x-y|}u_h(x)P_{<1}\mathcal{O}\left(u_l^2u_h\right)(x)dxdyds\right|\\&\lesssim \||u_h|^2*|x|^{-1}\|_{L^3L^{24}}\|u_h\|^2_{L^3L^{\frac{8}{3}}}\|u_l\|^2_{L^\infty L^{\frac{48}{5}}}\\&\lesssim \varepsilon^{\frac{10}{3}}\eta^2.\end{aligned}\tag{6.8.42}$$

When $k=2$, we use the Bernstein's properties (6.2.5), (6.8.3), (6.8.5), and (6.8.17) to get

$$\begin{aligned}&\left|\int_J\int_{\mathbb{R}^{2n}}|u_h(y)|^2\frac{(x-y)_j}{|x-y|}u_h(x)\partial_j\left(P_{<1}\mathcal{O}\left(u_lu_h^2\right)\right)(x)dxdyds\right|\\&\lesssim \|u_h\|^2_{L^\infty L^2}\|u_h\|_{L^3L^{\frac{8}{3}}}\|\partial_jP_{<1}\mathcal{O}\left(u_lu_h^2\right)\|_{L^{\frac{3}{2}}L^{\frac{8}{5}}}\\&\lesssim \|u_h\|^2_{L^\infty L^2}\|u_h\|_{L^3L^{\frac{8}{3}}}\|u_h^2u_l\|_{L^{\frac{3}{2}}L^{\frac{8}{6}}}\\&\lesssim \|u_h\|^2_{L^\infty L^2}\|u_h\|^3_{L^3L^{\frac{8}{3}}}\|u_l\|_{L^\infty L^\infty}\\&\lesssim \varepsilon^3\eta^2,\end{aligned}\tag{6.8.43}$$

and

$$\begin{aligned}&\left|\int_J\int_{\mathbb{R}^{2n}}\frac{|u_h(y)|^2}{|x-y|}u_h(x)P_{<1}\mathcal{O}\left(u_lu_h^2\right)(x)dxdyds\right|\\&\lesssim \||u_h|^2*|x|^{-1}\|_{L^3L^{24}}\|u_h\|_{L^3L^{\frac{8}{3}}}\|P_{<1}\mathcal{O}\left(u_lu_h^2\right)\|_{L^3L^{\frac{12}{7}}}\\&\lesssim \|u_h\|^2_{L^6L^{\frac{24}{11}}}\|u_h\|_{L^3L^{\frac{8}{3}}}\|P_{<1}\mathcal{O}\left(u_lu_h^2\right)\|_{L^3L^1}\\&\lesssim \|u_h\|^2_{L^6L^{\frac{24}{11}}}\|u_h\|^2_{L^3L^{\frac{8}{3}}}\|u_h\|_{L^\infty L^2}\|u_l\|_{L^\infty L^8}\\&\lesssim \varepsilon^{\frac{10}{3}}\eta^2.\end{aligned}\tag{6.8.44}$$

Finally, the case $k=3$ is treated as follows using the Bernstein's properties (6.2.5), (6.8.3), (6.8.5), and (6.8.17)

$$\begin{aligned}&\left|\int_J\int_{\mathbb{R}^{2n}}|u_h(y)|^2\frac{(x-y)_j}{|x-y|}u_h(x)\partial_j\left(P_{<1}\mathcal{O}\left(u_h^3\right)\right)(x)dxdyds\right|\\&\lesssim \|u_h\|^2_{L^\infty L^2}\|u_h\|_{L^3L^{\frac{8}{3}}}\|\nabla P_{<1}\mathcal{O}\left(u_h^3\right)\|_{L^{\frac{3}{2}}L^{\frac{8}{5}}}\\&\lesssim \|u_h\|^2_{L^\infty L^2}\|u_h\|_{L^3L^{\frac{8}{3}}}\|u_h^3\|_{L^{\frac{3}{2}}L^1}\\&\lesssim \|u_h\|^2_{L^\infty L^2}\|u_h\|_{L^3L^{\frac{8}{3}}}\|u_h\|^3_{L^{\frac{9}{2}}L^3}\\&\lesssim \varepsilon^2\eta^2\end{aligned}\tag{6.8.45}$$

and, similarly,

$$\begin{aligned}&\left|\int_J\int_{\mathbb{R}^{2n}}\frac{|u_h(y)|^2}{|x-y|}u_h(x)P_{<1}\mathcal{O}\left(u_h^3\right)(x)dxdyds\right|\\&\lesssim \||u_h|^2*|x|^{-1}\|_{L^3L^{24}}\|u_h\|_{L^3L^{\frac{8}{3}}}\|P_{<1}\mathcal{O}\left(u_h^3\right)\|_{L^3L^{\frac{12}{7}}}\\&\lesssim \|u_h\|_{L^6L^{\frac{24}{11}}}\|u_h\|_{L^3L^{\frac{8}{3}}}\|P_{<1}\mathcal{O}\left(u_h^3\right)\|_{L^3L^1}\\&\lesssim \|u_h\|^2_{L^6L^{\frac{24}{11}}}\|u_h\|^2_{L^3L^{\frac{8}{3}}}\|u_h\|^2_{L^\infty L^{\frac{16}{5}}}\\&\lesssim \varepsilon^{\frac{10}{3}}\eta^2.\end{aligned}\tag{6.8.46}$$

This finishes the analysis of $\tilde{\mathcal{R}}$. The first error term in (6.8.26) is now easy to treat. Indeed, integrating by parts,

$$\begin{aligned}
&\left|\int_J\int_{\mathbb{R}^{2n}} |u_h(s,y)|^2\frac{(x-y)}{|x-y|}\nabla\left(|u|^4-|u_l|^4-|u_h|^4\right)(s,x)dxdyds\right| \\
&\leq \sum_{k=1}^{3}\mathcal{O}\int_J\int_{\mathbb{R}^{2n}}\frac{|u_h(s,y)|^2|u_h(s,x)|^{4-k}|u_l(s,x)|^k}{|x-y|}dxdyds \\
&\lesssim (6.8.32)+(6.8.34)+(6.8.36) \\
&\lesssim \varepsilon^{\frac{7}{3}}\eta^{\frac{4}{3}}.
\end{aligned} \tag{6.8.47}$$

Finally, with (6.8.26)–(6.8.47), we obtain (6.8.25). As a consequence of (6.7.1), (6.8.18) and (6.8.25) on $J=J(2)$, we have that

$$\||\nabla|^{-\frac{3}{2}}|u_h|^2\|_{L^2L^2}^2 \lesssim \eta^2\left(\varepsilon^{\frac{7}{3}}\eta^{-\frac{2}{3}}+\varepsilon^2+\varepsilon^{\frac{16}{3}}\eta^{-\frac{4}{3}}\right)\leq \eta^2, \tag{6.8.48}$$

if $\varepsilon>0$ is sufficiently small, and $\eta>\varepsilon$. Letting $\eta=\varepsilon^{\frac{1}{2}}$, we obtain $J(2)\subset J(1)$. Finally, $J(1)$ is a closed, open nonempty subset of $\mathbb{R}$. Hence $J(1)=\mathbb{R}$, and this finishes the proof. □

It follows from Hölder's inequality that in the situation of Proposition 6.8.1, one also has the estimates (6.8.17) with $\eta=\varepsilon$.

6.9 The Soliton case

In this section, we deal with the first scenario in Theorem 18, namely the soliton case. We prove that the soliton scenario is inconsistent with the frequency-localized Morawtez interaction estimates developed in Section 6.7 and compactness up to rescaling.

Proposition 6.9.1. *Let $u\in C(\mathbb{R},\dot{H}^2)$ be a solution of* (6.1.1) *such that $K=\{u(t):t\in\mathbb{R}\}$ is precompact in $\dot{H}^2$ up to translation. If $n=8$, then $u=0$. In particular the soliton scenario in Theorem 18 does not hold true.*

Proof. Let $u\in C(\mathbb{R},\dot{H}^2)$ be a solution of (6.1.1) of energy $E(u)>0$ such that $K=\{g_{(1,y(t))}u(t):t\in\mathbb{R}\}$ is precompact in $\dot{H}^2$. In particular we can apply Proposition 6.8.1 with $\varepsilon>0$ and deduce that

$$\||\nabla|^{-\frac{3}{4}}P_{\geq 1}u\|_{L^4(\mathbb{R},L^4)}\lesssim 1. \tag{6.9.1}$$

Independently, by (6.8.1), we know that, for all t,

$$\|P_{\geq 1}u(t)\|_{\dot{H}^2}^2 \gtrsim_{E(u)} E(u)-\varepsilon^2>0, \tag{6.9.2}$$

if ε is sufficiently small. Then (6.9.2) implies that for all v in the $\dot{H}^2$-closure of K, $P_{\geq 1}v\neq 0$. Since K is precompact in $\dot{H}^2$ and the mapping $v\mapsto \||\nabla|^{-\frac{3}{4}}P_{\geq 1}v\|_{L^4}$ is continuous on $\dot{H}^2$, we get that there exists $\kappa>0$ such that

$$\forall v\in K,\ \||\nabla|^{-\frac{3}{4}}P_{\geq 1}v\|_{L^4}\geq\kappa. \tag{6.9.3}$$

Now, (6.9.1) and (6.9.3) imply that

$$\kappa^4 t\lesssim \||\nabla|^{-\frac{3}{4}}u_h\|_{L^4([0,t],L^4)}^4\lesssim 1. \tag{6.9.4}$$

Letting $t\to+\infty$, we get a contradiction in (6.9.4). This finishes the proof of Proposition 6.9.1. □

6.10 The Low-to-high cascade

Now, we are ready to deal with the last scenario, and to exclude the case of a low-to-high cascade solution. In order to do so, we use the estimates coming from the frequency-localized interaction Morawetz estimates developed in Section 6.7 to control the action of the high-frequency part of u. Then the low-frequency part obeys an analogue of (6.1.1) with initial data arbitrarily small. Hence one can make its $\dot{S}^2$-norm small, depending on the frequency, so as to prove that it is in fact small in L^2. Then the solution is an H^2 solution, and conservation of mass gives a contradiction. More precisely, we prove the following proposition.

Proposition 6.10.1. *Let $u \in C(I, \dot{H}^2)$ be a maximal lifespan solution of* (6.1.1) *such that $K = \{g_{(h(t),x(t))}u(t) : t \in I\}$ is precompact in $\dot{H}^2$ for some functions h, x such that $h(t) \leq h(0) = 1$, and*

$$\liminf_{t \to \sup I} h(t) = 0, \tag{6.10.1}$$

then if $n = 8$, we have that $u = 0$. In particular, the low-to-high cascade scenario does not hold true.

Proof. Let u be as above. Applying Proposition 6.6.1, we see that $I = \mathbb{R}$, and since $h \leq 1$, given $\varepsilon > 0$, we can apply Proposition 6.8.1 to get that (6.8.1) holds true. We may also suppose that (6.8.5) holds true. As a first step in the proof, we claim that if $\varepsilon > 0$ is suficiently small, the following holds true for all dyadic number $M \leq 1$:

$$\|P_{\leq M} u\|_{\dot{S}^2} \lesssim M^3. \tag{6.10.2}$$

Fix M_0, a dyadic number, let $m = M_0^{10}$ and let $\kappa > 0$ to be chosen later. Since we know that (6.10.1) holds true and that K is precompact, using (6.8.2) we get that there exists $t_0 > 0$ such that

$$\|P_{\leq 1} u(t_0)\|_{\dot{H}^2} \leq \kappa^2 m. \tag{6.10.3}$$

We claim that for any $C > 0$, if κ is sufficiently small, independently of m, then we have that, for all dyadic numbers $M \in [m, 1]$,

$$\|P_{\leq M} u\|_{\dot{S}^2(J)} \leq \kappa C \left(m + M^3\right) \tag{6.10.4}$$

when J is small. Indeed, using the Bernstein's properties (6.2.5), we get that, in J,

$$\begin{aligned}
\|P_{\leq M}u\|_{\dot{S}^2}^2 &\lesssim \sum_{N\leq M} N^4\|P_Nu\|_{L^\infty L^2}^2 + \sum_{N\leq M} N^6\|P_Nu\|_{L^2L^{\frac{8}{3}}}^2 \\
&\lesssim \sum_{N\leq M} N^4\|P_Nu(t_0)\|_{L^2}^2 + |J|^2\sum_{N\leq M} N^4\|\partial_t P_N\|_{L^\infty L^2}^2 \\
&+ |J|\sum_{N\leq M} N^6\|P_Nu\|_{L^\infty L^{\frac{8}{3}}}^2 \\
&\lesssim \kappa^4m^2 + |J|^2\sum_{N\leq M} N^4\|P_N\Delta^4u\|_{L^\infty L^2}^2 \\
&+ |J|^2\sum_{N\leq M} N^4\|P_N\left(|u|^2u\right)\|_{L^\infty L^2}^2 + |J|\sum_{N\leq M} N^8\|P_Nu\|_{L^\infty L^2}^2 \\
&\lesssim_{E(u)} \kappa^4m^2 + M^8|J|^2 + |J|^2\sum_{N\leq M} N^8\||u|^2u\|_{L^\infty L^{\frac{4}{3}}}^2 + |J|M^4 \\
&\lesssim_{E(u)} \kappa^4m^2 + M^8|J|^2 + M^4|J|,
\end{aligned}$$

and if $|J| \lesssim_{E(u),C} \kappa$, then (6.10.4) holds true. Now, let $J(C)$ be the maximum interval on which (6.10.4) holds true for the constant $C > 0$. We prove that $J(2) \subset J(1)$ if κ and ε are chosen sufficiently small, independently of m. Indeed, let

$$u_{vlow} = P_{\leq m}u \text{ , and } u_{med} = P_{m<\cdot<1}u.$$

In the following, all time integrals are taken on $J = J(2)$. Applying Strichartz estimates (6.3.3), we get that

$$\begin{aligned}
&\|P_{\leq M}u\|_{\dot{S}^2} \\
&\lesssim \|P_{\leq M}u(t_0)\|_{\dot{H}^2} + \|\nabla P_{\leq M}|u_{vlow}|^2u_{vlow}\|_{L^2L^{\frac{8}{5}}} \\
&+ \|\nabla P_{\leq M}\left(|u|^2u - |u_{vlow}|^2u_{vlow}\right)\|_{L^2L^{\frac{8}{5}}} \qquad (6.10.5)\\
&\lesssim \kappa m + \|P_{\leq m}u\|_{\dot{S}^2}^3 + M\|\tilde{P}_{\leq M}\left(|u_{vlow}|^2|u_{med}| + |u_{vlow}|^2|u_h|\right)\|_{L^2L^{\frac{8}{5}}} \\
&+ M\|\tilde{P}_{\leq M}|u_{med}|^3\|_{L^2L^{\frac{8}{5}}} + M\|\tilde{P}_{\leq M}|u_h|^3\|_{L^2L^{\frac{8}{5}}},
\end{aligned}$$

where $\tilde{P}_{\leq M}$ is the convolution operator whose kernel is

$$\tilde{k}(x) = M^8\hat{\psi}(Mx)^2,$$

where ψ is as in (6.2.4). We remark that $\tilde{P}_{\leq M}$ has nonnegative kernel and satisfies estimates similar to those of $P_{\leq M}$. In particular, (6.2.5) holds true with $\tilde{P}_{\leq M}$ in place of $P_{\leq M}$. By assumption we have that

$$\|P_{\leq m}u\|_{\dot{S}^2}^3 \leq (4\kappa)^3\, m^3. \qquad (6.10.6)$$

Besides, using the Bernstein's properties (6.2.5), and the assumption on J, we get that

$$\begin{aligned} M\|\tilde{P}_{\leq M}|u_{vlow}^2 u_{med}|\|_{L^2L^{\frac{8}{5}}} &\lesssim M\|u_{vlow}\|_{L^\infty L^4}^2 \|u_{med}\|_{L^2L^8} \\ &\lesssim M\left(4\kappa m\right)^2\left(\sum_{m<N<1} N^{-1}\||\nabla|P_N u\|_{L^2L^8}\right) \\ &\lesssim M\left(\kappa m\right)^2\left(\sum_{m<N<1} N^{-1}\|P_{\leq 2N}u\|_{\dot{S}^2}\right) \\ &\lesssim Mm^2\kappa^3\left(\sum_{m<N<1} N^{-1}\left(m+N^3\right)\right) \\ &\lesssim m^2\kappa^3 M \end{aligned} \tag{6.10.7}$$

and, similarly, using the Bernstein's properties, (6.2.5) and (6.8.17), we have that

$$\begin{aligned} M\|\tilde{P}_{\leq M}|u_{vlow}^2 u_h|\|_{L^2L^{\frac{8}{5}}} &\lesssim M^2\|u_{vlow}^2 u_h\|_{L^2L^{\frac{8}{6}}} \\ &\lesssim M^2\|u_{vlow}\|_{L^4L^8}^2\|u_h\|_{L^\infty L^2} \\ &\lesssim \kappa^2 M^2 m^2\varepsilon. \end{aligned} \tag{6.10.8}$$

Independently, using the Bernstein's properties (6.2.5) and the definition of J, we get that

$$\begin{aligned} M\|\tilde{P}_{\leq M}|u_{med}|^3\|_{L^2L^{\frac{8}{5}}} &\lesssim M^3\|u_{med}^3\|_{L^2L^{\frac{8}{7}}} \\ &\lesssim M^3\left(\sum_{m<N<1} N^{-1}\|\nabla P_N u\|_{L^6L^{\frac{24}{7}}}\right)^3 \\ &\lesssim M^3\left(\sum_{m<N<1} N^{-1}\|P_{\leq 2N}\|_{\dot{S}^2}\right)^3 \\ &\lesssim M^3\left(2\kappa\sum_{m<N\leq 1} N^{-1}m+N^2\right)^3 \\ &\lesssim \left(2\kappa\right)^3 M^3, \end{aligned} \tag{6.10.9}$$

and, using again the Bernstein's properties (6.2.5) and (6.8.1), we obtain that

$$\begin{aligned} M\|\tilde{P}_{\leq M}|u_h|^3\|_{L^2L^{\frac{8}{5}}} &\lesssim M^4\|u_h^3\|_{L^2L^1} \\ &\lesssim M^4\||\nabla|^{-\frac{1}{2}}u_h\|_{L^2L^{\frac{8}{3}}}\||\nabla|^{\frac{7}{4}}u_h\|_{L^\infty L^2}^2 \\ &\lesssim M^4\varepsilon^3. \end{aligned} \tag{6.10.10}$$

Finally, with (6.8.2)–(6.10.10), we get, if $\kappa=\varepsilon$ and ε is sufficiently small, that there holds that

$$\|P_{\leq M}u\|_{\dot{S}^2} \leq \kappa\left(m+M^3\right). \tag{6.10.11}$$

In particular, $J(2)\subset J(1)$. Consequently, $J(1)$ is a closed, open nonempty subset of $\mathbb{R}$. Hence $J(1)=\mathbb{R}$. Then (6.10.11) gives (6.10.2) for $M\in(M_0^{10},1)$.

Since M_0 can be chosen arbitrarily small, we get (6.10.2) for all $M \leq 1$. Now, we finish the proof of Proposition 6.10.1. A consequence of (6.10.2) is that $u \in L^\infty L^2$. Indeed, by the Bernstein's properties (6.2.5), $P_{\geq M} u \in L^\infty L^2$ for any dyadic M, and using (6.10.2), we get that, when $M \leq 1$,

$$\begin{aligned} \|P_{\leq M} u\|_{L^\infty L^2} &\leq \sum_{N \leq M} \|P_N u\|_{L^\infty L^2} \\ &\lesssim \sum_{N \leq M} N^{-2} \|P_{\leq N} u\|_{\dot{S}^2} \\ &\lesssim \sum_{N \leq M} N \lesssim M. \end{aligned} \tag{6.10.12}$$

Now, let $M > 0$ be an arbitrarily small dyadic number. Since (6.10.1) holds true, and since K is precompact in $\dot{H}^2$, we can find t_0 such that

$$\begin{aligned} \|P_{M < \cdot \leq M^{-1}} u(t_0)\|_{L^2} &\leq M^{-2} \|P_{M < \cdot \leq M^{-1}} u(t_0)\|_{\dot{H}^2} \\ &\leq M^{-2} \|P_{Mh(t_0) < \cdot \leq M^{-1} h(t_0)} \left(g(t_0) u(t_0)\right)\|_{\dot{H}^2} \\ &\leq M. \end{aligned} \tag{6.10.13}$$

Using conservation of mass, the Bernstein's properties (6.2.5), (6.10.12) and (6.10.13), we deduce that

$$\begin{aligned} \|u(0)\|_{L^2} &= \|u(t_0)\|_{L^2} \\ &\leq \|P_{> M^{-1}} u(t_0)\|_{L^2} + \|P_{M < \cdot \leq M^{-1}} u(t_0)\|_{L^2} + \|P_{\leq M} u\|_{L^\infty L^2} \\ &\leq M^2 E(u)^{\frac{1}{2}} + 2M. \end{aligned} \tag{6.10.14}$$

Since M is arbitrary, we get that $u(0) = 0$. This concludes the proof of Proposition 6.10.1. □

6.11 Analiticity of the flow map and scattering

In view of Theorem 17 and Corollary 6.5.1, we can finish the proof of Theorem 15 with Proposition 6.11.1 below.

Proposition 6.11.1. *Let $n \leq 8$. Then, for any $t > 0$, the mapping $u_0 \mapsto u(t)$, from H^2 into H^2, is analytic.*

Proof. We follow arguments developed in Pausader and Strauss [23] for the fourth-order wave equation. We use the implicit function theorem. In case $1 \leq n \leq 3$, the global bound on the energy gives a global bound on the L^∞-norm of u, and hence, the nonlinear term is lipschitz. In this case the problem can be solved with basic arguments. Now we treat the case $n \geq 4$. We divide $[0, t] = \cup_{j=1}^{k} I_j$ into subintervals $I_j = [a_j, a_{j+1}]$ such that

$$\|\nabla u\|_{L^{\frac{n+4}{2}}(I, L^{\frac{n(n+4)}{3n+4}})} \leq \delta. \tag{6.11.1}$$

First, if $I = I_0 = [0, a_1]$, we consider the mapping

$$\mathcal{T}_I : H^2 \times \dot{S}^0(I) \cap \dot{S}^2(I) \to H^2 \times \dot{S}^0(I) \cap \dot{S}^2(I)$$

defined by

$$\mathcal{T}(u_0, v) = \left(u_0, t \mapsto e^{it\Delta^2}u_0 + i\int_0^t e^{i(t-s)\Delta^2}|v|^2 v(s)ds\right).$$

The map $\mathcal{T}$ is well defined thanks to the Strichartz estimates (6.3.3). It is clearly analytic, and $u \in C(I, H^2)$ is a solution of (6.1.1) if and only if $\mathcal{T}(u(0), u) = (u(0), u)$. An application of Strichartz estimates gives that, if δ in (6.11.1) is sufficiently small, then

$$\|D_2\mathcal{T}(u(0), u)\|_{\dot{S}^0\cap\dot{S}^2\to\dot{S}^0\cap\dot{S}^2} < 1,$$

where D_2 denotes derivation with respect to the second argument. Consequently, $D_2(I - \mathcal{T})(u(0), u)$ is invertible, and the implicit function theorem ensures that $u_0 \mapsto u_{|I}$ is analytic. In particular, $u_0 \mapsto u(a_1)$, from H^2 into H^2, is analytic. By finite induction, we get that $u_0 \mapsto u(t)$ is analytic. □

Now, we turn to the proof of Theorem 16. The theorem is an easy consequence of Propositions 6.11.2 and 6.11.3 below.

Proposition 6.11.2. *Let $5 \le n \le 8$. For any $u^+ \in H^2$, respectively $u^- \in H^2$, there exists a unique $u \in C(\mathbb{R}, H^2)$, solution of (6.1.1) such that*

$$\|u(t) - e^{it\Delta^2}u^\pm\|_{H^2} \to 0 \tag{6.11.2}$$

as $t \to \pm\infty$. Besides, we have that

$$\begin{aligned} M(u(0)) &= M(u^\pm)\ ,\ \text{and} \\ 2E(u(0)) &= \|u^\pm\|^2_{\dot{H}^2}. \end{aligned} \tag{6.11.3}$$

This defines two mappings $\mathcal{W}_\pm : u^\pm \mapsto u(0)$ from H^2 into H^2, and $\mathcal{W}_+$ and $\mathcal{W}_-$ are continuous in H^2.

Proof. By time reversal symmetry, we need only to prove Proposition 6.11.2 for u^+. Let $\omega(t) = e^{it\Delta^2}u^+$. Then by the Strichartz estimates (6.3.3), $\omega \in \dot{S}^0(\mathbb{R}) \cap \dot{S}^2(\mathbb{R})$ and, given $\delta > 0$, there exists T_δ such that, on $I = [T_\delta, +\infty)$, (6.11.1) holds true with ω instead of u. For $u \in \dot{S}^0(I) \cap \dot{S}^2(I)$, we define

$$\Phi(u)(t) = \omega(t) - i\int_t^\infty e^{i(t-s)\Delta^2}|u(s)|^2u(s)ds. \tag{6.11.4}$$

For δ sufficiently small, Φ defines a contraction mapping on the set

$$\begin{aligned} X_{T_\delta} =&\{u \in \dot{S}^0(I) \cap \dot{S}^2(I); \|\nabla u\|_{L^{\frac{n+4}{2}}(I, L^{\frac{n(n+4)}{3n+4}})} \le 2\delta, \\ &\|u\|_{\dot{S}^0(I)} + \|u\|_{\dot{S}^2(I)} \lesssim \|u^+\|_{H^2}\}\ , \end{aligned}$$

equipped with the $\dot{S}^0(I)$-norm. Thus Φ admits a unique fixed point u. We observe that

$$u(T_\delta + t) = e^{it\Delta^2}u(T_\delta) + i\int_{T_\delta}^{T_\delta+t} e^{i(t-s)\Delta^2}|u(s)|^2u(s)ds$$

in H^2. Consequently, u solves (6.1.1) on $I = [T_\delta, +\infty)$. Hence, using Theorem 15 u can be extended for all times $t \in \mathbb{R}$. Now, (6.11.2) follows from (6.11.4) and the boundedness of u in $\dot{S}^2$ and $\dot{S}^0$-norms. Uniqueness follows from the fact that any solution of (6.1.1) has a restriction in X_T for some $T \geq T_\delta$, and uniqueness of the fixed point of Φ in such spaces. The continuity statements are easy adaptations of the proof of local well-posedness, see Pausader [21]. The first equality in (6.11.3) follows from conservation of Mass and convergence in L^2. For the second, we remark that since $\omega \in \dot{S}^0(\mathbb{R})$ there exists a sequence of times $t_k \to +\infty$ such that $\|\omega(t_k)\|_{L^4} \to 0$. Then, using conservation of energy, we compute

$$\begin{aligned} 2E(u(0)) &= 2E(u(t_k)) \\ &= 2E(\omega(t_k)) + o(1) \\ &= \|\omega(t_k)\|_{\dot{H}^2}^2 + o(1) = \|u^+\|_{\dot{H}^2}^2 + o(1), \end{aligned}$$

and letting $k \to +\infty$ we get that the second equation in (6.11.3) holds true. This finishes the proof of Proposition 6.11.2. □

Proposition 6.11.3. *Let* $5 \leq n \leq 8$. *Given any solution* $u \in C(\mathbb{R}, H^2)$ *of* (6.1.1)*, there exist* $u^\pm \in H^2$ *such that* (6.11.2) *holds true. In particular* $\mathcal{W}_\pm$ *are homeomorphisms of* H^2*.*

Proof. In case $5 \leq n \leq 7$, the equation is subcritical, and standard developments using the decay properties of the linear propagator, conservation of mass and the usual Morawetz estimates, give that for any solution $u \in C(\mathbb{R}, H^2)$ of (6.1.1), there exists $C > 0$ such that

$$\|u\|_{L^4(\mathbb{R}, L^4)} \leq C.$$

On such an assertion we refer to Cazenave [3] or Lin and Strauss [20] for the second order case, and to Pausader [21] for the classical Morawetz estimates in the case of the fourth-order Schrödinger equation. Consequently, applying Strichartz estimates, we get that

$$\|u\|_{\dot{S}^0(\mathbb{R})} + \|u\|_{\dot{S}^2(\mathbb{R})} \lesssim_u 1. \tag{6.11.5}$$

In case $n = 8$, as a consequence of Corollary 6.5.1, we get that any nonlinear solution u satisfies

$$\|u\|_{Z(\mathbb{R})} \lesssim_{E(u)} 1.$$

Using the work in Pausader [21, Proposition 2.6], we then get that (6.11.5) holds true also when $n = 8$. Since $e^{it\Delta^2}$ is an isometry on H^2, (6.11.2) is equivalent to proving that there exists $u^+ \in H^2$ such that

$$\|e^{-it\Delta^2}u(t) - u^+\|_{H^2} \to 0 \tag{6.11.6}$$

as $t \to +\infty$. Now we prove that $e^{-it\Delta^2}u(t)$ satisfies a Cauchy criterion. We note that Duhamel's formula gives that

$$e^{-it_1\Delta^2}u(t_1) - e^{-it_0\Delta^2}u(t_0) = i\int_{t_0}^{t_1} e^{-is\Delta^2}|u(s)|^2u(s)ds. \tag{6.11.7}$$

By duality, (6.3.3) gives that for any $s \in [0,2]$, and any $h \in \dot{\bar{S}}(\mathbb{R})$, we have that

$$\| \int_{\mathbb{R}} e^{-it\Delta^2} h(t)dt \|_{\dot{H}^s} \lesssim \|h\|_{\dot{\bar{S}}^s(\mathbb{R})}. \tag{6.11.8}$$

Now, (6.11.5) and (6.11.8) give that the right hand side in (6.11.7) is like $o(1)$ in H^2 as $t_0, t_1 \to +\infty$. In particular, $e^{-it\Delta^2}u(t)$ satisfies a Cauchy criterion, and there exists $u^+ \in H^2$ such that (6.11.6) holds true. We also get that

$$u^+ = u_0 + i \int_0^\infty e^{-is\Delta^2} |u(s)|^2 u(s) ds, \tag{6.11.9}$$

and u^+ is unique. The continuity statements are easy adaptations of the proof of local well-posedness, see Pausader [21]. Now, by uniqueness, we clearly have that $u(0) = \mathcal{W}_+(u^+)$, so that $\mathcal{W}_+$ is an homeomorphism. This ends the proof of Proposition 6.11.3. □

Proof of Theorem 16. Applying Propositions 6.11.2 and 6.11.3, we see that the scattering operator $S = \mathcal{W}_+ \circ \mathcal{W}_-^{-1}$ is an homeomorphism from H^2 into H^2. Using (6.11.4) and (6.11.9), and adapting slightly the proof of Proposition 6.11.1, we easily see that S is analytic. This ends the proof of Theorem 16. □

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